

Faculty of physics

# Theoretical particle physics

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# Contents


**1 Symmetries** **5**
- 1.1 Spacetime symmetries . . . 5
- 1.2 Internal symmetries . . . 9
  - 1.2.1 Spinors . . . 10
  - 1.2.2 Finding irreducible representations . . . 11
  - 1.2.3 Spinor Lagrangians . . . 13
  - 1.2.4 Massless spin-1 fields . . . 14
- 1.3 Lie groups . . . 16
- 1.4 Discrete symmetries . . . 19
  - 1.4.1 Parity transformation . . . 19
  - 1.4.2 Charge conjugation . . . 21
  - 1.4.3 Time reversal . . . 23
- 1.5 Gauge symmetry . . . 25
  - 1.5.1 Yang-Mills Lagrangian . . . 28

**2 Spontaneous symmetry breaking** **31**
- 2.1 Classical symmetry breaking . . . 31
- 2.2 Effective action . . . 37
- 2.3 Quantum Goldstone theorem . . . 40
- 2.4 General properties of the spontaneously broken symmetries . . . 43
- 2.5 The Anderson-Higgs mechanism . . . 44
- 2.6 Higgs' mechanism in non-Abelian gauge theory . . . 46
- 2.7 Coleman-Weinberg model . . . 49

**3 The strong interaction** **52**
- 3.1 Quantum chromodynamics . . . 52
- 3.2 Global symmetries of QCD . . . 55
- 3.3 Effective chiral Lagrangian . . . 59
- 3.4 Explicit breaking of chiral symmetry . . . 61

**4 Leptonic standard model** **68**
- 4.1 Fermi theory . . . 68
- 4.2 Gauge theory of weak interaction . . . 70
  - 4.2.1 Georgi-Glashow model . . . 70
  - 4.2.2 Glashow-Salam-Weinberg model . . . 71
  - 4.2.3 Coupling to fermions . . . 73
- 4.3 Masses for the matter fields and flavour mixing . . . 76

4.4 GIM mechanism . . . . . 82

**5 Beyond the Standard Model** **84**
5.1 Grand unified theories . . . . . 84
5.2 Neutrinos . . . . . 88
5.3 Outlook . . . . . 91

**Index** **94**

**Disclaimer:** these notes are not official and were not endorsed by professor. The content is almost entirely based on the lectures, but occasionally the notes were modified or extended slightly to improve clarity or add extra information. In particular, the mistakes are almost surely mine. If you find any, please let me know at simon.bukovsek@tum.de or Simon.Bukovsek@campus.lmu.de.

# Overview

The goal of this course is to understand the Standard model of particle physics.

Together with general relativity, it explains all known physics that we can test in laboratories. There is additional physics needed once we turn to astrophysics and the history of the universe (dark matter, matter-antimatter asymmetry, inflation ...), but apart from small anomalies it is a remarkable success.

Proposed structure of the course:

- **symmetries** (spacetime, discrete, gauge, global),
- **symmetry breaking** (Higgs, superconductors, non-Abelian Higgs),
- **strong force and EFT** (chiral perturbation theory, $\theta$-term),
- **anomalies** (anomaly matching),
- **electroweak sector**,
- **flavour physics**,
- **neutrino sector**.

All these topics are part of the Standard model of particle physics, the most accurate and most tested theory on physics. However, cosmological observations suggest that the Standard model is not the final theory. There are the dark matter, dark energy, cosmic inflation, matter-antimatter asymmetry, all of which cannot be adequately explained by the Standard model.

We will use outstanding SM puzzles as an introduction to BSM (beyond Standard model) ideas, providing occasional glimpses of concrete models.

# Standard model of particle physics

At the tree level, the "cartoon" version of the Standard model Lagrangian up to and including dimension 4 is

$$\mathcal{L} = \underbrace{-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}}_{\text{gauge}} + \underbrace{\overline{\psi}^i_{\mathrm{L}}\not{\mathrm{D}}\psi^i_{\mathrm{L}} + \overline{\psi}^j_{\mathrm{R}}\not{\mathrm{D}}\psi^j_{\mathrm{R}}}_{\text{matter}} + \underbrace{|\mathrm{D}_\mu\phi|^2 + \mu^2|\phi|^2 - \frac{\lambda}{4!}|\phi|^4}_{\text{Higgs/scalar}}$$
$$+ \underbrace{Y_u^{ij}\overline{Q}^i_{\mathrm{L}}\phi u^j_{\mathrm{R}} + Y_d^{ij}\overline{Q}^i_{\mathrm{L}}\phi^c d^j_{\mathrm{R}}}_{\text{Yukawa interaction (flavour)}} + (\text{neutrino masses}).$$

There is only one non-dimensionless parameter in the Lagrangian, the Higgs mass $\mu \approx 100\,\mathrm{GeV}$. When we include the loop corrections, more non-dimensionless parameters enter the game. At this point in the lecture, we don't need to understand any of the terms.

Standard model is a non-Abelian gauge theory with the symmetry group

$$G = \mathrm{SU}(3)_c \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}.$$

But this is far from the whole story. The SU(3) symmetry corresponds to a force that is so strong, that it is confined (isn't long range) and the other two s undergo a spontaneous symmetry breaking (SSB):

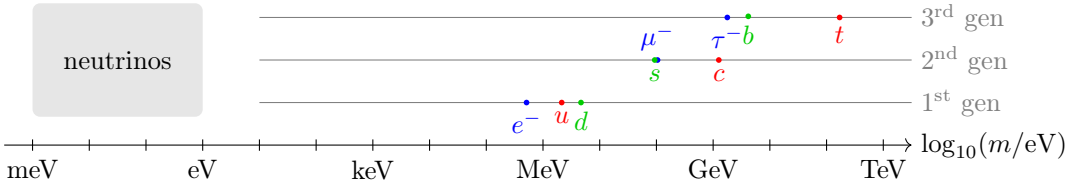
$$\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}} \xrightarrow{\text{SBB}} \mathrm{U}(1)_{\mathrm{em}}.$$

The electroweak gauge bosons are also screened and the remaining $\mathrm{U}(1)_{\mathrm{em}}$ is the only long-range interaction left.

There are three generations of chiral matter. The gauge forces interact with 15 Weyl fermions (per generation). For the first generation these are the electron (both chiralities), left-handed electron neutrino, up quark and down quark (both in two chiralities and three colours):

$$\begin{array}{rl} \text{leptons} & e_{\mathrm{L}}, e_{\mathrm{R}}, \nu_{\mathrm{L}} \\ \text{quarks} & d^i_{\mathrm{L}}, d^i_{\mathrm{R}}, u^i_{\mathrm{L}}, u^i_{\mathrm{R}} \text{ (colours } i = 1, 2, 3.) \end{array}$$

We need all 15 Weyl fermions because of the gauge redundancy (i.e. the possibility of a unitary, local, Lorentz-invariant theory of massless spin-1). Otherwise, anomalies occur. In total, it hence seems that there are 45 different fermions in three different generations.




Why is this pattern? Why does the order of masses change from generation to generation? And why are there exactly three generations?

There is also an additional scalar interaction (Higgs), that should be simple because the spin is zero. But it is also complicated, because of the lack of gauge restrictions. Gauge interactions are fixed by the representation and there is one gauge coupling for all fields. For scalar sector, however, we have a lot of free parameters. Each of $Y_u^{ij}$,$Y_d^{ij}$,$Y_\nu^{ij}$ and $Y_e^{ij}$ contains 18 parameters, though we can reduce the number of total physical parameters to 20 using some sleek group theory:

$$\begin{array}{rl} \text{quarks:} & 6 \text{ masses, } 3 \text{ mixing angles, } 1 \text{ CP phase,} \\ \text{leptons:} & 6 \text{ masses, } 3 \text{ mixing angles, } 1 \text{ CP phase,} \\ \text{Higgs:} & \lambda, \mu. \end{array}$$

Approximately 20 free parameters govern the scalar-matter interaction versus a single coupling constant for an Abelian gauge field.

Despite the seemingly overly-ornate structure, the standard model is astonishingly successful and similar to the periodic table. However, it holds the chess to what lies beyond.

# Chapter 1

# Symmetries

We can get far in understanding the standard model just by using symmetry arguments. We will talk about the following symmetries:

- *Poincaré symmetry*: Lorentz symmetry and translational symmetry. Particles and fields are the irreducible representations of the Poincaré group.
- Gauge redundancies: $A_\mu(x) \to A_\mu(x) - \partial_\mu \phi(x)$.
- Discrete symmetries: parity, time reversal, charge conjugation (none of which are respected by the standard model).
- Global symmetries: baryon, lepton number (probably accidental, broken at higher orders of perturbation theory).

## 1.1 Spacetime symmetries

We live in a spacetime with 3 spatial dimensions and one time dimension, $\mathbb{R}^{1,3}$ with the *Minkowski metric* $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The four vectors $x^\mu = (x_0, \mathbf{x})$ are transformed using *Lorentz transformations*

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu.$$

Such transformations leave expressions of type $x_\mu x_\nu \eta^{\mu\nu}$ invariant and obey the relation

$$\Lambda^T \eta \Lambda = \eta.$$

The Lorentz group has four connected components that are separated by the parity and time reversal transformations:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The part that is connected to the identity is called the *proper orthochronous Lorentz group* $\mathrm{SO}(1,3)$. For it, it holds $\det \Lambda = 1$ and $\Lambda^0{}_0 > 0$. Quantum mechanics allows for the projective representations (up to $\pm 1$):

$$\mathrm{SO}(1,3) \cong \mathrm{Spin}(1,3)/\mathbb{Z}_2.$$

The group $\mathrm{Spin}(1,3)$ is a double cover of the Lorentz group. They are locally the same, so they have the same Lie algebra $\mathfrak{so}(1,3)$. An element of a group can be expressed with its generators $J_{\mu\nu}$

$$\Lambda = \exp\left(-\frac{i}{2} w^{\mu\nu} J_{\mu\nu}\right).$$

These generators are the three boosts and three rotations, for example:

$$(J^{01})^{\rho}{}_{\sigma} = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (J^{12})^{\rho}{}_{\sigma} = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The first one is a boost in the $x^1$ direction, the second one is a rotation in the $x^1$-$x^2$ plane. The lie algebra that these generators form has the commutation relation

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\rho} J^{\mu\sigma} + \eta^{\nu\sigma} J^{\mu\rho} - \eta^{\mu\sigma} J^{\nu\rho}).$$

We will denote rotations and boosts with $J^i$ and $K^i$, respectively. It holds

$$J_i = \frac{1}{2}\varepsilon_{ijk} J_{ik}, \quad K_i = J_{0i}.$$

The rotations are Hermitian ($J_i^\dagger = J_i$) and the boosts are anti-Hermitian ($K_i^\dagger = -K_i$). These objects obey the following relations:

$$[J_i, K_j] = i\varepsilon_{ijk} J_k, \quad [J_i, K_j] = i\varepsilon_{ijk} K_k, \quad [K_i, K_j] = -i\varepsilon_{ijk} K_k.$$

The minus in the last expression might be a bit unexpected, but it makes sense because $\mathfrak{su}(2)/\mathbb{Z}_2 \cong \mathfrak{so}(3)$. Next, we introduce two new objects

$$A_i = \frac{1}{2}(J_i + iK_i), \qquad B_i = \frac{1}{2}(J_i - iK_i),$$

with commutation relations

$$[A_i, A_j] = i\varepsilon_{ijk} A_k, \qquad [B_i, B_j] = i\varepsilon_{ijk} B_k, \qquad [A_i, B_j] = 0.$$

These are two commuting $\mathfrak{su}(2)$ algebras. We will label each of them by $j \in \mathbb{Z}$, so each representation will be denoted by the tuple $(j_1, j_2)$ for $j_1, j_2 \in \{0, 1, 2, \dots\}$. The total spin of the particle is $j = j_1 + j_2$ of the representation. Here is a list of different representations and particles that they correspond to:

| $(j_1, j_2)$ | spin $j$ | interpretation | symbol |
|---|---|---|---|
| $(0,0)$ | 0 | scalar | $\phi$ |
| $(1/2,0)$ | 1/2 | LH Weyl spinor | $\psi_L$ |
| $(0,1/2)$ | 1/2 | RH Weyl spinor | $\psi_R$ |
| $(1/2,1/2)$ | 1 | vector | $A_\mu$ |
| $(1,0)$ | 1 | self-dual 2-form | $B^+_{\mu\nu}$ |
| $(0,1)$ | 1 | anti-self-dual 2-form | $B^-_{\mu\nu}$ |
| $(1,1)$ | 2 | graviton | $g_{\mu\nu}$ |

However, Lorentz group $\mathrm{SO}(1,3)$ is not isomorphic to $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$, but rather

$$\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$$

which then relates to the Euclidean rotation group as

$$\mathrm{Spin}(4)/\mathbb{Z}_2 \cong \mathrm{SO}(4).$$

The generators $A_i$ and $B_i$ are related through conjugation $(A_i)^* = -B_i$. The conjugation changes the order of the spin numbers in the representation notation

$$(j_1, j_2)^* = (j_2, j_1).$$

The symmetric (real) representations $(j, j)$ are not affected by conjugation, but the complex ones are:

$$(1/2, 0)^* \simeq (0, 1/2), \qquad \psi_\mathrm{L}^* \sim \psi_\mathrm{R}.$$

The Lorentz group can be expressed as

$$\mathrm{SO}(1,3) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)^*.$$

Now, we will add translations to the Lorentz group to get the Poincaré group. The generators of translations are $P_\mu$ together with the Lorentz generators they obey the commutation relations

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[J^{\mu\nu}, P^\sigma] &= i(\eta^{\nu\sigma} P^\mu - \eta^{\mu\sigma} P^\nu), \\
[J^{\mu\nu}, J^{\rho\sigma}] &= i(\eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\rho} J^{\mu\sigma} + \eta^{\nu\sigma} J^{\mu\rho} - \eta^{\mu\sigma} J^{\nu\rho}).
\end{aligned}$$

We want to classify the irreducible representations of the Poincaré group. Remember that SU(2) and SO(3) have the same Lie algebra ($\mathfrak{so}(3) \cong \mathfrak{su}(2)$). Our aim is to find *Casimirs*, the operators that commute with all generators of the group. The representations are identified by the maximally commuting set of generator and Casimirs. For warm-up, consider the group SO(3). We know that

$$C = \sum_{i=1}^{3} J_i^2, \qquad [C, J_i] = 0$$

so $\mathbf{J}^2 \left|j, m\right\rangle = j(j+1) \left|j, m\right\rangle$. Each representation has $(2j+1)$ entries labelled by number $m$, $|m| \leq j$.

For Poincaré group, we have two Casimirs. The first one is related to the *Pauli-Lubanski vector* $W^\mu$ defined as

$$W^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma}$$

which is some sort of relativistic angular momentum. The commutation relations are

$$\begin{aligned}
[W_\mu, P_\nu] &= 0, \\
[J^{\mu\nu}, W^\rho] &= i(\eta^{\nu\rho} W^\mu - \eta^{\mu\rho} W^\nu), \\
[W_\mu, W_\nu] &= -i\varepsilon_{\mu\nu\rho\sigma} W^\rho W^\sigma
\end{aligned}$$

The last relation is quadratic, which means that the operators $W^\mu$ do not close to a Lie algebra on their own. The two Casimirs are the

$$C_1 = P^\mu P_\mu, \qquad C_2 = W_\mu W^\mu.$$

They commute with all generators of the Poincaré group, $[C_i, P_\nu] = [C_i, J_{\mu\nu}] = 0$. Next, we need to find the maximally commuting set of operators. We distinguish two cases:

- **massive particles**: $C_1 > 0$, $C_2 < 0$,
- **massless particles**: $C_1 = 0$, $C_2 = 0$.

In massive case, the representative momentum (rest frame) is $k^\mu = (m, 0, 0, 0)$, so $W^0 = 0$ and $W^i = -mJ^i$ (or equivalently $J_i = W_i/m$). With this, the commutators become $[W^i, W^j] = i\varepsilon^{ijk} W^k$ which correspond to the $\mathfrak{su}(2)$ algebra. The *little group* in massive case is hence SO(3).

The massless case is more interesting. There is no obvious rest frame, so we choose $k^\mu = (E, 0, 0, E)$. We have

$$W^\mu = E \begin{pmatrix} -J_{12} \\ J_{23} - J_{02} \\ J_{31} - J_{01} \\ J_{12} \end{pmatrix} = \begin{pmatrix} -J_3 \\ J_1 - K_2 \\ J_2 - K_1 \\ J_3 \end{pmatrix}.$$

The commutations are

$$\begin{aligned} [W_1, W_2] &= 0 \qquad \text{(translation)} \\ [W_3, W_1] &= iEW_2 \qquad \text{(rotation)} \\ [W_3, W_2] &= -iEW_1 \qquad \text{(rotation)} \end{aligned}$$

These commutation relations represent (weirdly enough) the isometries of the Euclidean plane $\mathbb{R}^2$. The little group is then ISO(2). Since spin is not a continuous parameter we require the trivial representation only

$$W_1 |h\rangle = W_2 |h\rangle = 0.$$

The value $h$ is the eigenvalue of the state under rotation along the direction of motion. It is called *helicity* and it holds

$$e^{i\theta J_3} |h\rangle = e^{i\theta h} |h\rangle, \quad e^{\theta \hat{\mathbf{p}} \cdot \mathbf{J}} |p_\mu, h\rangle = e^{i\theta h} |h\rangle.$$

The values of helicity are integers or half-integers $h = 0, 1/2, 1, 3/2, \ldots$. Since QFT is CPT invariant, the helicity transforms as

$$|h\rangle \xrightarrow{\text{CPT}} |-h\rangle.$$

For $h > 0$ the helicity requires reducible representations with $\{|h\rangle, |-h\rangle\}$.

Assume $W_\pm = W_1 \pm iW_2$ and that $W_1$ and $W_2$ are in non-trivial representations. We have

$$[W_+, W_-] = 0, \qquad [J_3, W_\pm] = \pm W_\pm.$$

This is called the *ladder algebra*. The Casimir is

$$C_2 = W_\mu W^\mu = W_+ W_- = W_- W_+ = w^2.$$

The operators act on the states as

$$W_\pm |k, h\rangle_w = w |k, h \pm 1\rangle, \qquad J_3 W_\pm |k, h\rangle_w = (W_\pm J_3 \pm W_\pm) |k, h\rangle_w = (h \pm 1) W_\pm |k, h\rangle.$$

The normalization of the states is given by the inner product. The state is $|\psi\rangle = W_+ |k, h\rangle_w$ and the inner product is

$$\langle \psi | \psi \rangle = \langle k, h | W_- W_+ | k, h \rangle_w = w^2 \langle k, h | k, h \rangle_w = w^2.$$

For $w = 0$ we have the standard massless representation with one helicity. For $w \neq 0$ we have the *continuous spin representations*, where helicity can be any integer and half-integer (not to be confused with the word "continuous", it refers to the parameter $w$). It is an ongoing topic in research, but it is not clear if they are physical. Do they even exist in nature? How do they couple to ordinary matter? Is it consistent?

## 1.2 Internal symmetries

We want to include the internal symmetries to the Lorentz and translational symmetries. One such internal symmetry is the global phase rotation:

$$\phi(x) \to e^{i\alpha}\phi(x), \quad \alpha \in \mathbb{R}.$$

Consequently, there is a conserved current $\partial_\mu j^\mu = 0$ and a conserved charge

$$Q = \int \mathrm{d}^3x\, j^0, \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = 0.$$

There is also the Ward identity $\partial_\mu \langle\Omega|T\{j_\mu(x)\}|\Omega\rangle = 0$. We want to know whether there are any restrictions on these internal symmetries. Infinitesimally, the transformation is given by

$$\phi(x) \to (1 + i\alpha + \dots)\phi(x) = \phi(x) + \delta\phi(x).$$

The generator is Lorentz scalar! The *Coleman-Mandula theorem* says that there cannot be internal symmetries' generators with space-time indices. So the total symmetry must be the most trivial combination between the internal and space-time symmetries

$$G_{\text{total}} = \text{Poincaré} \otimes \text{internal}.$$

A quick sketch of the proof for Coleman-Mandula theorem goes like this.

The first conclusion is that if the scattering $2 \to 2$ is trivial, the theory is free. For non-trivial $2 \to 2$ amplitudes $\mathcal{A}(s, \cos\theta)$, any extra non-Poincaré-invariant generator (requiring some other current, e.g. $Q_\mu$ to be conserved, leading to algebraic relation involving momenta) would fix $(\cos\theta, s)$ to a discrete set of values. But the analyticity of amplitudes would then imply that the remaining $2 \to 2$ variables vanish and the theory again free.

However, there are loopholes to this theorem. Firstly, maybe there is a theory with no $2 \to 2$ interaction. But this is not the only possible exception:

1. **Supersymmetry.** Coleman-Mandula theorem assumed bosonic generators. If fermionic (spin 1/2) generators are permitted, a restricted set of additional symmetries becomes possible:
$$\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2(\sigma^\mu)_{\alpha\dot\alpha}P_\mu,$$
where curly brackets denote the anti-commutator. Indices $\alpha$ and $\dot\alpha$ denote the elements of the spinor, so they run from 1 to 2. This is the simplest version of supersymmetry with $N = 1$. The parameter $N$ represents the number of independent supersymmetry transformations, and it can go up to $N = 8$ in four-dimensional spacetime, corresponding to the maximal supersymmetry.

2. **Conformal invariance.** The proof used a *mass gap*, which implies a scaleless theory. If the theory instead possesses a dilatation symmetry
$$x^\mu \to \lambda x^\mu, \qquad \lambda \in \mathbb{R},$$
one is led (under suitable assumptions) to the full conformal group, including special conformal transformations:
$$x^\mu \longrightarrow \frac{x^\mu - a^\mu x^2}{1 - 2\,a\cdot x + a^2x^2}, \qquad a^\mu \in \mathbb{R}^4.$$
The standard model becomes scale invariant at large energies at least at the tree level. But this scale invariance might be broken by ROE.

### 1.2.1 Spinors

Scalar particles transform under $(0,0)$ representation which is trivial, so we begin the discussion with the first non-trivial Lorentz irreducible representation – the *spinors*. A quick reminder from QFT 1. The spinors that transform under $(0,1/2)$ or $(1/2,0)$ are called *Weyl spinors*. We distinguish left- and right-handed spinors:

$$\psi_{\mathrm{L}}: \ (0,1/2), \qquad \psi_{\mathrm{R}}: \ (1/2,0).$$

They transform independently under Lorentz transformations as:

$$\psi_{\mathrm{L}} \longrightarrow S\psi_{\mathrm{L}}, \quad S = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right),$$

$$\psi_{\mathrm{R}} \longrightarrow S\psi_{\mathrm{R}}, \quad S = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}\right),$$

with the generators

$$\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^{\mu}\overline{\sigma}^{\nu} - \sigma^{\nu}\overline{\sigma}^{\mu}\right), \qquad \overline{\sigma}^{\mu\nu} = \frac{i}{4}\left(\overline{\sigma}^{\mu}\sigma^{\nu} - \overline{\sigma}^{\nu}\sigma^{\mu}\right).$$

Here, we used the four-vectors made from *Pauli matrices* $\sigma^{\mu} = (1, \boldsymbol{\sigma})$, $\overline{\sigma}^{\mu} = (1, -\boldsymbol{\sigma})$ with $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$. Weyl's spinors are not invariant under parity symmetry. If we require this symmetry, it exchanges the two Weyl irreducible representations, and we have to join them into one four-dimensional representation. We form the *Dirac spinor* in the *Weyl basis* as

$$\Psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix}.$$

Dirac's spinors transform under the representation

$$(1/2,0) \oplus (0,1/2).$$

In this basis, Lorentz transformations act block-diagonally on $\Psi$, and the usual $\gamma$-matrices take the chiral form

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \overline{\sigma}^{\mu} & 0 \end{pmatrix}.$$

Recall the anti-Hermitian Lorentz generators in the Weyl basis $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$. They obey the *Lorentz algebra*:

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = i\left(\eta^{\nu\rho}\sigma^{\mu\sigma} - \eta^{\mu\rho}\sigma^{\nu\sigma} + \eta^{\mu\sigma}\sigma^{\nu\rho} - \eta^{\nu\sigma}\sigma^{\mu\rho}\right).$$

Similar holds for $\overline{\sigma}^{\mu\nu}$. Because $\mathrm{tr}(\sigma^{\mu\nu}) = 0$, any finite transformation

$$S = \exp(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu})$$

has unit determinant ($\det S = 1$) and it lies in the special linear group, $S \in \mathrm{SL}(2,\mathbb{C})$. Moreover, $\mathrm{Spin}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$. Earlier we introduced the compact generators $A_i = (J_i + iK_i)/2$, and $B_i = (J_i - iK_i)/2$ with $(A_i)^* = -B_i$. Hence, the $(1/2,0)$ (left-handed) and $(0,1/2)$ (right-handed) representations are related by complex conjugation, but they are not identical. Namely, $(\sigma^{\mu\nu})^* \neq \overline{\sigma}^{\mu\nu}$ so one cannot obtain the RH representation simply by conjugating the LH generators in the same two-component space Instead, LH and RH spinors live in inequivalent, complex-conjugate representations of $\mathrm{Spin}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$.

Parity transformation, however, does interchange the two representations $\psi_\mathrm{L} \leftrightarrow \psi_\mathrm{R}$. In the Standard Model, parity is maximally violated in the weak interactions, so $\psi_\mathrm{L}$ and $\psi_\mathrm{R}$ couple differently.

In order to build Lagrangians out of spinors, we need to contract them into Lorentz scalars. To do this, we first notice a useful identity for the Pauli matrices

$$\varepsilon^T (\sigma^{\mu\nu})^* \, \varepsilon = \overline{\sigma}^{\mu\nu}, \qquad \varepsilon = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Secondly, we see that a left-handed Weyl spinor can be expanded as $\psi_\mathrm{L} \sim a + b^\dagger$, and its Hermitian conjugate carries the opposite chirality $\psi_\mathrm{L}^\dagger \sim b + a^\dagger$, i.e. if $\psi_\mathrm{L}$ creates a left-handed particle, then $\psi_\mathrm{L}^\dagger$ creates the corresponding right-handed particle.

The next step in achieving a general procedure to generate Lorentz scalars is to introduce the spinor index notation. Write

$$(\psi_\mathrm{L})_\alpha = \begin{pmatrix} (\psi_\mathrm{L})_1 \\ (\psi_\mathrm{L})_2 \end{pmatrix}, \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For the dot product between two spinors, define

$$\psi_\mathrm{L}\chi_\mathrm{L} \coloneqq \varepsilon^{\alpha\beta} (\psi_\mathrm{L})_\alpha (\chi_\mathrm{L})_\beta = (\psi_\mathrm{L})_2 (\chi_\mathrm{L})_1 - (\psi_\mathrm{L})_1 (\chi_\mathrm{L})_2,$$

which is antisymmetric in spinor space (because $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$). This product is also Lorentz invariant:

$$\psi_\mathrm{L}\chi_\mathrm{L} \longrightarrow (S\psi_\mathrm{L})(S\chi_\mathrm{L}) = S^\alpha{}_\rho S^\beta{}_\sigma \, \varepsilon^{\rho\sigma} \, (\psi_\mathrm{L})_\alpha (\chi_\mathrm{L})_\beta = \underbrace{\det S}_{=1} \, \varepsilon^{\alpha\beta} \, (\psi_\mathrm{L})_\alpha (\chi_\mathrm{L})_\beta = \psi_\mathrm{L}\chi_\mathrm{L},$$

because $\det S = 1$ for $S \in \mathrm{SL}(2, \mathbb{C})$.

The main take-away is that the antisymmetric $\varepsilon$-tensor allows us to contract two same-handed Weyl spinors into a Lorentz scalar, a building block ubiquitous in constructing Lorentz invariant Lagrangians.

### 1.2.2 Finding irreducible representations

We have seen that Weyl spinors transform under irreducible representations. But when we join both spinors into one Dirac spinor, we multiply those two representations together. The newly formed representation is no longer irreducible, and we would like to find a general way to split any representation in a direct sum of irreducible representations. In particular, for Dirac spinors it holds

$$(1/2, 0) \otimes (1/2, 0) = (0, 0) \oplus (1, 0),$$

i.e. the product of two left-handed Weyl spinors splits into a scalar and a self-dual antisymmetric tensor. The singlet $(0, 0)$ is Lorentz invariant. Building higher irreducible representations from tensor products follows two basic steps.

1. Separate symmetric and antisymmetric combinations of indices – these subspaces are preserved by the group action.
2. Subtract pieces that are themselves group invariants (constructed with invariant tensors such as $\varepsilon^{\alpha\beta}$ or $\delta_{ij}$).

The invariant tensors for different groups are listed in the table below.

| Group | Invariant tensor |
|---|---|
| $\mathrm{SO}(N)$ | $\delta_{ij}$ |
| $\mathrm{SO}(1,3)$ | $g_{\mu\nu}$ |
| $\mathrm{SU}(2)$ | $\varepsilon^{\alpha\beta}$ |
| $\mathrm{SU}(3)$ | $\varepsilon^{ijk}$ |
| $\mathrm{SU}(N)$ | $\varepsilon^{abc\ldots N}$ |

**Example 1.** Let's consider the group SO(3). The fundamental (vector) representation, $v_i$ with $i = 1, 2, 3$, transforms as

$$v_i \longrightarrow O_{ij} v_j, \qquad O^T O = \mathbb{1}, \ \det O = 1.$$

Next, consider a two-index (tensor) representation. Which irreducible representations does $T_{ij}$ belong to? Firstly, we apply step 1 and split it into symmetric and antisymmetric part

$$T_{ij} = S_{ij} + A_{ij}, \qquad S_{ij} \coloneqq \frac{1}{2}\left(T_{ij} + T_{ji}\right), \ A_{ij} \coloneqq \frac{1}{2}\left(T_{ij} - T_{ji}\right).$$

The antisymmetric part $A_{ij}$ has three independent components and is isomorphic to the fundamental representation (via the Levi-Civita tensor). The symmetric part $S_{ij}$ still contains the invariant trace $S_{ii}$. Remove it according to step 2:

$$S_{ij} = \left(S_{ij} - \frac{1}{3}\delta_{ij} S_{kk}\right) + \left(\frac{1}{3}\delta_{ij} S_{kk}\right),$$

giving a 5-dimensional traceless symmetric irreducible representation and a singlet. For SO(3) the invariant rank-2 tensor is the *Kronecker delta* $\delta_{ij}$, satisfying

$$\delta_{ij} \longrightarrow O^T \delta_{ij} O = O^T O = \delta_{ij}.$$

Combining our previous result we can rewrite the two-index tensor as

$$T_{ij} = \underbrace{\left(\delta_{kl} T_{kl}/3\right)\delta_{ij}}_{\mathbf{1}} + \underbrace{\left(S_{ij} - \left(\operatorname{tr} S\right)\delta_{ij}/3\right)}_{\mathbf{5}} + \underbrace{A_{ij}}_{\mathbf{3}}.$$

Using the notation of direct sums and products, the equation above reads

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1}_{\mathrm{S}} \oplus \mathbf{5}_{\mathrm{S}} \oplus \mathbf{3}_{\mathrm{A}},$$

where A and S denote the symmetric and antisymmetric part. One should recognize this counting from the multipole expansion in electrodynamics (monopole: 1 parameter, dipole: 3 parameters, quadrupole: 5 parameters). Why does it appear here? Same dimensions, same group theory. All we're doing – whether you call it "multipole moments" or "irreducible tensors" – is decomposing a function or object into pieces that transform irreducibly under rotations. The numbers match because they must match: SO(3) has only one irreducible representation of each odd dimension, and anything built from vectors will break up into that set.

These decomposition techniques generalize to any Lie group and underpin the classification of operator multiplets in both particle physics and conformal field theory.

For homework try to find the decomposition the $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ of SO(3) into irreducible representations.

We can use these findings and apply them to the SU(2) group. In the Weyl (two-component) formalism the invariant tensor is

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon^{\alpha\beta} \longrightarrow S^{\alpha}{}_{\rho} S^{\beta}{}_{\sigma} \varepsilon^{\rho\sigma} = (\det S)\varepsilon^{\alpha\beta} = \varepsilon^{\alpha\beta}.$$

This dovetails neatly with the antisymmetry of *Grassmann* fields:

$$\psi_{\mathrm{L}}\chi_{\mathrm{L}} = (\psi_{\mathrm{L}})_{\alpha}\varepsilon^{\alpha\beta}(\chi_{\mathrm{L}})_{\beta} = \chi_{\mathrm{L}}\psi_{\mathrm{L}}, \quad \text{(fermionic sign)},$$

$$\psi_{\mathrm{L}}\chi_{\mathrm{L}} = (\psi_{\mathrm{L}})_2(\chi_{\mathrm{L}})_1 - (\psi_{\mathrm{L}})_1(\chi_{\mathrm{L}})_2 = -(\chi_{\mathrm{L}})_1(\psi_{\mathrm{L}})_2 + (\chi_{\mathrm{L}})_2(\psi_{\mathrm{L}})_1 = \chi_{\mathrm{L}}\psi_{\mathrm{L}}.$$

Thus, invariant spinor bilinears are automatically antisymmetric and perfectly compatible with the Grassmann nature of quantum-field operators.

### 1.2.3 Spinor Lagrangians

For details see any QFT textbook, but let us recall the essentials. The free *Dirac equation* reads

$$(i\gamma^{\mu}\partial_{\mu} - M)\Psi = 0,$$

and follows from the action[1]

$$S_{\mathrm{Dirac}} = \int \mathrm{d}^4x \, \left(i\overline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - M\overline{\Psi}\Psi\right), \qquad \overline{\Psi} \equiv \Psi^{\dagger}\gamma^0.$$

Using the Weyl decomposition we can write the four-component spinor in the chiral form

$$\Psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix}, \qquad \overline{\Psi} = \left(\overline{\psi}_{\mathrm{L}}, \, \overline{\psi}_{\mathrm{R}}\right), \quad \overline{\psi}_{\mathrm{L}} \equiv \psi_{\mathrm{L}}^{\dagger}, \; \overline{\psi}_{\mathrm{R}} \equiv \psi_{\mathrm{R}}^{\dagger}.$$

Using the Weyl representation for $\gamma^{\mu}$ one obtains

$$S_{\mathrm{Dirac}} = \int \mathrm{d}^4x \, \left(i\overline{\psi}_{\mathrm{L}}\overline{\sigma}^{\mu}\partial_{\mu}\psi_{\mathrm{L}} + i\overline{\psi}_{\mathrm{R}}\sigma^{\mu}\partial_{\mu}\psi_{\mathrm{R}} - M\left(\overline{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} + \overline{\psi}_{\mathrm{L}}\psi_{\mathrm{R}}\right)\right).$$

Let's consider global internal symmetries of this action.

- For $M \neq 0$, the action is invariant under a single phase rotation

  $$\Psi \longrightarrow e^{i\alpha}\Psi, \qquad \mathrm{U}(1)_{\mathrm{L+R}}.$$

- In the massless limit ($M \to 0$) the left- and right-handed spinors decouple, enlarging the symmetry to

  $$\psi_{\mathrm{L}} \to e^{i\alpha}\psi_{\mathrm{L}}, \qquad \psi_{\mathrm{R}} \to e^{i\beta}\psi_{\mathrm{R}}, \qquad \mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}.$$

  This is now a *chiral symmetry*, as left- and right-handed spinors might have different charges. Turning the mass back on forces $\alpha = \beta$. In this sense, mass is a *spurion*[2] of chiral symmetry breaking. Imagine that mass transformed as $M_0 \longrightarrow e^{i(\beta-\alpha)}M_0$ (as a non-dynamical scalar field). Then the simultaneous transformation of $\psi_{\mathrm{L}}$, $\psi_{\mathrm{R}}$ and $M_0$ would conserve the chiral symmetry.

[1] $M \in \mathbb{R}$ to ensure Hermiticity and a positive Hamiltonian.

[2] Definition from Wikipedia: a spurion is a fictitious, auxiliary field in a quantum field theory that can be used to parameterize any symmetry breaking and to determine all operators invariant under the symmetry.

A purely left-handed Weyl field can acquire a mass even if no right-handed partner exists. In two-component notation the action is

$$S_{\text{Majorana}} = \int \mathrm{d}^4x \left( i\bar{\psi}_{\mathrm{L}}\bar{\sigma}^\mu \partial_\mu \psi_{\mathrm{L}} + \frac{m}{2}\psi_{\mathrm{L}}^T \varepsilon \psi_{\mathrm{L}} + \frac{m^*}{2}\bar{\psi}_{\mathrm{L}}\varepsilon \bar{\psi}_{\mathrm{L}}^T \right), \qquad \varepsilon = i\sigma^2.$$

The kinetic term is the usual U(1)-invariant piece; the second and third terms are the *Majorana mass*. Because $\psi_{\mathrm{L}} \to e^{i\alpha}\psi_{\mathrm{L}}$ yields an overall phase $e^{2i\alpha}$ in the bilinear $\psi_{\mathrm{L}}^T\varepsilon\psi_{\mathrm{L}}$, any non-zero $m \in \mathbb{C}$ breaks the global U(1) symmetry (e.g. lepton number in the SM), whereas a Dirac mass conserves it.[3]

In the language of the representation theory, if $\psi_{\mathrm{L}}$ transforms in a representation $R$ of a group $G$,

$$\psi_{\mathrm{L}} \longrightarrow D_R(g)\psi_{\mathrm{L}}, \qquad g \in G,$$

then the Majorana term exists if and only if $R$ is *real* or *pseudo-real*. The criterion is[4]

$$D_R(g)^* = S^{-1} D_R(g) S \quad \Longrightarrow \quad (S\varepsilon)^T = \pm(S\varepsilon).$$

The product $S\varepsilon$ is denoted by $\Gamma$. Let's see a few examples

| Representation type | example | $SD_R(g)^* \stackrel{?}{=} D_R(g)S$ | sign |
|---|---|---|---|
| real | **3** of SO(3) | yes | $S^T = +S$ |
| pseudo-real | **2** of SU(2) | yes | $S^T = -S$ |
| complex | **3** of SU(3) | no | |

Hence, for fields in complex reps a Majorana mass is forbidden.[5]

### 1.2.4 Massless spin-1 fields

Massless spin-1 particles (and in fact all higher-spin fields) must be described by redundant variables in order to maintain both Lorentz invariance and unitarity.[6]

The gauge potential $A_\mu(x)$ is not unique. Its action must be invariant under

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \alpha(x), \qquad \alpha(x) \text{ any smooth function.}$$

Hence, field $A_\mu$ cannot appear alone in the Lagrangian and not even in the contraction $A_\mu A^\mu$. However, we have shown in QFT 1 that the combination

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is gauge invariant. It is called the *field strength tensor*. The unique Lorentz- and gauge-invariant free action is the *Maxwell action*

$$S = -\frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu},$$

[3] For a single Weyl field the phase of $m$ can always be shifted into $\psi_{\mathrm{L}}$; one may therefore take $m \in \mathbb{R}^+$.

[4] $S$ is a similarity matrix, possibly depending on $R$; $^*$ denotes complex conjugation.

[5] This is why SM neutrinos (which are in a complex weak-isospin representation) cannot get a Majorana mass without introducing a gauge-singlet (right-handed) neutrino or via a higher-dimensional operator.

[6] See any advanced QFT course for the no-go theorems that motivate this statement.

leading to the equations of motion (in units where $c = 1$)

$$\partial_\mu F^{\mu\nu} = 0 \quad \Longleftrightarrow \quad \nabla \cdot \mathbf{E} = 0, \qquad \nabla \times \mathbf{B} - \dot{\mathbf{E}} = 0$$

The first equation on the right removes one parameter and the second one prescribes a time evolution. Two physical polarizations remain after quotienting by the gauge redundancy – exactly the helicity $\pm 1$ states of a massless spin-1 particle.

The main point of this discussion it that gauge redundancy is not an optional complication; it is the price we pay for a consistent, Lorentz-invariant description of massless vectors.

To make the theory more interesting, we can add a conserved current to which the vector field couples. The part in the Lagrangian has the form $j_\mu A^\mu$ and the equation of motion is $\partial_\mu F^{\mu\nu} = j^\nu$. We could obtain another tensor by multiplying the existing field strength tensor with the invariant tensor of $\mathrm{SO}(1,3) \cong \mathrm{SO}(4)$: this is the Levi-Civita tensor with 4 indices $\varepsilon^{\mu\nu\rho\sigma}$. What we get is also the Hodge dual of the field strength tensor:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}.$$

The Hodge dual changes $F_{\mu\nu} \to \tilde{F}_{\mu\nu}$, $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$. In the free theory, it still holds $\partial^\mu \tilde{F}_{\mu\nu} = 0$ (which is called the *Bianchi identity*), but it can also couple to a conserved current $\partial_\mu \tilde{F}^{\mu\nu} = j^\nu_{\mathrm{mag}}$.

Another astonishing result is that a commutator of covariant derivatives is no derivative at all, but rather

$$[\mathrm{D}_\mu, \mathrm{D}_\nu] = -ieF_{\mu\nu}.$$

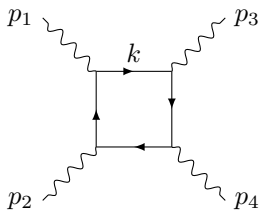
With the help of *Jacobi identity*

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$$

we can show

$$[\mathrm{D}^\mu, [\mathrm{D}^\nu, \mathrm{D}^\lambda]] = 0 \quad \Longleftrightarrow \quad \varepsilon^{\mu\nu\sigma\rho}\partial_\nu F_{\sigma\rho} = 0.$$

As mentioned before, gauge redundancy prohibits terms such as $A_\mu A^\mu$ in the action, and it also makes certain diagrams more convergent. Consider, for example, the following diagram.




The superficial degree of divergence of this diagram is 0, so it is at most logarithmically divergent. But when we consider what type of term this diagram would correspond to, we find that the only option is $(F_{\mu\nu}F^{\mu\nu})^2 = 4(\mathbf{B}^2 - \mathbf{E}^2)^2$ (there could also be $(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 = 16(\mathbf{B}\cdot\mathbf{E})^2$). With these terms, the degree of divergence is actually $-4$ which is very much convergent.

Consider a naive term in Lagrangian $\overline{\Psi}\gamma_\mu\Psi A^\mu$. This is not gauge invariant and hence not allowed. In order to gauge the matter Lagrangians we start with the simplest case: Abelian U(1) invariant Lagrangian and postulate a global U(1) symmetry to a local U(1) symmetry,

$\psi(x) \to e^{i\alpha(x)q}\phi(x)$. The field $\alpha(x)$ is real and $q$ represents the charge of the field $\psi(x)$ under the U(1) symmetry. We could have different fields with different charges, which is very common in the Standard Model:

$$\psi_1 \longrightarrow e^{iq_1\alpha}\phi_1, \qquad \psi_2 \longrightarrow e^{iq_2\alpha}\phi_2.$$

In the case $q_1 = 2$, $q_2 = 1$ the Lagrangian term could look something like $\mu\phi_1^\dagger\phi_2^2 + \text{h.c.}$ To enforce the gauge invariance to the kinetic term, we change the normal derivative $\partial_\mu$ to the *covariant derivative* $\mathrm{D}_\mu$:

$$\mathrm{D}_\mu\phi_i(x) = (\partial_\mu - iq_iA_\mu(x))\phi_i(x).$$

This product transforms nicely with the symmetry transformations (as we have shown in QFT1):

$$\mathrm{D}_\mu\phi_i(x) \xrightarrow{\mathrm{U}(1)} e^{iq_i\alpha(x)}\phi_i(x).$$

The Hermitian conjugate is $(\mathrm{D}_\mu\phi_i(x))^\dagger = (\partial_\mu + iq_iA_\mu)\phi_i^\dagger(x)$ and the term transforms with opposite sign in the exponent. A gauge redundant term in the Lagrangian is

$$S = \int \mathrm{d}^4x\ (\mathrm{D}^\mu\phi)^\dagger\mathrm{D}_\mu\phi + V(\phi^\dagger\phi).$$

It works exactly the same way for spinors: $\psi_\mathrm{L} \to e^{i\alpha(x)q}\psi_\mathrm{L}$ and a possible Lagrangian is

$$\mathcal{L} = i\overline{\psi}_\mathrm{L}\overline{\sigma}^\mu\mathrm{D}_\mu\psi_\mathrm{L}.$$

## 1.3 Lie groups

A *Lie group* is a continuous group, characterized by a set of continuous parameters. Lie groups are also smooth manifolds. For example, $\mathrm{U}(1) \cong S^1$ and $\mathrm{SU}(2) \cong S^3$. All the elements of a (simple) Lie group are continuously connected to the identity element $\mathbb{1}$. Any element $\mathcal{U}$ of the group $G$ can be expressed as

$$\mathcal{U} = e^{i\theta^aT^a} \approx \mathbb{1} + i\theta^aT^a, \qquad \theta^a \in \mathbb{R}^{\dim G}.$$

The objects $T^a$ are the *generators* of the group. Locally, close to the identity, the group is isomorphic to the *Lie algebra* $\mathfrak{g}$ of the group. The commutation relations of the generators are given by the *structure constants* $f^{abc}$:

$$[T^a, T^b] = if^{abc}T^c.$$

Two groups that have the same Lie algebra have the same local behaviour, but they need not be globally isomorphic. For example, SU(2) and SO(3) have the same Lie algebra, but they are not isomorphic as groups because one is a double cover of the other.

Let's quickly remind ourselves of the basic properties of groups. A group $G$ is a set of elements with a binary operation $\cdot$ such that the following properties hold:

- **Closure:** For any two elements $g_1, g_2 \in G$, the product $g_1 \cdot g_2$ is also in $G$.
- **Associativity:** For any three elements $g_1, g_2, g_3 \in G$, $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$.
- **Identity:** There exists an element $e \in G$ such that for any element $g \in G$, $e \cdot g = g \cdot e = g$.
- **Inverse:** For each element $g \in G$, there exists an element $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

Often times, we will omit the multiplication sign and write $g_1 g_2$ instead of $g_1 \cdot g_2$. With these rules, we can explain why the commutation relations for Lie algebras have such structure – why is the commutator in the span of the generators? The product of two elements in the group is also an element of the group, so using the *Baker-Campbell-Hausdorff formula* we find

$$gh = \exp(i\epsilon X)\exp(i\epsilon Y) = \exp(i\epsilon(X+Y) - \epsilon^2[X,Y] + \dots).$$

Moreover, consider a small square loop path in the group space

$$\gamma(t) = e^{itX}e^{itY}e^{-itX}e^{-itY} = \mathbb{1} - t^2[X,Y] + \mathcal{O}(t^3) \in G.$$

Writing $\gamma$ as another element of $G$, $\gamma(t) = e^{icZ}$, we find that it must hold $icZ = -t^2[X,Y]$. Hence, the commutator $[X,Y]$ must be an element of the Lie algebra, so it must be a linear combination of the generators $T^a$.

If the Lie group is Abelian, all the generators commute, and the structure constants vanish, $f^{abc} = 0$.

**Example 2.** Consider the group $\mathfrak{su}(2)$ algebra. The generators are the Pauli matrices $T^a = \sigma^a/2$ for $a = 1,2,3$ and the structure constants are given by the Levi-Civita symbol $f^{abc} = \varepsilon^{abc}$.

Generators will always obey the *Jacobi identity*:

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0.$$

Writing this in terms of structure constants, we find

$$[T^a,[T^b,T^c]] = [T^a,(if^{bcd})T^d] = if^{bcd}if^{ade}T^e.$$

We do the same for both other terms, and we get

$$\boxed{f^{bcd}f^{ade} + f^{cde}f^{bde} + f^{abd}f^{cde} = 0.}$$

An *ideal* is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $[g,i] \in \mathfrak{h}$ for all $g \in \mathfrak{g}$ and $i \in \mathfrak{h}$. The identity element $\{0\}$ and the whole algebra $\mathfrak{g}$ are always ideals. An example of a non-trivial ideal is $\mathfrak{u}(1)$ of the algebra $\mathfrak{u}(N)$: $\mathfrak{u}(1) : i_\alpha = e^{i\alpha}\mathbb{1}_{N\times N}$. In general, the group $\mathrm{U}(N)$ decomposes into

$$\mathrm{U}(N) = (\mathrm{U}(1)/\mathbb{Z}_N) \otimes \mathrm{SU}(N).$$

Usually, people don't bother dividing by $\mathbb{Z}_N$, but if we want to be pedantic, we should do that. If there is no non-trivial ideal, the group is called *simple* (and so is the algebra). For a simple algebra, it holds $\mathrm{tr}(T^a) = 0$. A slight generalization of simple Lie algebra is *semisimple Lie algebra*. It is a direct sum of simple Lie algebras:

$$\mathfrak{g}_{\mathrm{semi}} = \bigoplus_k \mathfrak{g}_k^{\mathrm{simple}}.$$

A Lie algebra is called *compact* (also the group) if the group manifold is compact. It holds for $\mathrm{U}(N)$ and $\mathrm{SU}(N)$, but not for the Lorentz group $\mathrm{SO}(1,3)$.

> *All finite-dimensional representations of compact Lie groups are equivalent to unitary representations.*

Unitarity implies $\mathcal{U}^{-1} = \mathcal{U}^\dagger$, which can be expressed as

$$\mathbb{1} - i\alpha^a (T^a)^\dagger = \mathbb{1} - i\alpha^a T^a.$$

This implies $T^a = (T^a)^\dagger$, so the generators are Hermitian. For Hermitian generators, the structure constants are real, which can be shown by comparing

$$([T^a, T^b])^\dagger = -if^{abc*}T^c, \qquad [T^b, T^a] = -if^{abc*}T^c$$

and concluding that $f^{abc} = f^{abc*}$. The generators $T^a$ can be abstract objects, but they can also be represented as matrices. The smallest non-trivial representation is called the *fundamental representation* (e.g. a rotation matrix for SO(3)). A *representation* is a homomorphism $D$ from the group to the group of invertible matrices (preserving the group structure, $D(g_1)D(g_2) = D(g_1 g_2)$). There is always a *trivial representation* $D(g) = \mathbb{1}$ for all $g \in G$. It is also called the *singlet representation*. In the Standard Model, fermions and Higgs fields are in the fundamental representation of the interaction symmetry $G_{\text{SM}}^{\text{int}}$. An example is the doublet $Q_{\text{L}}^i = (u_{\text{L}}^i \; d_{\text{L}}^i)^T$, where $i = 1, 2, 3$ is the colour index. This is a doublet in the SU(2) group (weak isospin) and a triplet in the SU(3) group (colour).

A *conjugate representation* is a representation $D^*$ such that $D^*(g) = D(g)^\dagger$ (can be also written as $T_{R^c} = -T_R^*$, where $R$ denotes the representation and $R^c$ the conjugate representation).

Another very important representation is the *adjoint representation*. It corresponds to the action of the Lie group on the Lie algebra of the fundamental representation $\mathbf{v} \to D_{\text{adj}}(g)\mathbf{v}$ (for $\mathbf{v}$ in the fundamental representation). The generators transform as

$$T^a \longrightarrow D(g)T^a D(g)^{-1}.$$

Very importantly, the gauge fields in the Standard Model transform in the adjoint representation of the gauge group. Denote $T = A^a T^a$ for $A^a \in \mathbb{R}^{\dim G}$. Then

$$\begin{aligned} D(g)A^b T^b D(g)^{-1} &\sim (\mathbb{1} + i\alpha T)(A^b T^b)(\mathbb{1} - i\alpha T) \\ &= A^b T^b + i\alpha^a A^b [T^a, T^b] \\ &= A^b T^b + i\alpha^a A^b (if^{abc}) T^c \\ &= A^b T^b + i\alpha^a A^b (-if^{abc}) T^b \end{aligned}$$

Under the adjoint representation, the coefficients $A^b$ transform as $A^b + i\alpha^a(-if^{abc})A^c$, so

$$\boxed{(T^a_{\text{adj}})_{bc} = -if^{abc}.}$$

In the exercises we will show that the structure constants are fully antisymmetric for the groups in interest (simple and compact). Namely, for semisimple Lie groups it holds $\text{tr}(T^a) = 0$ and for compact Lie groups it holds

$$\text{tr}(T^a T^b) = \delta_{ab}/2$$

in the fundamental representation.

A counterexample is the Lorentz group SO(1, 3), which is not compact. The generators are $K_i$ and $J_i$ (boosts and rotations). The traces of the pairs of them are $\text{tr}(K_i K_j) = -\delta_{ij}/2$ and $\text{tr}(J_i J_j) = +\delta_{ij}/2$, but the combinations are zero $\text{tr}(J_i K_j) = 0$.

Later we will see that vector fields transform as $\mathbf{A}_\mu(x) = A^a_\mu(x)T^a$ and $\mathbf{F}_{\mu\nu} = F^a_{\mu\nu}T^a$. The field strength tensor will be slightly redefined as $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + if^{abc}A^b_\mu A^c_\nu$ and the action will be

$$S = -\frac{1}{2}\int \mathrm{d}^4x \ \mathrm{tr}(F^a_{\mu\nu}F^{\mu\nu b}T^aT^b) = -\frac{1}{4}\int \mathrm{d}^4x \ F^a_{\mu\nu}F^{\mu\nu a}.$$

Every Lie algebra carries a bilinear form, called the *Killing form*. In the matrix representation it is proportional to the trace

$$\kappa^{ab} \propto \mathrm{tr}(T^aT^b)$$

which is named *Cartan-Killing form*.

Cartan classified all semisimple compact Lie algebras.

| $G$ | $\dim G$ | $\dim F$ (fund. rep.) |
|---|---|---|
| $\mathrm{SU}(N)$ | $N^2-1$ | $N$ |
| $\mathrm{SO}(N)$ | $N(N-1)/2$ | $N$ |
| $\mathrm{Sp}(2N)$ | $N(2N+1)$ | $2N$ |
| $E_6$ | 78 | 27 |
| $E_7$ | 133 | 56 |
| $E_8$ | 248 | 248 |
| $F_4$ | 52 | 26 |
| $G_2$ | 14 | 7 |

The last column is the dimension of the fundamental representation.

## 1.4 Discrete symmetries

Discrete symmetries are important for development and structure of the Standard Model. There are three important discrete symmetries: parity $\mathcal{P}$, charge conjugation $\mathcal{C}$ and time reversal $\mathcal{T}$. Any of them can be broken individually (the weak interaction with great pleasure) and even combinations of any two of them. However, the full $\mathcal{CPT}$ symmetry is respected by any local, Lorentz-invariant quantum field theory. As a motivation, an interesting consequence of the $\mathcal{CPT}$ invariance is the fact that massless particles of spin $s$ must always occur in both helicities $h = \pm s$ since the helicity transforms with negative sign under $\mathcal{CPT} : h \to -h$.

### 1.4.1 Parity transformation

The *parity transformation* is a discrete symmetry that flips the sign of the spatial coordinates. It is defined as

$$\mathcal{P} : (t, \mathbf{x}) \to (t, -\mathbf{x}).$$

More precisely, it is better to flip only one spatial coordinate if we work in $D$ spatial dimensions (any odd number of flips works). The Standard Model is *not* invariant under parity transformation. The asymmetry was first noticed by Lee and Yang in 1956 and confirmed in 1957 by Wu and co-workers. Let's see how the gauge field transforms under parity. The covariant derivative is

$$\mathrm{D}_\mu = \partial_\mu - ieA_\mu$$

and the components transform:

$$\begin{aligned}\mathcal{P}: &\quad \partial_\mu = (\partial_0, \partial_i) \to (\partial_0, -\partial_i),\\ \mathcal{P}: &\quad A_0(t,\mathbf{x}) \to A_0(t,-\mathbf{x}),\\ \mathcal{P}: &\quad A_i(t,\mathbf{x}) \to -A_i(t,-\mathbf{x}).\end{aligned}$$

The classical fields transform according to equations

$$\mathbf{E} = -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

Hence, electric field is a (polar) vector $\mathcal{P}: \mathbf{E} \to \mathbf{E}$ and magnetic field is an (axial) pseudo-vector $\mathcal{P}: \mathbf{B} \to -\mathbf{B}$. Another axial vector common in physics is angular momentum. Continuing with the transformation of the spin-1 field, we find

$$\begin{aligned}\mathcal{P}\hat{A}_0(t,\mathbf{x})\mathcal{P}^\dagger &= \hat{A}_0(t,-\mathbf{x}),\\ \mathcal{P}\hat{A}_i(t,\mathbf{x})\mathcal{P}^\dagger &= -\hat{A}_i(t,-\mathbf{x}).\end{aligned}$$

The situation with spinors is a bit more complicated. The equation of motion for Weyl spinors is (Weyl equation)

$$\begin{aligned}\overline{\sigma}^\mu \partial_\mu \psi_{\mathrm{L}}(x) = 0, &\qquad \text{left-handed Weyl}\\ \sigma^\mu \partial_\mu \psi_{\mathrm{R}}(x) = 0, &\qquad \text{right-handed Weyl.}\end{aligned}$$

Both equations individually violate parity. To compensate for the transformation of the derivative during the parity transformation, we also switch between the left- and right-handed Weyl spinors:

$$\begin{aligned}\mathcal{P}\psi_{\mathrm{L}}(t,\mathbf{x})\mathcal{P}^\dagger &= \psi_{\mathrm{R}}(t,-\mathbf{x}),\\ \mathcal{P}\psi_{\mathrm{R}}(t,\mathbf{x})\mathcal{P}^\dagger &= \psi_{\mathrm{L}}(t,-\mathbf{x}).\end{aligned}$$

For a parity-invariant Lagrangian we need both left- and right-handed Weyl spinors, which we stack in a Dirac bispinor $\Psi_{\mathrm{Dirac}} = (\psi_{\mathrm{L}}, \psi_{\mathrm{R}})$ which transforms as

$$\mathcal{P}\Psi(t,\mathbf{x})\mathcal{P}^\dagger = \gamma^0 \Psi(t,-\mathbf{x})$$

where

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

In the limit where the momentum is zero ($p^\mu = (E, \mathbf{0})$), fermion and anti-fermion can be described simply by

$$\Psi_{\mathrm{particle}} = \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \qquad \Psi_{\mathrm{anti-particle}} = \begin{pmatrix} \eta \\ -\eta \end{pmatrix}$$

for $\eta, \xi \in \mathbb{C}^2$. The Dirac equation $(\not{p} - m)\Psi = 0$ becomes

$$(\gamma^0 E - m\mathbb{1})\Psi = 0, \quad \implies \quad \begin{pmatrix} -m & E \\ E & -m \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix} = 0.$$

The combined equation for left-handed spinor gives $E^2\psi_{\mathrm{L}} = m^2\psi_{\mathrm{L}}$ so $E = \pm m$. Hence, we see

$$\begin{aligned}\text{particle: } E = +m &\qquad \psi_{\mathrm{L}} = \psi_{\mathrm{R}}\\ \text{anti-particle: } E = -m &\qquad \psi_{\mathrm{L}} = -\psi_{\mathrm{R}}.\end{aligned}$$

This implies the selection rules in parity invariant theory. The intrinsic parity of different particles is

$$\text{particle: } \gamma^0 \begin{pmatrix} \xi \\ \xi \end{pmatrix} = +1 \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$
$$\text{anti-particle: } \gamma^0 \begin{pmatrix} \eta \\ -\eta \end{pmatrix} = -1 \begin{pmatrix} \eta \\ -\eta \end{pmatrix}.$$

In the action, spinors appear as bilinears:

$$\begin{aligned}
&\mathcal{P}: \quad \overline{\Psi}\Psi \to \overline{\Psi}\Psi, && \text{scalar,}\\
&\mathcal{P}: \quad \overline{\Psi}\gamma^5\Psi \to -\overline{\Psi}\gamma^5\Psi, && \text{pseudo-scalar,}\\
&\mathcal{P}: \quad \overline{\Psi}\gamma^\mu\Psi \to \begin{cases} \overline{\Psi}\gamma^0\Psi \\ -\overline{\Psi}\gamma^i\Psi \end{cases}, && \text{vector,}\\
&\mathcal{P}: \quad \overline{\Psi}\gamma^\mu\gamma^5\Psi \to \begin{cases} -\overline{\Psi}\gamma^0\gamma^5\Psi \\ \overline{\Psi}\gamma^i\gamma^5\Psi \end{cases}, && \text{pseudo-vector.}
\end{aligned}$$

For example, the Dirac action is invariant under parity transformation (the factor of $\gamma^0$ compensates the $\mathrm{d}^4x$ transformation in the mass term):

$$S_{\text{Dirac}} = \int \mathrm{d}^4x\, \overline{\Psi}\left(i\gamma^\mu\partial_\mu - M\right)\Psi.$$

Scalar fields can have different intrinsic parities and can also couple to spinors. The transformation is $\mathcal{P}\phi(t,\mathbf{x})\mathcal{P}^\dagger = \pm\phi(t,-\mathbf{x})$. Depending on the sign, such field can couple to either Dirac scalar or pseudo-scalar bilinears:

$$\phi\overline{\Psi}\Psi \text{ (scalar)}, \qquad \phi\overline{\Psi}\gamma^5\Psi \text{ (pseudo-scalar)}.$$

### 1.4.2 Charge conjugation

The *charge conjugation* operator $\mathcal{C}$ transforms a particle into its antiparticle. In scalar QED a scalar field $\phi$ couples to a gauge field $A_\mu$. The field transforms as

$$\mathcal{C}: \phi \to \pm\phi^\dagger.$$

In order to keep the covariant derivative invariant,

$$\mathrm{D}_\mu\phi = (\partial_\mu - ieA_\mu)\phi, \qquad (\mathrm{D}_\mu\phi)^\dagger = (\partial_\mu + ieA_\mu)\phi^\dagger$$

the gauge field must transform as

$$\mathcal{C}: A_\mu(x) \to -A_\mu(x).$$

For the complex gauge fields $W_\mu^\pm = W_\mu^1 \pm iW_\mu^2$, the transformation is

$$\mathcal{C}W_\mu\mathcal{C}^\dagger = -W_\mu^\dagger.$$

Dirac's spinors have the general transformation

$$\mathcal{C}: \Psi \to C\Psi^*$$

where $C$ is a matrix in spinor space. To determine $C$ let's look at the Dirac equation:

$$i\gamma^\mu(\partial_\mu - ieA_\mu)\Psi - M\Psi = 0$$

Transforming the equation with $\mathcal{C}$, we find

$$i\gamma^\mu(\partial_\mu + ieA_\mu)C\Psi^* - MC\Psi^* = 0$$

and multiplying with $C^\dagger$ from the left, we get

$$iC^\dagger\gamma^\mu C(\partial_\mu + ieA_\mu)\Psi^* - M\Psi^* = 0.$$

Comparing this to the complex conjugate Dirac equation,

$$-i\gamma^{\mu *}(\partial_\mu + ieA_\mu)C\Psi^* - M\Psi^* = 0$$

we find

$$C^\dagger\gamma^\mu C = -(\gamma^\mu)^*.$$

In our choice of basis, where $\gamma^2$ is completely imaginary, and all other matrices are real, a possible solution to the above equation is $C = \pm i\gamma^2$ and

$$\mathcal{C} : \Psi \to \pm i\gamma^2\Psi^*$$

where

$$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}.$$

As a convention, we choose the positive sign. The charge conjugation is a powerful tool in analysing possible processes. As with parity transformation, different particle species have different intrinsic $\mathcal{C}$-parity ($\pm 1$). For example, the $\mathcal{C}$-parity for neutral pion $\pi^0$ is $C(\pi^0) = +1$ and for the photon $\gamma$ it is $C(\gamma) = -1$. By charge symmetry, the decay $\pi^0 \to \gamma\gamma$ is allowed (and indeed the main decay channel with branching ratio around 99 %), but the decay $\pi^0 \to \gamma\gamma\gamma$ is forbidden (this is true for $\mathcal{C}$-invariant theory such as QED). However, it is allowed in the parity violating theories such as the theory of weak interactions (due to the high mass of $W$ boson the branching ratio is unmeasurably small). The difference in the decay rates can also be seen in the results of Feynman diagrams:

$$\Gamma_{\gamma\gamma} \sim \left(\frac{m_\pi}{\Lambda}\right)^2 m\pi, \qquad \Gamma_{\gamma\gamma\gamma} \sim \left(\frac{m_\pi}{m_W}\right)^6 m_\pi.$$

From our discussion of the Dirac spinor transformations, we can immediately infer the Weyl spinor transformations:

$$\mathcal{C} : \psi_\mathrm{L} \to \pm i\sigma^2\psi_\mathrm{L}^*, \qquad \mathcal{C} : \psi_\mathrm{R} \to \mp i\sigma^2\psi_\mathrm{R}^*.$$

If the theory only includes one of the Weyl spinors or if it is chiral ($Q_\mathrm{L} \neq Q_\mathrm{R}$) then $\mathcal{C}$ and $\mathcal{P}$ are individually broken. However, the combined symmetry $\mathcal{CP}$ can still be conserved:

$$\mathcal{CP} : \psi_\mathrm{L}(t, \mathbf{x}) \to \pm i\sigma_2\psi_\mathrm{L}(t, -\mathbf{x}),$$

$$\mathcal{CP} : \psi_\mathrm{R}(t, \mathbf{x}) \to \pm i\sigma_2\psi_\mathrm{R}(t, -\mathbf{x}).$$

In the exercises we will show that the kinetic term $\psi_\mathrm{L}^\dagger i\overline{\gamma}^\mu \mathrm{D}_\mu\psi_\mathrm{L}$ is invariant under the combined transformation $\mathcal{CP}$. Nevertheless, it is still possible to construct a Lagrangian that breaks the combined $\mathcal{CP}$ symmetry using the Yukava coupling.

### 1.4.3 Time reversal

The time reversal operator $\mathcal{T}$ is defined as

$$\mathcal{T} : (t, \mathbf{x}) \to (-t, \mathbf{x}).$$

For example, the diffusion equation

$$\partial_t T = \nabla^2 T$$

is not $\mathcal{T}$-invariant (since the entropy can only increase). However, we can force the Schrödinger equation (which differs from the diffusion equation by the imaginary unit $i$)

$$i\partial_t \psi = -\nabla^2 \psi$$

to be invariant under time reversal by requiring that if $\psi(t)$ is a solution then $\psi(-t)$ is not the solution, but rather $\psi^*(-t)$ is the solution. This implies that $\mathcal{T}$ is *anti-unitary* (unitary and anti-linear) operator:

$$\mathcal{T}(\alpha\psi_1 + \beta\psi_2) = \alpha^*\mathcal{T}(\psi_1) + \beta^*\mathcal{T}(\psi_2).$$

In quantum mechanics, Wigner showed that

$$\langle \mathcal{T}\psi_1 | \mathcal{T}\psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle^*$$

is the same observable (since the physical observable is the absolute value). We can check this for the plane waves:

$$\psi(t) = e^{i(\mathbf{k}\cdot\mathbf{x} - Et)} \xrightarrow{\mathcal{T}} \psi^*(-t) = e^{i(-\mathbf{k}\cdot\mathbf{x} - Et)}.$$

The energy is conserved, but the momentum flips the sign.

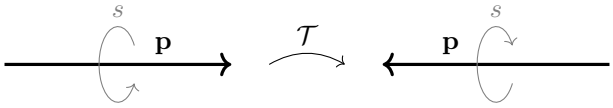
The transformation of the gauge field can again be inferred from the covariant derivative (keeping in mind that all of $t$, $\partial_0$ and $i$ change sign under $\mathcal{T}$):

$$\mathcal{T} : A_0(t, \mathbf{x}) \to +A_0(-t, \mathbf{x}),$$

$$\mathcal{T} : A_i(t, \mathbf{x}) \to -A_i(-t, \mathbf{x})$$

and accordingly, the electric and magnetic fields as $\mathbf{E} \to \mathbf{E}$ and $\mathbf{B} \to -\mathbf{B}$. This makes intuitive sense, since the electric field is generated by (stationary) charges which shouldn't change under time reversal, while the magnetic field is generated by currents which are reversed under time reversal.

When we transform fermions, we have to change the direction of the momentum and the spin.




We can again ask what matrix we have to multiply the spinor with, so that it reverses spin and momentum. The derivation is in Peskin & Schröder and the result is

$$\mathcal{T}\Psi(t, \mathbf{x})\mathcal{T}^\dagger = \gamma^1\gamma^3\Psi(-t, \mathbf{x}).$$

As you can see, there is no conjugation involved. For positive- and negative-energy Dirac spinors one finds, e.g.

$$v^{-s}(\tilde{p}) = \gamma^1\gamma^3 \left(v_s(p)\right)^*, \qquad u^{-s}(\tilde{p}) = -\gamma^1\gamma^3 \left(u_s(p)\right)^*,$$

where $\tilde{p} = (p^0, -\mathbf{p})$. For Weyl spinors, the transformation is

$$\mathcal{T} : \psi_{\mathrm{L}}(t, \mathbf{x}) = i\sigma^2 \psi_{\mathrm{L}}(-t, \mathbf{x}),$$

$$\mathcal{T} : \psi_{\mathrm{R}}(t, \mathbf{x}) = i\overline{\sigma}^2 \psi_{\mathrm{R}}(-t, \mathbf{x}).$$

The same as $\mathcal{CP}$ transformation, the time reversal keeps the handedness of the spinors. However, not all the Standard model is $\mathcal{CP}$ invariant. There is a rather weak breaking of the $\mathcal{CP}$ symmetry in the weak interaction.

To get the true symmetry of the SM, we need to combine all three. The combined symmetry $\mathcal{CPT}$ is always present in local, Lorentz-invariant QFT, $\mathcal{CPT} = \mathbb{1}$, so $\mathcal{CP} = \mathcal{T}^{-1}$. This is called the $CPT$ theorem. In plain language: if you look at a process in a mirror and run the film backwards, what you see obeys exactly the same dynamics once particles are swapped with their antiparticles.

For a Dirac field one convenient convention is

$$\mathcal{CPT} : \Psi(x) \longrightarrow -\gamma^5 \Psi^*(-x).$$

The transformation of the simplest Dirac bilinear is

$$\mathcal{CPT} : \overline{\Psi}\Psi \longrightarrow \Psi^\dagger \gamma^5 \gamma^0 \gamma^5 \Psi^* = -\Psi^\dagger \gamma^0 \Psi^* = \overline{\Psi}\Psi.$$

All other bilinears can can be checked in the same way. A full proof is technical (see, e.g. Streater & Wightman or Bogoliubov et al.) but the core ingredients are:

- **Locality**. Fields commute or anti-commute at space-like separation. This lets one rotate correlation functions from real to imaginary time without bumping into singularities.
- **Lorentz covariance**. Wightman's functions are analytic in the complexified Lorentz group; combining this with locality yields the so-called *edge-of-the-wedge* theorem, which in turn implies $\mathcal{CPT}$ invariance.
- **Spin-statistics**. The proof also uses the fact that half-integer spin fields anti-commute and integer-spin fields commute – a statement already enforced by locality and positivity of energy.

Once a QFT respects causality and special relativity, $\mathcal{CPT}$ symmetry is not optional, and we are stuck with it. Any observed violation would announce physics outside the remit of ordinary local quantum field theory.

## Summary of discrete symmetries

The transformation properties of the various fermion bilinears under $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ are summarized in the following table. The factors of $i$ have been added to some bilinears in order to ensure Hermiticity. Throughout the shorthand notation is used: $(-1)^\mu = 1$ for $\mu = 0$ and $(-1)^\mu = -1$ for $\mu = 1, 2, 3$.

| | $\overline{\Psi}\Psi$ | $i\overline{\Psi}\gamma^5\Psi$ | $\overline{\Psi}\gamma^\mu\Psi$ | $\overline{\Psi}\gamma^\mu\gamma^5\Psi$ | $\overline{\Psi}\sigma^{\mu\nu}\Psi$ | $\partial_\mu$ |
|---|---|---|---|---|---|---|
| $\mathcal{P}$ | $+1$ | $-1$ | $(-1)^\mu$ | $-(-1)^\mu$ | $(-1)^\mu(-1)^\nu$ | $(-1)^\mu$ |
| $\mathcal{T}$ | $+1$ | $-1$ | $(-1)^\mu$ | $(-1)^\mu$ | $-(-1)^\mu(-1)^\nu$ | $-(-1)^\mu$ |
| $\mathcal{C}$ | $+1$ | $+1$ | $-1$ | $+1$ | $-1$ | $+1$ |
| $\mathcal{CPT}$ | $+1$ | $+1$ | $-1$ | $-1$ | $+1$ | $-1$ |

In proving these transformation properties, the following properties of $\gamma$-matrices are important:

- Squares of matrices: $(\gamma^\mu)^2 = (-1)^\mu \mathbb{1}$, $(\gamma^5)^2 = \mathbb{1}$.
- Anti commutation relations: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$, $\{\gamma^5, \gamma^\mu\} = 0$.
- Transposition: $(\gamma^\mu)^T = \gamma^\mu$ (for $\mu = 0, 2, 5$), $(\gamma^\mu)^T = -\gamma^\mu$ (for $\mu = 1, 3$).
- Complex conjugation: $(\gamma^\mu)^* = \gamma^\mu$ (for $\mu = 0, 1, 3, 5$), $(\gamma^\mu)^* = -\gamma^\mu$ (for $\mu = 2$).

General Dirac bilinear $\overline{\Psi}\Gamma\Psi$ transforms as

$$\begin{aligned}
\mathcal{P}: &\quad \overline{\Psi}\Gamma\Psi \to \overline{\Psi}\gamma^0\Gamma\gamma^0\Psi, \\
\mathcal{T}: &\quad \overline{\Psi}\Gamma\Psi \to \overline{\Psi}\gamma^1\gamma^3\Gamma^*\gamma^3\gamma^1\Psi, \\
\mathcal{C}: &\quad \overline{\Psi}\Gamma\Psi \to \overline{\Psi}\gamma^0\gamma^2\Gamma^T\gamma^0\gamma^1\Psi.
\end{aligned}$$

## 1.5 Gauge symmetry

In the Abelian case, the gauge invariance is simple. For the complex-valued Dirac field $\Psi \in (1/2, 0) \oplus (0, 1/2)$ we require the local U(1) invariance $\Psi(x) \to e^{i\alpha(x)}\Psi(x)$. The Lagrangian should be gauge-invariant, so how can we construct it? One option is to use the product of the field and its Hermitian conjugate $\overline{\Psi}\Psi$. This works for example for the mass term , $m\overline{\Psi}(x)\Psi(x)$. However, it fails for the naive kinetic term $\overline{\Psi}\gamma^\mu\partial_\mu\Psi$. Let's try to fix it. The (regular) derivative in the direction of $n^\mu$ is

$$n^\mu\partial_\mu\Psi(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\Psi(x+\epsilon n) - \Psi(x)\right].$$

However, in the theory with local gauge invariance this construction is not sensible, as $\Psi(x)$ and $\Psi(x+\epsilon n)$ have different phase transformations. To mend this situation, we introduce the *compensating factor*, a scalar function $U(x, y)$ which transforms as

$$U(x, y) \to e^{i\alpha(x)}U(x, y)e^{-i\alpha(y)}.$$

It should hold $U(x, x) = 1$ and in general, $U$ should be a pure phase $U(x, y) = e^{i\phi(x,y)}$. With the compensator, $\Psi(y)$ and $U(x, y)\Psi(x)$ transform the same way, so a sensible way to define a derivative is

$$n^\mu \mathrm{D}_\mu\Psi(x) = \lim_{\epsilon\to 0}\frac{1}{\varepsilon}\left[\Psi(x+\epsilon n) - U(x+\epsilon n, x)\Psi(x)\right].$$

This is called the *covariant derivative*. The operator $U(x+\epsilon n, x)$ can be expanded infinitesimally as

$$U(x+\epsilon n, x) = 1 - ig\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2),$$

where $g$ is a constant (for later convenience) and $A_\mu(x)$ is some continuous function. This expansion defines the vector field $A_\mu(x)$ which is called a *connection* on the principle bundle. We can now write the covariant derivative of the Dirac field as

$$\mathrm{D}_\mu\Psi(x) = \partial_\mu\Psi(x) - igA_\mu(x)\Psi(x).$$

By using the transformation $U(y, x) \to e^{i\alpha(y)}U(y, x)e^{-i\alpha(x)}$, we find

$$\begin{aligned}
U(x+\epsilon n, x) &\to e^{i\alpha(x+\epsilon n)}\, U(x+\epsilon n, x)e^{-i\alpha(x)} \\
&= e^{i\alpha(x)}(1 + i\epsilon n^\mu\partial_\mu\alpha(x))\left[1 - ig\cdot\epsilon n^\mu A_\mu(x)\right]e^{-i\alpha(x)} \\
&= 1 - ig\cdot\epsilon n^\mu\left(A_\mu - \frac{1}{g}\partial_\mu\alpha\right) + \mathcal{O}(\epsilon^2).
\end{aligned}$$

And therefore, the vector field $A_\mu(x)$ transforms as

$$A_\mu(x) \to A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x)$$

which is exactly how we know U(1) gauge field should transform. The transformation of the derivative $\mathrm{D}_\mu\Psi(x)$ is

$$\begin{aligned}\mathrm{D}_\mu\psi(x) \to &\left[\partial_\mu + ig\left(A_\mu - \frac{1}{g}\partial_\mu\alpha\right)\right] e^{i\alpha(x)}\Psi(x)\\ =&e^{i\alpha(x)}(\partial_\mu + igA_\mu)\Psi(x)\\ =&e^{i\alpha(x)}\mathrm{D}_\mu\Psi(x).\end{aligned}$$

This is exactly what we wanted: the covariant derivative transforms in the same way as the field $\Psi(x)$. This insight allows us to quickly build locally invariant Lagrangians, including the fields' derivatives, such as $\overline{\Psi}\not{\mathrm{D}}\Psi$. Note that the covariant derivative is not invariant under the gauge transformation, but it transforms *covariantly*.

We want to generalize the Abelian case to any local symmetry $\Psi(x) \to V(x)\Psi(x)$. To talk about the derivative, we will need a notion of a parallel transport, the equivalent of the compensating factor:

$$\psi(y) = U(y,x)\Psi(x).$$

Here, $U$ is called the *parallel transport matrix*. If $\Psi_{x\to y}$ is to transform as a field at $y$ under local gauge transformation, then $U(y,x)$ must transform as

$$U(y,x) \to V(y)U(y,x)V^{-1}(x)$$

and $U(x,x) = \mathbb{1}$. Following the same logic as in the Abelian case, we will expand this operator around the identity in terms of a local vector fields:

$$U(x+\mathrm{d}x, x) = \mathbb{1} - i\mathbf{A}_\mu \mathrm{d}x^\mu.$$

The connection (or the gauge field) $\mathbf{A}_\mu$ is an element of the Lie algebra and can be expanded in terms of generators:

$$\mathbf{A}_\mu(x) = A^a_\mu(x)T^a.$$

$\mathbf{A}^a_\mu$ is a matrix field, and it will transform under the adjoint representation of the gauge group. The local transformation of the field $A_\mu$ is

$$V(x+dx)(\mathbb{1} - i\mathbf{A}_\mu \mathrm{d}x^\mu)V^{-1}(x) = \left(V(x) + \partial_\mu V(x)\mathrm{d}x^\mu\right)(\mathbb{1} - i\mathbf{A}_\mu \mathrm{d}x^\mu)V^{-1}(x) = \mathbb{1} - i\mathbf{A}^V_\mu(x)\mathrm{d}x^\mu.$$

This shows that the gauge field transforms as

$$\mathbf{A}^V_\mu(x) = V(x)\mathbf{A}_\mu(x)V^{-1}(x) - i\left(\partial_\mu V(x)\right)V^{-1}(x).$$

The first part transforms as adjoint and survives even in the case of global symmetry ($V$ = const.) The second term is specific to the gauge field (connection). For U(1) we see that choosing $V(x) = e^{i\alpha(x)}$ gives the expected result.

Next, we will consider infinitesimal gauge transformations:

$$V(x) = \mathbb{1} + i\omega^a(x)T^a + \ldots,$$

where $\omega^a(x)$ are infinitesimal parameters. The transformation of the gauge field is then

$$\mathbf{A}^V_\mu(x) = \mathbf{A}_\mu(x) + i[T^a\omega^a(x), \mathbf{A}_\mu(x)] - \partial_\mu\omega^a(x)T^a + \dots$$

This equation is written in matrix form (remember that $\mathbf{A}_\mu$ are matrices). We can project the matrix form to a single component using $2\,\mathrm{tr}[T^c\mathbf{A}_\mu] = A^c_\mu$ (we used the identity $2\,\mathrm{tr}[T^aT^b] = \delta^{ab}$). We find

$$A^c_\mu(x) \to A^c_\mu(x) - \partial_\mu\omega^c(x) + 2i\omega^a\,\mathrm{tr}\,[[T^a, \mathbf{A}_\mu(x)]\,T^c] + \dots$$

Using the fact that generators obey the Lie algebra

$$[T^a, T^b] = if^{abc}T^c$$

we obtain

$$A^c_\mu \to A^c_\mu - \partial_\mu\omega^c - f^{abc}\omega^a A^b_\mu.$$

Denote $\boldsymbol{\omega} = \omega^aT^a$. Under gauge transformation, it transforms to $V\boldsymbol{\omega}V^{-1}$ (in the adjoint representation). The covariant derivative for a field in the adjoint representation is

$$\mathrm{D}^{\mathrm{adj}}_\mu\boldsymbol{\omega} = \partial_\mu\boldsymbol{\omega} + i[\mathbf{A}_\mu, \boldsymbol{\omega}].$$

Under the gauge transformation, the gauge field gets an additional term:

$$\delta\mathbf{A}_\mu = \mathbf{A}^\omega_\mu - \mathbf{A}_\mu = i\,[T^a\omega^a, \mathbf{A}_\mu(x)] - \partial_\mu(\omega_aT^a) = -\mathrm{D}^{\mathrm{adj}}_\mu\boldsymbol{\omega}.$$

The connection allows us to define a covariant derivative for non-Abelian fields:

$$\mathrm{D}_\mu\Psi(x) = (\partial_\mu + i\mathbf{A}_\mu(x))\,\Psi(x).$$

It transforms as

$$\begin{aligned}\mathrm{D}^V_\mu V(x)\Psi(x) =& (\partial_\mu + i\mathbf{A}^V_\mu(x))V(x)\Psi(x)\\ &= (\partial_\mu + iV\mathbf{A}_\mu V^{-1} - (\partial_\mu V)V^{-1})V\Psi\\ &= V(x)\mathrm{D}^V_\mu\Psi.\end{aligned}$$

Just as in the Abelian case, the covariant derivative has nice transformation properties and behaves like the $\Psi$ field itself, i.e. $\mathrm{D}_\mu\Psi(x) \to V(x)\mathrm{D}_\mu\Psi(x)$. Somewhat miraculously, the commutator of two covariant derivatives isn't derivative any more:

$$[\mathrm{D}_\mu, \mathrm{D}_\nu] = i\mathbf{F}_{\mu\nu}$$

where $\mathbf{F}_{\mu\nu}$ is the field strength tensor. It transforms the same as the derivative:

$$\mathbf{F}_{\mu\nu}(x) \to V(x)\mathbf{F}_{\mu\nu}(x)V^{-1}(x).$$

Unlike the Abelian field strength, the non-Abelian field strength is *not* gauge-invariant. You need at least two powers of $\mathbf{F}_{\mu\nu}$ and trace over this to find something gauge invariant (or couple it to matter forming a gauge singlet).

The field strength $\mathbf{F}_{\mu\nu}$ transforms covariantly as an element of the adjoint representation. The infinitesimal transformation is

$$F^a_{\mu\nu}(x) \to F^a_{\mu\nu}(x) - f^{abc}\omega^bF^c_{\mu\nu}(x) + \dots$$

With the covariant derivative in the fundamental representation, we find

$$\mathbf{F}_{\mu\nu} = -i\left[\mathrm{D}_\mu, \mathrm{D}_\nu\right] = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + i\left[\mathbf{A}_\mu, \mathbf{A}_\nu\right]$$

or in component form

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu.$$

We know that non-Abelian field strength is not gauge-invariant and to form an invariant function, we must take the trace over the group indices of $\mathbf{F}_{\mu\nu}$ or the trace of products of field strengths. We can easily generalize the Abelian procedure to arrive at locally invariant Lagrangians: $\partial_\mu \to \mathrm{D}_\mu$, e.g.

$$\mathcal{L}_{\text{matter}} = \overline{\Psi}(i\not{\mathrm{D}} - m)\Psi = \overline{\Psi}(i\not{\partial} - \not{A}^a T^a - m)\Psi$$

is invariant under local SU($N$) (or any Lie group). An invariant kinetic term quadratic in derivatives is

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right] + \mathcal{L}_{\text{matter}}$$

with the normalization $2\,\mathrm{tr}\left[T^a T^b\right] = \delta^{ab}$,

$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \mathcal{L}_{\text{matter}}(\Psi, \mathrm{D}_\mu \Psi).$$

Most commonly, however, we use a different rescaling convention, $\mathbf{A}_\mu \to g\mathbf{A}_\mu$. With this modification, the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \mathcal{L}_{\text{matter}}(\Psi, \mathrm{D}_\mu \Psi)$$

and

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + ig[\mathbf{A}_\mu, \mathbf{A}_\nu].$$

Note that all of $F^a_{\mu\nu}$ are not independent due to the *Bianchi identity*:

$$D_\mu \mathbf{F}_{\rho\sigma} + \mathrm{D}_\rho \mathbf{F}_{\sigma\mu} + \mathrm{D}_\sigma \mathbf{F}_{\mu\rho} = 0$$

which is a direct consequence of the Jacobi identity

$$0 = \left[\mathrm{D}_\mu, \left[\mathrm{D}_\rho, \mathrm{D}_\sigma\right]\right] + \left[\mathrm{D}_\rho, \left[\mathrm{D}_\sigma, \mathrm{D}_\mu\right]\right] + \left[\mathrm{D}_\sigma, \left[\mathrm{D}_\mu, \mathrm{D}_\rho\right]\right].$$

### 1.5.1 Yang-Mills Lagrangian

The Yang-Mills action without any fermionic matter in the less common normalization is

$$S_{\text{YM}} = -\frac{1}{2g^2}\int \mathrm{d}^4x\ \mathrm{tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu})$$

where $\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + i[\mathbf{A}_\mu, \mathbf{A}_\nu]$ is the field strength tensor and $\mathbf{A}_\mu = A^a_\mu T^a$. Rescaling the field with $A \to gA$ and writing out the terms explicitly, we find

$$\begin{aligned} S_{\text{YM}} = -\int \mathrm{d}^4x\ &\left(-\frac{1}{2}\partial_\mu A^a_\nu \partial^\mu A^{\nu a} + \frac{1}{2}\partial_\mu A^a_\nu \partial^\nu A^{\mu a}\right) + \\ &+ \left(g f^{abc} A^b_\mu A^c_\nu \partial^\mu A^{\nu a} - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e}\right). \end{aligned}$$

The term in the first parentheses is the same as in the Abelian case (leads to linear equations of motion). The second term is non-linear self-interaction of the gauge fields (cubic and quartic). The equations of motion are obtained by varying the action with respect to the gauge field:

$$\delta S = -\frac{1}{g^2}\int \mathrm{d}^4x \ \mathrm{tr}(\mathbf{F}_{\mu\nu}\delta\mathbf{F}^{\mu\nu}).$$

The used variation of the field strength is

$$\delta\mathbf{F}_{\mu\nu} = \partial_\mu\delta\mathbf{A}_\nu + i\delta\mathbf{A}_\mu\mathbf{A}_\nu + i\mathbf{A}_\mu\delta\mathbf{A}_\nu + (\mu \leftrightarrow \nu).$$

Using the antisymmetry of the field strength, we can write

$$\delta S = -\frac{2}{g^2}\int \mathrm{d}^4x \ \mathrm{tr}\left(\mathbf{F}_{\mu\nu}(\partial_\mu\delta\mathbf{A}_\nu + i\delta\mathbf{A}_\mu\mathbf{A}_\nu + i\mathbf{A}_\mu\delta\mathbf{A}_\nu)\right).$$

To simplify the expression, we integrate the first term by parts and use the cyclicity of the trace:

$$\delta S = \frac{2}{g^2}\int \mathrm{d}^4x \ \mathrm{tr}\left((\partial^\mu\mathbf{F}_{\mu\nu} + i[\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}])\delta\mathbf{A}^\nu\right).$$

The equations of motion in matrix form are thus

$$\partial^\mu\mathbf{F}_{\mu\nu} + i[\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] = \mathrm{D}^\mu_{\mathrm{adj}}\mathbf{F}_{\mu\nu} = 0.$$

The last equation shows that the equations of motion are covariant under the gauge transformation. The Bianchi identity can be expressed compactly as

$$\mathrm{D}^\mu_{\mathrm{adj}}\tilde{\mathbf{F}}_{\mu\nu} = 0.$$

This is not the equations of motion since it is trivially satisfied by the Jacobi identity. The conserved Noether current $\partial_\mu j^\mu = 0$ is (in matrix form)

$$\mathbf{j}_\nu = -\partial^\mu\mathbf{F}_{\mu\nu} = i[\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}],$$

where $\mathbf{j}_\mu = j^a_\mu T^a$.

We will now discuss the renormalization of non-Abelian gauge theories, such as QCD. In QCD the gauge group is $G = \mathrm{SU}(3)$ and the matter fields are $n_f$ Dirac fermions in the fundamental representation of $G$. Perturbation theory only makes sense after we eliminate the redundant gauge degrees of freedom in the path integral. We therefore impose the covariant gauge condition $\partial_\mu A^{a\mu} = 0$ (generalized to $\partial_\mu A^{a\mu} = B^a$ in the BRST formalism) and add the gauge-fixing term

$$-\frac{1}{2\xi}(\partial_\mu A^{a\mu})^2$$

to the action. The *Faddeev-Popov procedure* introduces an anticommuting *scalar* doublet $(c^a, \overline{c}^a)$ whose quadratic part $-\overline{c}^a\Box c^a$ and cubic interaction $gf^{abc}(\partial_\mu\overline{c}^a)A^{b\mu}c^c$ exactly cancel the unphysical longitudinal and time-like gluon modes in loop diagrams, ensuring unitarity and gauge parameter independence of physical observables.

The parameter $\xi$ labels a single-parameter family of gauges; in practice, we set $\xi = 1$ (Feynman gauge) to streamline propagators, but any choice gives the same on-shell results once ghosts are included. The final Lagrangian thus contains the canonical Yang-Mills and matter pieces, the covariant gauge fixing term, and the ghost sector, all of which are required for a consistent and renormalizable perturbative expansion. For more details, see QFT 2.

## Feynman rules

The kinetic term of the Yang-Mills Lagrangian is

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + \overline{\Psi}_i(i\partial\!\!\!/ - m)\Psi_i - \overline{c}^a \Box c^a.$$

The propagators are

$$\nu, b \sim\sim\sim \mu, a \quad = i\frac{-\eta^{\mu\nu} + (1-\xi)p^\mu p^\nu/p^2}{p^2 + i\varepsilon}\delta^{ab}, \qquad \text{gauge},$$

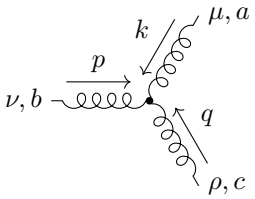
$$i \longrightarrow j \quad = \frac{i\delta^{ij}}{p\!\!\!/ - m + i\varepsilon}, \qquad \text{fermion},$$

$$b \text{ - - - -} \blacktriangleright \text{- - - - } a \quad = \frac{i\delta^{ab}}{p^2 + i\varepsilon}, \qquad \text{ghost}.$$

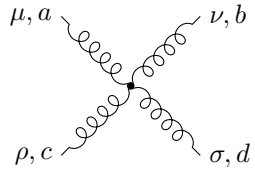
Most frequently, we use the Feynman gauge $\xi = 1$. The interaction part of the Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{int}} = &- g f^{abc}(\partial_\mu A_\nu)^a A^b_\mu A^c_\nu - \frac{1}{4}g^2(f^{gab}A^a_\mu A^b_\nu)(f^{gcd}A^c_\mu A^d_\nu) + \\ &+ g f^{abc}(\partial_\mu \overline{c}^a)A^b_\mu c^c + g\overline{\Psi}_i\gamma^\mu T^a_{ij}\Psi_j A^a_\mu.\end{aligned}$$

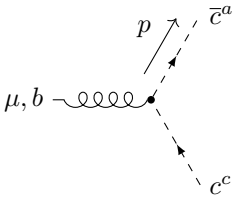
There are four types of vertices:



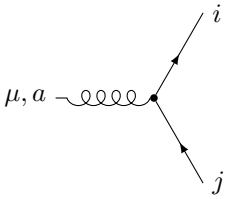

$$= g f^{abc}\left(\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu\right) \qquad \text{triple gauge},$$




$$\begin{aligned}&-ig^2[f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ = \quad &+ f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ &+ f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})]\end{aligned} \qquad \text{quadruple gauge},$$




$$= -g f^{abc} p^\mu \qquad \text{ghost vertex},$$




$$= ig\gamma^\mu T^a_{ij} \qquad \text{fermion vertex}.$$

# Chapter 2

# Spontaneous symmetry breaking

This is one of the most important concepts in particle physics. It occurs when the symmetry of system is not the same as the symmetry of the ground state. It can occur in a classical system, but not in a quantum system (but then again it can appear in QFT or statistical physics).

An analogy in real world is driving direction: globally it doesn't matter which side we drive on, as long as we all agree on the same side. So the minimum danger state is if we all drive on the same side, but which side is completely arbitrary. Hence, continuously connected regions (i.e. countries) decide on one side or the other.

A more physical example is the ferromagnet. The magnet has a global SO(3) symmetry (by Ising model), which breaks spontaneously to SO(2) at a particular temperature (called critical temperature). Crystals also have SSB, as Hamiltonian is translationally invariant, but ground state has discrete shift symmetry. In particle physics, one example is chiral symmetry breaking in QCD. However, the most important example is the Higgs mechanism with massive gauge bosons. A non-relativistic equivalent to Higgs is the superconductor (Bardeen-Cooper-Schrieffer (BCS) theory) and related Meissner effect. Many beyond the Standard Model theories also have SSB.

## 2.1 Classical symmetry breaking

Let's start with an easy example, the O($N$)-model. In this model there is an $N$-tuple of real scalar fields $\phi^i \in \mathbb{R}^N$ with a Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2}\mu^2 \phi^i \phi^i - \frac{\lambda}{4}(\phi^i \phi^i)^2.$$

Note the positive sign in front of the mass term. This model possesses a global O($N$) symmetry: $\phi^i \to V^{ij}\phi^j$ for orthogonal transformations, $VV^T = \mathbb{1}$. This symmetry is broken if we assign a slightly different mass to one of the components, $\Delta\mathcal{L} = \delta m^2 \phi_1^2$. In this case the symmetry breaks from O($N$) to O($N-1$). However, this is not the only way to break the symmetry. By adding an odd power of the field, $\Delta\mathcal{L}' = \phi^2\phi_1$, we break the $\phi \to -\phi$ symmetry.

We can also break symmetry through interactions with other sectors. Consider the interaction

$$\Delta\mathcal{L}_{\text{int}} = \lambda_1 \phi_1 \overline{\Psi}\Psi + (\mathrm{D}_\mu \phi^i)^2.$$

We can also have a situation, where only one part of O($N$) is gauged:

$$\mathrm{D}_\mu \phi^i = (\partial_\mu - ieA_\mu \lambda^1)\phi^1,$$

where $\lambda^1$ is the first of the Gell-Mann matrices.

Pions $(\pi^0, \pi^\pm)$ form a SU(2) triplet, but they don't have the same mass due to symmetry breaking, $\delta_{m_\pm^2 - m_0^2} \sim e^2\Lambda^2/16\pi^2$.

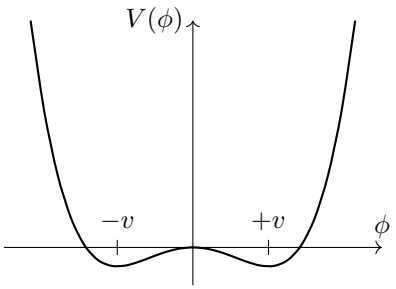
We will now look at the example of $N = 1$, where the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4.$$

Assuming that $\mu^2 > 0$ we define the potential

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4.$$

This is called the *double-well potential*.




The minima of the potential are at

$$\phi_0 = \pm v = \pm\sqrt{\frac{\mu^2}{\lambda}}.$$

Expressing the potential with minimal value, we get

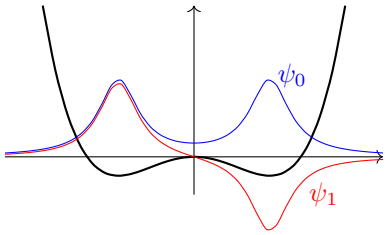
$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 + \text{const.}$$

Expanding the potential around the minimum $\phi(t) = v + \sigma(t)$, we find

$$V(\sigma) = \frac{\lambda}{4}(2v\sigma + \sigma^2)^2 = \lambda\left(v^2\sigma^2 + v\sigma^3 + \sigma^4/4\right).$$

The potential is no longer symmetric under $\sigma \to -\sigma$! The odd terms $\sigma^3$ seem to break the $\mathbb{Z}_2$ symmetry. And classically, they do. However, in quantum mechanics, we need to be a bit more careful. The particle that is in the ground state can tunnel though the potential barrier and end up in the other minimum. So the ground state is symmetric again (but not the first excited state):

$$\psi_0 \approx \exp\left(-\frac{\sqrt{\lambda}v}{2}(v+\phi)^2\right), \qquad \psi_1 \approx \exp\left(-\frac{\sqrt{\lambda}v}{2}(v-\phi)^2\right).$$




Since the wave function can "leak" between the two minima it is a sum of the two states, $\psi_0 = \psi_{\text{left}} + \psi_{\text{right}}$. The first excited state is the other linear combination of the non-symmetric states, $\psi_1 = \psi_{\text{left}} - \psi_{\text{right}}$.

We will calculate the amplitude $v \to -v$ in the Wick-rotated space, $\tau = it$:

$$\langle v|e^{-H\tau}|-v\rangle = \int \mathcal{D}\phi \, \exp(-S_{\text{E}}(\phi))$$

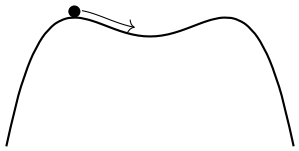
with

$$S_{\text{E}}(\phi) = \int \mathrm{d}\tau \left(\frac{\phi^2}{2} + V(x)\right).$$

Using the saddle point approximation, we first find the classical solution $\phi_{\text{cl}}(t)$, which is the solution of the equation of motion $\delta S_{\text{E}}/\delta\phi = 0$. The action can be Taylor expanded around the classical solution:

$$S_{\text{E}}(\phi) = S_{\text{E}}(\phi_{\text{cl}}) + \int \underbrace{\frac{\delta S_{\text{E}}}{\phi_{\text{cl}}}}_{0} \phi + \underbrace{\int \frac{\delta^2 S_{\text{E}}}{\delta \phi_{\text{cl}}^2}\phi^2}_{\ll 1} + \cdots \approx S_{\text{E}}(\phi_{\text{cl}}).$$

Due to the Wick rotation, the classical potential is inverted.



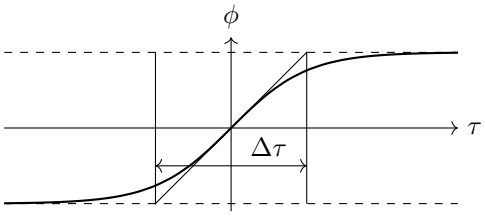
The classical trajectories in the inverted potential obey the equation of motion

$$\ddot{\phi} = \lambda\phi(\phi^2 - v^2).$$

The solution is

$$\phi_{\text{cl}} = v \tanh\left(\sqrt{\frac{\lambda v^2}{2}}(\tau - \tau_0)\right),$$

where the tunnelling time is $\Delta\tau \approx 1/\sqrt{\lambda v^2} \approx 1/\mu$.



The classical action is then exactly

$$S_{\mathrm{E}} = \int_{-\infty}^{\infty} \mathrm{d}\tau \left( \frac{1}{2}\dot{\phi}_{\mathrm{cl}}^2 + \lambda(\phi_{\mathrm{cl}}^2 - v^2)^2 \right) = \frac{\lambda v^4}{2} \int_{-\infty}^{\infty} \mathrm{d}\tau \ \mathrm{sech}^4 \left( \sqrt{\frac{\lambda v^2}{2}} \tau \right) = \frac{2}{3}\sqrt{2\lambda} v^3.$$

The tunnelling is suppressed by a large barrier ($\lambda \to \infty$) and the distance between the minima ($v \to \infty$). Using this, we recover the WKB approximation form the quantum mechanics:

$$\lim_{\mu \to \infty} \langle +v | e^{-H\tau} | -v \rangle = K e^{-S_{\mathrm{cl}}}.$$

The energy split between the ground and first excited state is

$$\Delta E \approx \sqrt{\lambda v^2} e^{-S_{\mathrm{cl}}^{\mathrm{E}}}.$$

In QFT, the action looks exactly the same

$$S_{\mathrm{E}} = \int \mathrm{d}\tau \, \mathrm{d}^3 x \left( \frac{1}{2}(\partial_\mu^{\mathrm{E}} \phi)^2 + V(\phi) \right).$$

The states will be denoted as follows: $\phi_0(x, \tau_1) = -v$, $\phi_0(x, \tau_2) = +v$ and $\tau = \tau_1 - \tau_2$. The transition amplitude $\langle +v | e^{-H\tau} | +v \rangle$ will again be dominated by Euclidean action

$$(\partial_\tau^2 - \nabla^2)\phi = \mathrm{D}_{\mathrm{E}}\phi = \lambda\phi(\phi^2 - v^2).$$

Choose $\phi_{\mathrm{cl}}(x, \tau) = \phi_{\mathrm{cl}}(\tau)$, which minimizes the action $(\nabla\phi)^2 > 0$. Take the QM solution $\phi_{\mathrm{cl}}(\tau) = v \tanh(\sqrt{\lambda/2} v\tau)$. The classical Euclidean action is

$$S_{\mathrm{E}}(\phi_{\mathrm{cl}}) = \int \mathrm{d}\tau \, \mathrm{d}^3 x \left( \frac{1}{2}(\partial_\mu^{\mathrm{E}} \phi_{\mathrm{cl}})^2 + V(\phi_{\mathrm{cl}}) \right) = \theta S_{\mathrm{cl}}^{\mathrm{QM}}.$$

The tunnelling in QFT vanishes, $e^{-\theta S_{\mathrm{cl}}} \to 0$, since the space is infinite and $\theta \to \infty$. Hence, in QFT the tunnelling requires that the transition $v \to -v$ happens at every point in space, which is the spontaneous symmetry breaking. It can also occur in large systems like statistical mechanics.

Let's return to ferromagnet example. Below the critical temperature, the system spontaneously orders itself into one of the ground states. However, the regions of alignment can be finite, they are called domains. The domains are separated by *domain walls*, which are the regions of the system where the order parameter changes from one value to another. The analogy with driving direction would be country borders, where the direction of driving changes. In QFT, the energy density is $\mathcal{E} = 2\sqrt{2\lambda} v^3/3$. In $d = 1 + 1$, the total energy is finite, which leads to pseudo-particles and solitons.

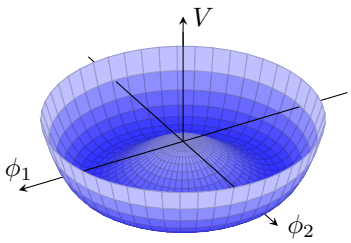
Let's see what happens in the case O(2). In this case, we can write

$$\Phi = \frac{1}{2}(\phi_1 + i\phi_2)$$

which transforms under U(1): $\Phi \to e^{i\alpha}\Phi$. The Lagrangian is

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi + \frac{1}{2}\mu^2 \Phi^* \Phi - \frac{\lambda}{4}(\Phi^* \Phi)^2.$$

The potential in this case is the famous Mexican hat.




The vacuum manifold is a circle, $S^1$. Namely, if we require $\partial V/\partial|\Phi| = 0$, the solution is $\Phi_0 = \mu/\sqrt{\lambda}e^{i\alpha}$. The kinetic energy required for a transition between any two minima (at all points in space) is infinite, hence the spontaneous symmetry breaking still occurs. We expand the field around the minimum (so-called *linear sigma model*), where $\alpha = 0$:

$$\phi_1(x) = \phi_0(x) + \chi(x)$$
$$\phi_2(x) = \theta(x).$$

The potential is

$$V(\theta, \chi) = -\frac{\mu^2}{2}((\phi_0 + \chi)^2) + \frac{\lambda}{4}((\phi_0 + \chi)^2 + \theta^2)^2.$$

In terms of Lagrangian, the modification is

$$\mathcal{L}^{(2)} = \frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{2}(\partial_\mu \theta)^2 - \mu^2\chi^2 + \mathcal{O}(\chi^3, \theta^3).$$

From this, it follows that $\theta$ is massless and $\chi$ has mass $m_\chi = \sqrt{2}\mu$. In general, masses are related by the sum rules:

$$m_{\phi_1}^2 + m_{\phi_2}^2 = m_\chi^2 + m_\theta^2 = 2m^2.$$

The massless particle is called the *Goldstone boson*. We can also perform a different parametrization of the field,

$$\phi(x) = \frac{1}{\sqrt{2}}(v + r(x))e^{i\pi(x)/v}.$$

Don't confuse $\pi$ (the number, which is not in this equation) and $\pi(x)$ (the field component that characterizes the phase in the minimum). For the kinetic term we get

$$\partial_\mu \phi^* \partial^\mu \phi = \frac{1}{2}\partial_\mu r \partial^\mu r + \frac{1}{2}\left(1 + \frac{r(x)}{v}\right)^2 \partial_\mu \pi \partial^\mu \pi$$

and the full Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu r \partial^\mu r + \frac{1}{2}\left(1 + 2\frac{r}{v} + \frac{r^2}{v^2}\right)\partial_\mu \pi \partial^\mu \pi - V(r).$$

Goldstone bosons satisfy a shift symmetry $\pi(x) \to \pi(x) + \alpha v$ since only the derivative of $\pi$ appears in the Lagrangian. We say that the symmetry is non-linearly realized. Spontaneous symmetry breaking leads to a shift-symmetric realization of the symmetry.

Now, we want to generalize the symmetry to any group $G$. After the spontaneous symmetry breaking (SBB), the remaining symmetry is only a subgroup $H \leq G$.

**Example 3.** Consider $n$ real scalar fields $\phi_i$. The Lagrangian could be

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i + \frac{m^2}{2}\phi_i\phi_i - \frac{\lambda}{16}(\phi_i\phi_i)^2.$$

This Lagrangian is invariant under SO($N$). The extremum of the potential $\partial V/\partial\phi_i = 0$ gives the condition $\phi_i\phi_i = 4m^2/\lambda = v^2$. For the vacuum state we can choose $\langle\phi_i\rangle = (0,\dots,0,v)^T$ which is still invariant under SO($N-1$). Hence, SBB broke the SO($N$) symmetry to SO($N-1$).

For a theory with scalar fields $\phi^a$ we define a *vacuum expectation value* (VEV) as

$$\psi_0^a = \langle\Omega|\psi^a|\Omega\rangle$$

such that

$$\left.\frac{\partial V}{\partial\phi_a}\right|_{\phi^a=\phi_0^a} = 0.$$

We also define the *mass matrix* as

$$m_{ab}^2 = \left.\frac{\partial V}{\partial\phi_a\partial\phi_b}\right|_{\phi^a=\phi_0^a}.$$

At the minimum of $V$ we expect no negative eigenvalues.

**Theorem 2.1** (*Goldstone*)**.** *Every continuous symmetry of $\mathcal{L}$ that is not a symmetry of the vacuum leads to zero eigenvalue of $m_{ab}^2$, i.e., a massless scalar boson for each broken generator.*

*Proof.* In a general continuous system, we can write the transformation in terms of operators $O^a$:

$$\phi^a \longrightarrow \phi^a + \varepsilon O^a(\phi) = \phi^a + \varepsilon\lambda^i T_{ab}^i\phi_b.$$

The potential should be invariant under this transformation, $V(\phi^a) = V(\phi^a + \varepsilon O^a(\phi))$, thus it should hold

$$0 = O^a(\phi)\frac{\partial V}{\partial\phi^a} + \mathcal{O}(\varepsilon).$$

Taking another derivative and evaluating it at the vacuum expectation value, we get

$$0 = \frac{\partial O^a}{\partial\phi^b}\underbrace{\frac{\partial V}{\partial\phi^a}}_{0}\Bigg|_{\phi^a=\phi_0^a} + O^a(\phi)\left.\frac{\partial V}{\partial\phi^a\partial\phi^b}\right|_{\phi^a=\phi_0^a}.$$

The first term is zero by the observation that the derivative of a potential is zero for vacuum state. The second term gives $O^a(\phi)m_{ab}^2 = 0$. If $O^a = 0$, then the symmetry is not broken ($\phi^a \to \phi^a$). If $O^a(\phi) \neq 0$ it must be the eigenvector of $m_{ab}^2$ with zero eigenvalue. In general, we thus have one zero eigenvalue for every broken symmetry. □

Hence, in order to find the Goldstone bosons, we have to find the broken symmetries. When we have the symmetry group $G$ of the full Lagrangian and the subgroup $H$ of the vacuum state, the number of Goldstone bosons is

$$n(\text{Goldstone}) = n(G) - n(H).$$

In the previous example, we had

$$n(\text{Goldstone}) = n(\text{SO}(N)) - n(\text{SO}(N-1)) = \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N-1.$$

Sadly, the Goldstone theorem doesn't hold for other field types, as in that case the vacuum expectation value would need to have a preferred direction.

## 2.2 Effective action

Our analysis in the previous section was entirely classical. We would like to construct a function that captures also the quantum corrections to the vacuum state. We will see that effective potential is what we are looking for. In order to include quantum effects, we need to consider the path integral:

$$Z[\mathcal{J}] = \frac{\int \mathcal{D}[\phi] \exp\left(iS[\phi] + i\int \mathrm{d}^4x\, \mathcal{J}(x)\phi(x)\right)}{\int \mathcal{D}[\phi] \exp\left(iS[\phi]\right)}.$$

This is the generating functional for all Feynman diagrams. We defined the $n$-point correlation Green function as

$$G(x_1,\ldots,x_n) = \langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle = \left(\prod_{i=1}^{n}\frac{1}{i}\frac{\delta}{\delta\mathcal{J}(x_i)}\right) Z[\mathcal{J}]\big|_{\mathcal{J}=0}.$$

From this generating functional, we constructed another generating functional $W[\mathcal{J}]$ (defined implicitly as $Z[\mathcal{J}] = e^{iW[\mathcal{J}]}$), that generated all *connected* diagrams:

$$G^{\text{c}}(x_1,\ldots,x_n) = \left(\prod_{i=1}^{n}\frac{1}{i}\frac{\delta}{\delta\mathcal{J}(x_i)}\right) W[\mathcal{J}]\big|_{\mathcal{J}=0}.$$

At this point, it is worth stopping a bit and reflecting on what we have constructed. All constructions, we are dealing with here are extremely reminiscent to the statistical physics. Let's look for example at a magnetic thermodynamic system. There we have a local spin orientation $s(x)$, the spin energy density $\mathcal{H}(s)$ and the external magnetic field $H$. The partition function $Z$ is a function of external field as

$$Z(H) = e^{-\beta F(H)} = \int \mathcal{D}s \exp\left(-\beta\int \mathrm{d}^4x\, \left(\mathcal{H}(s) - Hs(x)\right)\right).$$

The partition function relates to the Helmholtz free energy $F(H)$ as seen in this equation. The average over all possible spin orientations is given by the magnetization $M$. All these quantities are related to equivalent quantities in QFT, as listed in the following table.

| Magnetic System | Quantum Field Theory |
|---|---|
| $\mathbf{x}$ | $x = (t, \mathbf{x})$ |
| $s(\mathbf{x})$ | $\phi(x)$ |
| $H(\mathbf{x})$ | $\mathcal{J}(x)$ |
| $\mathcal{H}(s)$ | $\mathcal{L}(\phi)$ |
| $Z[H]$ | $Z[\mathcal{J}]$ |
| $F(H)$ | $W[\mathcal{J}]$ |
| $M(\mathbf{x})$ | $\phi_{\text{cl}}(x)$ |
| $G(M)$ | $-\Gamma(\phi_{\text{cl}})$ |

In the magnetic system, we can perform the Legendre transformation to get from Helmholtz free energy the Gibbs free enthalpy. We can do exactly the same in QFT. The only difference is in the name: the *effective action*

$$\Gamma[\phi] = W[\phi] - \int \mathrm{d}^4x\ \phi(x)\mathcal{J}(x).$$

We can expand the effective action in two ways.

- Around the vacuum expectation value $\phi_0$ as $\hat{\phi} = \phi - \phi_0$. The expansion is

$$\Gamma[\phi] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}^4x_1 \dots \mathrm{d}^4x_n\, \Gamma^{(n)}_{\phi_0(x)}(x_1, \dots, x_n)\hat{\phi}_1(x_1) \dots \hat{\phi}_n(x_n),$$

  where

$$\Gamma^{(n)}_{\phi_0(x)}(x_1, \dots, x_n) = \left( \prod_{i=1}^{n} \frac{\delta}{\delta\phi(x_i)} \right) \Gamma(\phi)\big|_{\phi=\phi_0}.$$

- Assuming that $\phi(x)$ is slowly varying, we can perform a *gradient expansion*:

$$\Gamma(\phi) = \int \mathrm{d}^4x\ \left( -V_{\mathrm{eff}}(\phi) + \frac{1}{2} Z[\phi]\partial_\mu\phi\partial^\mu\phi + \dots \right).$$

  The potential $V_{\mathrm{eff}}$ is called the *effective potential*.

There are four important properties of the effective action.

1. The effective action generates all one-particle-irreducible (1PI) Green's functions. Following the Weinberg QFT2 book, chapter 16, we can see that for $n = 2$ it holds

$$\begin{aligned} \delta(x-y) = \frac{\delta\phi(x)}{\delta\phi(y)} &= \frac{\delta}{\delta\phi(y)} \frac{\delta W}{\delta\mathcal{J}(x)} \\ &= i \int \mathrm{d}^4z\ \frac{\delta^2(iW)}{i\delta\mathcal{J}(x) i\delta\mathcal{J}(z)} \frac{\delta\mathcal{J}(x)}{\delta\phi(x)} \\ &= -\int \mathrm{d}^4z\ G^{(2)\mathrm{c}}(x,z) \frac{\delta^2\Gamma}{\delta\phi(x)\delta\phi(z)}. \end{aligned}$$

   It follows that the second derivative of the effective action is the inverse of the connected two-point function (for zero current it equals propagator), $\Gamma^{(2)}(z,y) = iG^{(2)\mathrm{c}}_{\mathcal{J}}(z,y)^{-1}$. Iteratively, we can write any derivative of the effective action, and we can see that they equal 1PI $n$-point functions. For the 3-point function, we have to use some tricks. Firstly, the functional derivative with respect to $\mathcal{J}$ can be expanded using the chain rule

$$\frac{\delta}{\delta\mathcal{J}(z)} = \int \mathrm{d}^4w\ \frac{\delta\phi_0(w)}{\delta\mathcal{J}(z)} \frac{\delta}{\delta\phi_0(w)} = i \int \mathrm{d}^4w\ G^{(2)}_{\mathcal{J},\mathrm{c}}(z,w) \frac{\delta}{\delta\phi_0(x)}.$$

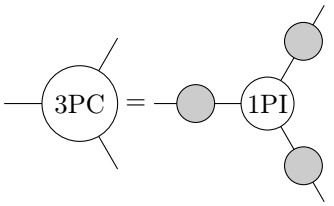
   Another useful trick is the derivative of the inverse operator:

$$\frac{\partial}{\partial\alpha} M^{-1}(\alpha) = -M^{-1} \frac{\partial M}{\partial\alpha} M^{-1}.$$

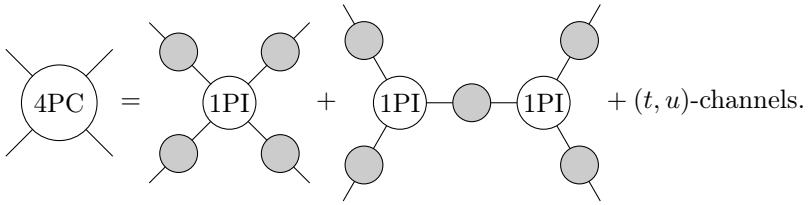
Using these two tricks, we can write the 3-point connected function as

$$\begin{aligned}
G^{(3)}_{\mathrm{c}}(x_1,x_2,x_3) &= \frac{\delta^3 W[\mathcal{J}]}{\delta\mathcal{J}(x_1)\delta\mathcal{J}(x_2)\delta\mathcal{J}(x_3)} \\
&= i\int \mathrm{d}^4 z_1\; G^{(2)}_{\mathcal{J},\mathrm{c}}(x_1,z_1)\frac{\delta}{\delta\phi_0(z_1)}\left(\frac{\delta^2\Gamma}{\delta\phi_0(x_2)\delta\phi_0(x_3)}\right)^{-1} \\
&= i\int \mathrm{d}^4 z_1\; G^{(2)}_{\mathcal{J},\mathrm{c}}(x_1,z_1)\int \mathrm{d}^4 z_2\;\int \mathrm{d}^4 z_3 \\
&\qquad\qquad (-iG^{(2)}_{\mathcal{J},\mathrm{c}}(x_2,z_2))\frac{\delta^3\Gamma}{\delta\phi_0(z_1)\delta\phi_0(z_2)\delta\phi_0(z_3)}(-iG^{(2)}_{\mathcal{J},\mathrm{c}}(x_3,z_3)) \\
&= \int \mathrm{d}^4 z_1\;\mathrm{d}^4 z_2\,\mathrm{d}^4 z_3\;\frac{i\delta^3\Gamma}{\delta\phi_0(z_1)\delta\phi_0(z_2)\delta\phi_0(z_3)}G^{(2)}_{\mathcal{J},\mathrm{c}}(x_1,z_1)G^{(2)}_{\mathcal{J},\mathrm{c}}(x_2,z_2)G^{(2)}_{\mathcal{J},\mathrm{c}}(x_3,z_3).
\end{aligned}$$

A way to interpret this result is the following: a three-point connected function equals the third derivative of the effective action and two-point connected diagrams on each of the external legs. This is exactly the structure of the 1PI diagrams, where each external leg absorbs a maximal connected two-point diagram. Below is a pictorial representation of this statement: each grey circle represents a connected function and the label 3PC means three-point connected function.




The diagram below shows the decomposition of the connected 4-point function.




2. The stationary points of effective action/potential will describe the vacuum expectation value. Denote the vacuum expectation value at some source $\mathcal{J}$ as

$$\phi_{\mathrm{cl}} = \langle\Omega|\phi(x)|\Omega\rangle_{\mathcal{J}} = \left.\frac{\delta W[\mathcal{J}]}{\delta\mathcal{J}(x)}\right|_{\mathcal{J}}.$$

If $\phi_{\mathrm{cl}} \neq 0$ for $\mathcal{J} = 0$ then this corresponds to a field configuration where $\Gamma[\phi]$ is stationary:

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_{\mathrm{cl}}} = 0.$$

To show this, write the derivative of the effective action

$$\begin{aligned}
\frac{\delta\Gamma[\phi]}{\delta\phi_{\mathrm{cl}}(y)} &= \frac{\delta}{\delta\phi_{\mathrm{cl}}(y)}\left(W[\mathcal{J}] - \int [4]\phi_{\mathrm{cl}}(x)\mathcal{J}(x)\right) \\
&= \int \mathrm{d}^4 x\;\frac{\delta W}{\delta\mathcal{J}(x)}\frac{\delta\mathcal{J}(x)}{\delta\phi_{\mathrm{cl}}(y)} - \int \mathrm{d}^4 x\;\left(\frac{\delta\mathcal{J}(x)}{\delta\phi\mathrm{cl}(y)}\phi_{\mathrm{cl}}(x) - \mathcal{J}(x)\right).
\end{aligned}$$

Using $\phi_{\text{cl}} = \langle\Omega|\phi(x)|\Omega\rangle = \delta W/\delta\mathcal{J}(x)$, we get

$$\frac{\delta\Gamma[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(y)} = -\mathcal{J}(y).$$

In the absence of the external source $\mathcal{J} = 0$, the derivative of the effective action is zero at $\phi_{\text{cl}}$. If we assume that the vacuum is Poincaré invariant, then the vacuum expectation value $\phi_{\text{cl}}$ is constant in space and time, i.e., $\phi_{\text{cl}}(x) = \phi_0 = \text{const}$. From the gradient expansion of the effective action, we get

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_0} = -\left.\frac{\partial V_{\text{eff}}}{\partial\phi}\right|_{\phi=\phi_0} = 0.$$

Hence, the vacuum expectation value $\phi_0$ is a stationary point of the effective potential $V_{\text{eff}}(\phi)$. The symmetry is spontaneously broken if the vacuum expectation is not equal to zero.

3. The classical limit ($\hbar \to 0$) is $\Gamma[\phi] = S[\phi]$. This means $\Gamma[\phi]$ contains the classical action and includes quantum corrections when $\hbar \neq 0$. To show this, we introduce $\hbar$ into the generating functional

$$Z[\mathcal{J}] = N\int\mathcal{D}[\phi]\exp\left(\frac{i}{\hbar}S[\phi] + \frac{i}{\hbar}\int \mathrm{d}^4x\ \mathcal{J}(x)\phi(x)\right).$$

Every propagator is proportional to $\hbar$, and every vertex is proportional to $\hbar^{-1}$. The connected diagram is then proportional to $\hbar^{P-V} = \hbar^{L-1}$, where $P$ is the number of propagators, $V$ is the number of vertices and $L$ is the number of loops. For more details, see notes for QFT1. When $\hbar \to 0$ the only diagrams that survive are the tree-level diagrams, hence the effective action is equal to the classical action (no quantum corrections).

4. The effective potential $V_{\text{eff}}(\phi_0)$ is the minimum energy density under consraint that $\phi = \langle\Omega|\phi(x)|\Omega\rangle$ takes a vacuum expectation value. The equation $\delta\Gamma/\delta\phi = 0$ (for $\mathcal{J} = 0$) is the quantum version of the Euler-Lagrange equation. Indeed, for $\hbar \to 0$, we get $\delta S/\delta\phi = 0$.

## 2.3 Quantum Goldstone theorem

The symmetry of the effective action tightly relates to the symmetry of the action.

> *If $S[\phi]$ is invariant under a non-anomalous linear symmetry, then the effective action is also invariant.*

To show this, we will need to use the *Ward-Takahashi identity*. We will generally describe the symmetry of the action by

$$\phi_n(x) \longrightarrow \phi'_n(x) = \phi_n(x) + \varepsilon^a F^a_n[\phi_m],$$

and we will denote $\delta\phi = \varepsilon^a F^a_n[\phi_m]$. For a linear symmetry, we can write

$$F^a_n[\phi_m] = iT^a_{nm}\phi_m(x)$$

where $T^a_{nm}$ are the generators of the symmetry group. To show the Ward identity, we start with the generating functional:

$$\begin{aligned}
Z[\mathcal{J}] =& N \int \mathcal{D}[\phi_n] \exp\left( iS[\phi_n] + i \int \mathrm{d}^4x \; \mathcal{J}_n(x)\phi_n(x) \right) \\
=& N \int \mathcal{D}[\phi'_n] \exp\left( iS[\phi'_n] + i \int \mathrm{d}^4x \; \mathcal{J}_n(x)\phi'_n(x) \right) \\
=& N \int \mathcal{D}[\phi_n] \det\left( \frac{\delta\phi'_n}{\delta\phi_m} \right) \exp\left( iS[\phi_n] + i \int \mathrm{d}^4x \; \frac{\delta S}{\delta\phi_n}\delta\phi_n + i \int \mathrm{d}^4x \; \mathcal{J}_n \cdot (\phi_n + \delta\phi_n) \right).
\end{aligned}$$

The determinant is equal to 1 since the symmetry is non-anomalous and the term $\delta S/\delta\phi_n(x)$ is zero due to the invariance of the action. What remains can be expanded as

$$Z[\mathcal{J}] = N \int \mathcal{D}[\phi_n] \exp\left( iS[\phi_n] + i \int \mathrm{d}^4x \; \mathcal{J}_n(x)\phi_n(x) \right) \left[ 1 + i \int \mathrm{d}^4x \; \mathcal{J}_n(x)\varepsilon^a F^a_n[\phi_m] + \dots \right].$$

The first term is already the same as the original generating functional $Z[\mathcal{J}]$, hence all other terms have to vanish for each power of $\varepsilon^a$ independently. The Ward identity then reads

$$\int \mathcal{D}[\phi_n] \left( e^{is[\phi_n] + i\int \mathrm{d}^4x \; \mathcal{J}_n(x)\phi_n(x)} \int \mathrm{d}^4x \; \mathcal{J}_n(x) F^a_n[\phi_m](x) \right) = 0.$$

This identity implies

$$\begin{aligned}
0 &= \int \mathrm{d}^4x \; \mathcal{J}_n(x) i T^a_{nm} \int \mathcal{D}[\phi_n] e^{iS[\phi_n] + i\int \mathrm{d}^4x \; \mathcal{J}_n(x)\phi_n(x)} \phi_m(x) \\
&= i \int \mathrm{d}^4x \; \mathcal{J}_n(x) T^a_{nm} \left\langle \Omega | \phi_m(x) | \Omega \right\rangle_{\mathcal{J}} \\
&= i \int \mathrm{d}^4x \; \left( -\frac{\delta\Gamma}{\delta\phi^{\mathrm{cl}}_n(x)} \right) T^a_{nm} \phi^{\mathrm{cl}}_m(x) \\
&= -\int \mathrm{d}^4x \; \frac{\delta\Gamma}{\delta\phi_n(x)} F^a_n[\phi_m].
\end{aligned}$$

But this is precisely the change of $\Gamma[\phi]$ under the symmetry transformation $\delta^a\phi_n(x) = iT^a_{nm}\phi_m(x)$:

$$\Gamma[\phi'] = \Gamma[\phi_n + \varepsilon^a F^a_n(\phi_m)] = \Gamma[\phi_n] + \varepsilon^a \int \mathrm{d}^4x \; \frac{\delta\phi\Gamma}{\delta\phi_n(x)} F^a_n(\phi_m) = \Gamma[\phi].$$

This completes the proof that the effective action is invariant under the same symmetry as the classical action.

Equipped with this knowledge, we can now prove the quantum version of the Goldstone theorem using the mass matrix

$$m^2_{ab} = \left. \frac{\partial V}{\partial\phi_a \partial\phi_b} \right|_{\phi_0}.$$

We will assume that the theory is invariant under a non-anomalous symmetry group $G$. The effective action is hence also invariant under $G$-action

$$\int \mathrm{d}^4x \; \frac{\delta\Gamma[\phi]}{\delta\phi_n} T^a_{nm}\phi_m(x) = 0, \qquad \forall T^a \in \mathfrak{g},$$

where $\mathfrak{g}$ is the Lie algebra of the group $G$. Due to Poincaré invariance of the vacuum state ($\phi_m(x) = \phi_0 = \langle\Omega|\phi(x)|\Omega\rangle$) we can replace $V$ with $V_{\text{eff}}$ in the mass matrix definition and show the Goldstone's theorem for quantum systems. We will show that

$$\left.\frac{\partial^2 V_{\text{eff}}}{\partial\phi_a\partial\phi_b}\right|_{\phi=\phi_0} = M^2_{ab}$$

is the mass matrix. The effective potential can be expanded as

$$V_{\text{eff}} = -\sum_{n=0}^{\infty}\frac{1}{n!}\phi_{k_1}\dots\phi_{k_n}\tilde{\Gamma}^{(n)}_{k_1\dots k_n}(p_i = 0)$$

where $\tilde{\Gamma}^{(n)}_{k_1\dots k_n}(p_1,\dots,p_n)$ is the momentum space $n$-point one-particle irreducible (1PI) function. It is related to the regular $n$-point 1PI function by

$$(2\pi)^4\delta^{(4)}(p_1+\dots+p_n)\tilde{\Gamma}^{(n)}_{k_1\dots k_n}(p_1,\dots,p_n) = \int \mathrm{d}^4p_1\ \dots\ \mathrm{d}^4p_n\, e^{ip_1x_1+\dots+ip_nx_n}\Gamma^{(n)}_{k_1\dots k_n}(x_1,\dots,x_n).$$

To show that the expansion holds, we write (bear in mind that $\phi = \text{const.}$)

$$\begin{aligned}
-\int \mathrm{d}^4x\ V_{\text{eff}}(\phi) = \Gamma[\phi] &= \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}^4x_1\ \dots\ \mathrm{d}^4\,\phi_{k_1}\dots\phi_{k_n}\Gamma^{(n)}_{k_1\dots k_n}(x_1,\dots,x_n)\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\phi_{k_1}\dots\phi_{k_n}(2\pi)^4\delta^4(0)\tilde{\Gamma}^{(n)}_{k_1\dots k_n}(p_1=0,\dots,p_n=0)\\
&= \int \mathrm{d}^4x\ \sum_{n=0}^{\infty}\frac{1}{n!}\phi_{k_1}\dots\phi_{k_n}\tilde{\Gamma}^{(n)}_{k_1\dots k_n}(\mathbf{p}=0).
\end{aligned}$$

In the last step, we used that $(2\pi)^4\delta^{(4)}(0)$ equals the integral over the whole space. Since everything is constant, we can pull the integral out of the sum. The second derivative of the effective potential is then

$$\left.\frac{\partial^2 V_{\text{eff}}}{\partial\phi_a\partial\phi_b}\right|_{\phi=\phi_0} = -\tilde{\Gamma}^{(2)}_{ab}(\mathbf{p}=0) = -iG^{(2),\mathrm{c}}_{ab}(\mathbf{p}=0)^{-1}.$$

The last term is the inverse of the connected two-point function $\langle\Omega|T\{\phi_a\phi_b\}|\Omega\rangle$ in the momentum space. It holds

$$G^{(2),\mathrm{c}}_{ab}(p)^{-1} = p^2 - m^2_{ab} - \Pi_{ab}(\mathbf{p}) \overset{\mathbf{p}=0}{=} -(m^2+\Pi(0))_{ab} = iM^2_{ab}.$$

We used the tree-level mass matrix $m^2_{ab}$ and the quantum correction $\Pi_{ab}(0)$. The classical argument can then be repeated verbatim, and we conclude

$$M^2_{ab}O_b(\phi_0) = 0.$$

Since the poles of the Green's function correspond to the one-particle states (with $p^2 = m^2$) and since $\phi$ has to be a scalar, we conclude that there must be at least $n(G) - n(H)$ massless spin-0 particles, the Goldstone bosons.

## 2.4 General properties of the spontaneously broken symmetries

As we have seen, the vacuum state is not unique. Any element of the group $g \in G$ acting on a vacuum state yields another vacuum state, $\phi_0' = g\phi_0$. Indeed, the possible vacuum states span an entire submanifold in the space of all possible field configurations. This is called the *vacuum manifold*:

$$\mathcal{M}_0 = \{\phi_0 \,|\, V(\phi_0) = V_{\min}\}.$$

**Example 4.** In O($N$) case from before, the potential was $V(\phi) = \lambda(\phi^2 - v^2)^2/4$. For the vacuum state we chose $\phi_0 = (0, \ldots, 0, v)^T$, however $\phi_0 = (v, 0, \ldots, 0)$ is also a valid option. Any two vacuum states are related by an orthogonal transformation (rotation), hence the vacuum manifold is the $(N-1)$-sphere, $\mathcal{M}_0 \cong S^{N-1}$.

The symmetry of the vacuum state $H$ can be thought of as the *stabilizer subgroup* of the vacuum state, i.e., the subgroup of $G$ that leaves the vacuum state invariant:

$$H = \{h \in G \,|\, h\phi_0 = \phi_0 \ \forall \phi_0 \in \mathcal{M}_0\} \leq G.$$

The sign $\leq$ means that it is a subgroup. In the example of O($N$) the unbroken subgroup was $H = \mathrm{O}(N-1)$. For the choice of vacuum state $\phi_0 = (0, \ldots, 0, v)^T$, the stabilizer subgroup is

$$H = \left\{ \begin{bmatrix} X & 0 \\ 0 & 1 \end{bmatrix} ; \ X \in \mathrm{O}(N-1) \right\} \leq \mathrm{O}(N).$$

For a different vacuum state $\phi_0' = (v, 0, \ldots, 0)^T$, the corresponding stabilizer subgroup is conjugate to the previous one. Hence, all stabilizers are isomorphic. The set of all degenerate vacua can be written as a left coset space

$$\mathcal{M}_0 \cong G/H.$$

In this quotient group, two elements $g_1, g_2 \in G$ are identified if there exists $h \in H$ such that $g_1 = hg_2$. The dimension of the vacuum manifold follows from the first isomorphism theorem:

$$\dim(G/H) = \dim(G) - \dim(H).$$

**Example 5.** The dimension of the quotient between O($N$) and O($N-1$) is

$$\dim(\mathrm{O}(N)/\,\mathrm{O}(N-1)) = \dim(\mathrm{O}(N)) - \dim(\mathrm{O}(N-1)) = \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N-1.$$

For a more general treatment of spontaneously broken symmetry we consider the conserved charges (generators) of the symmetry group $G$:

$$Q^a = \int \mathrm{d}^3x \ J_0^a(x)$$

together with associated variation of any operator $\mathcal{O}$:

$$\delta_a \mathcal{O} = i[Q^a, \mathcal{O}].$$

If the operator is the field itself, we have

$$\delta^a \Phi^i = i[Q^a, \Phi^i(x)] = i(T^a)^i{}_j \Phi^j(x)$$

where $T^a$ are the representation matrices of the corresponding symmetry group. The vacuum expectation value (VEV, sometimes called the order parameter) is defined as before, $\langle\Phi\rangle = \langle\Omega|\Phi|\Omega\rangle$. The symmetry is broken spontaneously if $\langle\Phi\rangle \neq 0$. Consequently,

$$\langle\Omega|[Q^a, \Phi^i]|\Omega\rangle = (T^a)^i{}_j \langle\Phi^j\rangle \neq 0$$

which immediately implies $Q^a|\Omega\rangle \neq 0$ for at least one generator $Q^a$. Thus, the symmetry generators fail to annihilate the vacuum whenever spontaneous symmetry breaking occurs. If $\langle\Phi\rangle = 0$ then also $Q^a|\Omega\rangle = 0$.

We can construct a one-particle Goldstone state entirely from symmetry considerations by Fourier-projecting the charge density on the vacuum:

$$|\pi^a(\mathbf{p})\rangle \propto \int \mathrm{d}^3x\, e^{-i\mathbf{x}\cdot\mathbf{p}} J_0^a(\mathbf{x})|\Omega\rangle\,.$$

From this, we can make two observations. First, the state clearly has a (spatial) momentum $\mathbf{p}$. To see this, look at the action of the momentum on this state

$$\begin{aligned}
\hat{P}^i|\pi^a(\mathbf{p})\rangle &\propto \int \mathrm{d}^3x\, e^{i\mathbf{p}\cdot\mathbf{x}}[\hat{P}^i, J_0^a(\mathbf{x})]|\Omega\rangle \\
&= \int \mathrm{d}^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}(-\partial_i J_0^a(\mathbf{x}))|\Omega\rangle \\
&\propto \mathbf{p}\int \mathrm{d}^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} J_0^a(\mathbf{x})|\Omega\rangle \\
&= \mathbf{p}\,|\pi^a(\mathbf{p})\rangle\,.
\end{aligned}$$

The last step was done by the integration by parts. The second observation is that when the momentum goes to zero, the energy also goes to zero. It can be shown by noting that in the soft-momentum limit, we get

$$\lim_{\mathbf{p}\to 0}|\pi^a(\mathbf{p})\rangle \propto Q^a|\Omega\rangle\,.$$

Since $Q^a|\Omega\rangle \neq 0$ in the spontaneously broken phase and $[Q^a, H] = 0$, this state is degenerate with the vacuum, $HQ^a|\Omega\rangle = Q^a H|\Omega\rangle = 0$, hence $E_{\min} = 0$. The Goldstone boson created by $J_0^a$ is thus exactly massless – a key result form general symmetry principles alone.

Goldstone bosons can be discussed at quantum level by using the effective action.

## 2.5 The Anderson-Higgs mechanism

The next question one might ask is how does spontaneous symmetry breaking manifest itself in gauge theories. First, we will recall the main results from the chapter about gauge theories. A global symmetry is promoted to local one $\phi(x) \to e^{i\alpha(x)}\phi(x)$. If we require the Lagrangian to be invariant under this transformation, we need to fix the derivative to be covariant, $\partial_\mu\phi \to \mathrm{D}_\mu\phi = \partial_\mu\phi + igA_\mu\phi$. The field $A_\mu$ is called the gauge field and transforms as $A_\mu \to A_\mu + \partial_\mu\alpha(x)/g$. Using this, we showed that the derivative of a field transforms exactly he same as the field alone $\mathrm{D}_\mu\phi \to e^{i\alpha(x)}\mathrm{D}_\mu\phi$. After adding the kinetic term for the gauge field, the invariant Lagrangian becomes (in the Abelian case)

$$\mathcal{L} = (\mathrm{D}_\mu\phi)^\dagger(\mathrm{D}^\mu\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

To get the energy of the ground state, we can split the kinetic terms to temporal and spatial part and write

$$E(A_\mu, \phi) = \int \mathrm{d}^3x \, \left( \frac{1}{2}(F_{0i})^2 + \frac{1}{4}(F_{ij})^2 + (\mathrm{D}_0\phi)^\dagger(\mathrm{D}_0\phi) + (\mathrm{D}_i\phi)^\dagger(\mathrm{D}_i\phi) + V(\phi^\dagger, \phi) \right).$$

The goal is to find the combinations of $\phi$ and $A_\mu$ that minimize the energy in the ground state. Due to gauge invariance we know that if $(A_\mu^{(0)}, \phi^{(0)})$ is a ground state, then $(A_\mu^{(0)} + \partial_\mu\alpha/g, e^{i\alpha}\phi^{(0)})$ is also one. The gauge kinetic terms $(F_{0i})^2/2 + (F_{ij})^2/4$ are minimized if the gauge field is "pure gauge"

$$A_\mu = \frac{1}{g}\partial_\mu\alpha(x).$$

The scalar kinetic terms $(\mathrm{D}_0\phi)^\dagger(\mathrm{D}_0\phi) + (\mathrm{D}_i\phi)^\dagger(\mathrm{D}_i\phi)$ are minimized by the condition

$$\mathrm{D}_\mu\phi = (\partial_\mu - i\partial_\mu\alpha(x))\phi(x) = 0$$

which has a solution

$$\phi(x) = e^{i\alpha(x)}\frac{\phi_0}{\sqrt{2}}$$

where $\phi_0$ is some constant. We can get this constant from minimizing the potential $V(\phi^\dagger, \phi)$, $\phi_0 = \mu/\sqrt{\lambda}$. Let's choose a convenient value for $\alpha$, say $\alpha(x) = 0$. The minimizing fields are then

$$A_\mu^{(0)} = 0, \qquad \phi^{(0)} = \frac{\phi_0}{\sqrt{2}} = \frac{\mu}{\sqrt{2\lambda}}.$$

The small excitations around the ground state can be parametrized as

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_0 + \chi(x) + i\theta(x))$$

where $\chi$ and $\theta$ are real fields. The potential up to quadratic order (and ignoring any constant contributions) is

$$V(\chi, \phi) = \mu^2\chi^2 + \mathcal{O}(\text{field}^3).$$

The field derivative is now slightly easier

$$\mathrm{D}_\mu\phi = \frac{1}{\sqrt{2}}(\partial_\mu\chi - i\partial_\mu\theta - ie\phi_0 A_\mu) + \mathcal{O}(\text{field}^2).$$

The gauge coupling was renamed to $e$. Writing out the Lagrangian to quadratic order in fields, we find

$$\mathcal{L}^{(2)} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu\chi)^2 - \mu^2\chi^2 + \frac{e^2\phi_0^2}{2}\left(A_\mu - \frac{1}{e\phi_0}\partial_\mu\theta\right)^2.$$

Since we don't like the mixed term $A_\mu\partial^\mu\theta$ we can introduce a new field

$$B_\mu = A_\mu - \frac{1}{e\phi_0}\partial_\mu\theta$$

to get a more familiar version of Lagrangian

$$\mathcal{L}^{(2)} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{e^2\phi_0^2}{2}B_\mu B^\mu + \frac{1}{2}(\partial_\mu\chi)^2 - \mu^2\chi^2.$$

The kinetic term didn't change, because the new field $B_\mu$ looks exactly as a gauge transformation of $A_\mu$. The Goldstone boson $\theta$ is now absorbed into the gauge field $B_\mu$, which has suddenly acquired a mass $m_B = e\phi_0$. The same works in the non-linear sigma model parametrization,

$$\phi(x) = \frac{\rho}{\sqrt{2}} e^{i\beta(x)}.$$

The covariant derivative is

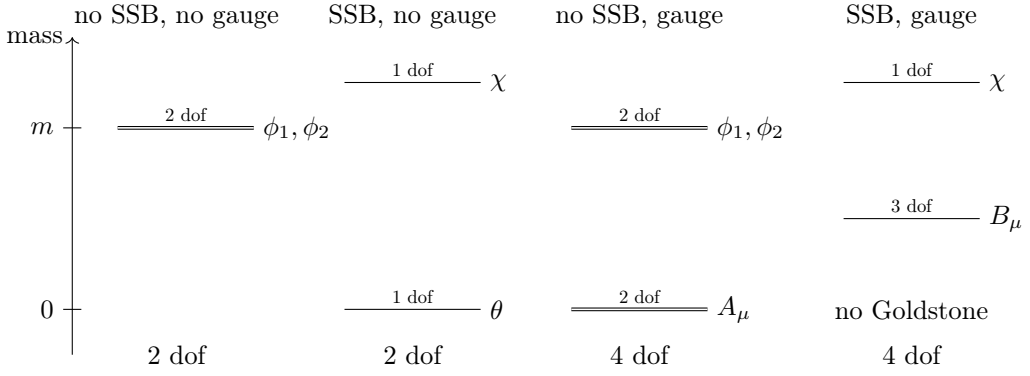
$$\mathrm{D}_\mu \phi = \frac{1}{\sqrt{2}}(\partial_\mu + i\rho(\partial_\mu \beta - eA_\mu))e^{i\beta}$$

and the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu \rho)^2 + \frac{e^2\rho^2}{2}\left(A_\mu - \frac{1}{e}\partial_\mu \beta\right)^2 - V(\rho).$$

Reparametrizing again, $B_\mu = A_\mu - \partial_\mu \beta / e$, yields a Lagrangian that is independent of $\beta$:

$$\mathcal{L} - \frac{1}{4}B_{\mu\nu}^2 + \rho^2 \frac{e^2}{2} B_\mu^2 + \frac{1}{2}(\partial_\mu \rho)^2 - V(\rho).$$




## 2.6 Higgs' mechanism in non-Abelian gauge theory

Consider the easiest example, SU(2). We have two complex fields $\phi_1, \phi_2$ in the fundamental representation. Denoting $\phi = (\phi_1, \phi_2)^T$, the Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu,a} + (\mathrm{D}_\mu \phi)^\dagger (\mathrm{D}^\mu \phi) - (\lambda(\phi^\dagger \phi)^2 - \mu^2 \phi^\dagger \phi).$$

The energy functional is similar to what we had in the Abelian case

$$E(A_\mu, \phi) = \int \mathrm{d}^3 x \, \left(\frac{1}{2}(F_{\mu\nu}^a) + (\mathrm{D}_\mu \phi)^\dagger (\mathrm{D}^\mu \phi) + V(\phi^\dagger, \phi)\right).$$

As before, the ground state satisfies $F_{\mu\nu}^a = 0$, which means that the gauge field is pure gauge, $A_\mu(x) = \omega(x)\partial_\mu \omega^{-1}(x)$ with $\omega(x) \in \mathrm{SU}(2)$. The minimization condition for the scalar field is $\mathrm{D}_\mu \phi(x) = 0$, so $\phi(x) = \omega(x)\phi^{(0)}$, where $\phi^{(0)}$ is a complex (constant) vector that minimizes the potential. The condition for the minimization of potential is

$$\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial \phi^\dagger} = 0$$

and the solution is already well known $\phi^\dagger\phi = \mu^2/2\lambda$. For convenience, we choose a particular $\phi^{(0)}$ that minimizes the potential:

$$(A^a_\mu)^{(0)} = 0, \qquad \phi^{(0)} = \begin{pmatrix} 0 \\ \phi_0/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix},$$

where $\phi_0 = \mu/\sqrt{\lambda} = \sqrt{2}v$. The expansion of the scalar field around the ground state gives

$$\phi(x) = \omega(x) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(\phi_0 + \chi(x)) \end{pmatrix}.$$

Can we really write this? Is this enough general. Well, the most general expansion of the scalar field around the ground state would be

$$\phi(x) = \begin{pmatrix} \xi_1(x) + i\xi_2(x) \\ v + \xi_3(x) + i\xi_4(x) \end{pmatrix}$$

where all fields $\xi_i(x)$ are real. The SU(2) operator field $\omega(x)$ takes the general form

$$\omega(x) = \exp(i\sigma^i\beta^i(x)/2) \approx \mathbb{1} + \begin{pmatrix} \beta^3(x) & \beta^1(x) - i\beta^2(x) \\ \beta^1(x) + i\beta^2(x) & -\beta^3(x) \end{pmatrix}.$$

Multiplying the ground state with the SU(2) field gives

$$\phi(x) = \begin{pmatrix} iv\beta^1(x) + v\beta^2(x) \\ v + \chi(x)/2 - iv\beta^3(x) \end{pmatrix}.$$

Hence, the expression we have is indeed the most general expansion possible. Now, in order to simplify our lives, we can choose the most convenient gauge, say $\omega(x) = \mathbb{1}$. This is called the *unitary gauge* and gives the following expansion around the ground state:

$$\phi(x) = \begin{pmatrix} 0 \\ (\phi_0 + \chi(x))/\sqrt{2} \end{pmatrix}.$$

We expand the Lagrangian to the quadratic order using

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu$$

and

$$\mathrm{D}_\mu\phi = \partial_\mu\phi - ig\frac{\sigma^a}{2}A^a_\mu\phi.$$

Remember that for SU(2) the generators are $\sigma^a/2$ (Pauli matrices) and the structure constants are $\varepsilon^{abc}$ (Levi-Civita symbols). The Lagrangian to the quadratic order in fields is

$$\mathcal{L}^{(2)} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + \frac{1}{2}(\partial_\mu\chi)^2 - \mu^2\chi^2 + \frac{g^2\phi_0^2}{2}A^a_\mu A^{\mu,a}.$$

We have seen that we got three massive vector fields $M^a_\mu$ with mass $m_A = g\phi_0$ and one massive scalar field $\chi$ with mass $m_\chi = \sqrt{2}\mu$. In other words, the gauge fields have eaten the Goldstone mass, and they got fat in the process (acquired mass). The degrees of freedom are conserved in the process as seen on the following table

| Before | #DOF | After | #DOF |
|---|---|---|---|
| 2 complex scalars $\phi_1, \phi_2$ | 4 | 1 massive scalar $\chi$ | 1 |
| 3 massless gauge fields $A^a_\mu$ | $2 \cdot 3$ | 3 massive vector fields $M^a_\mu$ | $3 \cdot 3$ |
| | 10 | | 10 |

In general, if the symmetry is gauged, instead of getting $n(G) - n(H)$ massless scalars ($G$ and $H$ being the symmetry group and the stabilizer subgroup), we get $n(G) - n(H)$ *massive* gauge bosons. An overview of common types of spontaneous symmetry breaking is given in the following table.

| $G$ | $H$ | $n(G)$ | $n(H)$ | #GB |
|---|---|---|---|---|
| U(1) | $\varnothing$ | 1 | 0 | 1 |
| SU(2) | $\varnothing$ | 3 | 0 | 1 |
| SO(3) | SO(2) | 3 | 1 | 2 |
| SU(5) | SM = SU(3) × SU(2) × U(1) | 24 | 12 | 12 |

The last case is one particular example of Grand Unified Theories (GUTs).

Vector bosons acquiring mass presents a new problem. We have seen in QFT1 that massive vector field are not renormalizable (using, for example, the superficial degree of divergence). This can be most easily seen from the propagator

$$\frac{i\delta^{ab}}{p^2 - m_A^2 + i\varepsilon}\left(-\eta_{\mu\nu} + \frac{p^\mu p^\nu}{m_A^2}\right).$$

However, we still have a choice of gauge. When we change it, the Goldstone boson also contributes to the propagating degrees of freedom (and this is common to do in loop calculations). A natural choice of the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \frac{m_A^2}{2}\left(A^a_\mu - \frac{1}{m_A^2}\partial_\mu \pi^a + \dots\right)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu - \xi m_A \pi^a)^2 + \dots$$

where the Goldstones are in the non-linear sigma parametrization. The Faddeev-Popov procedure yields the following propagator

$$\frac{i}{p^2 - m_A^2}\left(-\eta_{\mu\nu} + \frac{p^\mu p^\nu}{p^2 - \xi m_A^2}(1-\xi)\right)\delta^{ab}.$$

This propagator has the desired $p^{-2}$ scaling in the large momentum limit. The unitary gauge is recovered in the limit $\xi \to \infty$. However, we still have the field $\pi(x)$ that needs a propagator and Faddeev-Popov procedure also introduces ghosts. Propagators for $\pi$ and ghosts are the same

$$\frac{i\delta^{ab}}{p^2 - \xi m_A^2}.$$

Ghosts have a mass that depends on the gauge parameter $\xi$, but the unitary gauge removes both Goldstone $\pi$ and ghosts. Note that one can show that a theory with massive vector bosons, whose mass is generated by a Higgs mechanism or a linear sigma model, is renormalizable. In the UV ($E \gg m_A$) spontaneous symmetry breaking becomes a small effect and the linear sigma model is just a gauge theory coupled to a linearly transforming matter field and is renormalizable for the same reason that non-Abelian gauge theories are renormalizable.

## 2.7 Coleman-Weinberg model

As we have seen, the vacuum state $\phi_0$ must be constant in order to be Poincaré invariant. Classically, it takes such value to minimize the tree-level potential $V(\phi_0) = V_{\min}$. In order to find the potential that is corrected by quantum fluctuations, we introduced the effective action $\Gamma[\phi]$. We started with the generating functional of Green's functions

$$Z[\mathcal{J}] = \frac{\int \mathcal{D}\phi \, \exp(iS[\phi] + i\int \mathrm{d}^4x \, \mathcal{J}\phi)}{\int \mathcal{D}\phi \, \exp(iS[\phi])},$$

converted it into the generating functional of connected Green's functions $W[\mathcal{J}] = -i \ln Z[\mathcal{J}]$ and then performed the Legendre transformation to get the effective action

$$\Gamma[\phi] = W[\mathcal{J}] - \int \mathrm{d}^4x \, \phi(x)\mathcal{J}(x).$$

Assuming that the field $\phi$ is slowly varying, we performed the gradient expansion

$$\Gamma(\phi) = \int \mathrm{d}^4x \, (-V_{\text{eff}} + Z[\phi]\partial_\mu\phi\partial^\mu\phi/2 + \dots)$$

where $V_{\text{eff}}(\phi)$ is the *effective potential*. The main goal of this section is to show how to calculate the effective action. To do this, we employ the background field method, where we expand the field $\phi$ around a background field $\phi_S$: $\phi(x) = \phi_S(x) + \tilde{\phi}(x)$. The background field (sometimes also denoted as $\phi_{\text{cl}}$) solves the classical equations of motion, $\delta S/\delta\phi = \mathcal{J}$. The action expressed in terms of the fluctuations around the background field $\phi_S$ is

$$S[\phi] = S[\phi_S] + \int \mathrm{d}^4x \, \frac{1}{2}\tilde{\phi}(-\partial^2 - V''(\phi_S)/2)\tilde{\phi} + \int \mathrm{d}^4x \, \partial_\mu\phi_S\partial^\mu\phi_S + \dots$$

Inserting this into the path integral and replacing $\mathcal{D}[\tilde{\phi}]$ with $\mathcal{D}[\phi]$ (they differ only by a constant shift), we find

$$\begin{aligned} Z[\mathcal{J}] &= e^{iW[\mathcal{J}]} \\ &= \int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar}(S[\phi] + \int \mathrm{d}^4x \, \mathcal{J}\phi)\right) \\ &= \exp\left(\frac{i}{\hbar}S[\phi_S] + \frac{i}{\hbar}\int \mathrm{d}^4x \, \mathcal{J}\phi\right) \int \mathcal{D}[\tilde{\phi}] \exp\left(\frac{i}{\hbar}\left(\int \mathrm{d}^4x \, \frac{1}{2}(\partial^\mu\phi)^2 - \frac{1}{2}V''(\phi_S)\tilde{\phi}^2\right)\right). \end{aligned}$$

The Gaussian integration on the path integral and denoting $V''(\phi_S) = m_{\text{eff}}$ gives

$$\begin{aligned} I &= \int \mathcal{D}[\tilde{\phi}] \exp\left(\frac{i}{\hbar}\int \mathrm{d}^4x \, \frac{1}{2}\left((\partial^\mu\tilde{\phi})^2 - m_{\text{eff}}^2\tilde{\phi}^2\right)\right) \\ &= C\left(\frac{1}{\det(\partial^2 + m_{\text{eff}}^2)}\right)^{1/2} \\ &= C\exp\left(-\frac{\hbar}{2}\operatorname{tr}\ln(\partial^2 + m_{\text{eff}}^2)\right). \end{aligned}$$

Finally, we find

$$W[\mathcal{J}] = S[\phi_S] + \int \mathrm{d}^4x \, \partial_\mu\phi_S + \frac{i\hbar}{2}\operatorname{tr}\ln(\partial^2 + V''(\phi_S)) + \dots$$

The Legendre transformation gives

$$\phi = \frac{\delta W}{\delta \mathcal{J}} = \underbrace{\frac{\delta(S[\phi_S] + \int \mathcal{J}\phi_S)}{\delta \phi_S}}_{0} \frac{\delta \phi_S}{\delta \mathcal{J}} + \phi_S + \mathcal{O}(\hbar) = \phi_S + \mathcal{O}(\hbar).$$

This means that the full field $\phi$ is just the background field $\phi_S$ corrected to the order of $\hbar$. To the leading order in $\hbar$, we can write the effective action as

$$\Gamma[\phi] = S[\phi_S] + \frac{i\hbar}{2} \operatorname{tr} \ln(\partial^2 + V''(\phi(x))) + \mathcal{O}(\hbar^2).$$

This is not possible to calculate in general, but we are in fact interested in the case $\phi(x) = \phi_0 =$ const. (as we want the vacuum to be Poincaré invariant). In this case, the operator $\partial^2 + V''(\phi)$ is translation invariant, and we can calculate the effective action in the momentum space. Denote the kernel $K(x, y) = (\partial^2 + V''(\phi_0))\delta^{(4)}(x - y)$. The kernel in the momentum space is

$$K(p, q) = \int \frac{\mathrm{d}^4 x}{(2\pi)^4} e^{-ipx} \int \frac{\mathrm{d}^4 y}{(2\pi)^4} e^{-iqy} K(x, y) = (-p^2 - V''(\phi_0))\delta^{(4)}(p - q).$$

The trace can be computed as

$$\begin{aligned} \operatorname{tr} \ln(K) &= \int \mathrm{d}^4 p \ \ln(K(p, p)) \\ &= \delta^{(4)}(0) \int \mathrm{d}^4 p \ \ln(-p^2 + V''(\phi_0)) \\ &= \frac{VT}{(2\pi)^4} \int \mathrm{d}^4 p \ \ln(-p^2 + V''(\phi_0)). \end{aligned}$$

When we plug this into the effective action, we can extract the effective potential

$$V_{\text{eff}}(\phi_0) = V(\phi_0) - \frac{i\hbar}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \ln\left(\frac{k^2 - V''(\phi_0)}{k^2}\right) + \mathcal{O}(\hbar^2).$$

We added the term $k^2$ in the denominator of the logarithm, which just changes the value of the potential by a constant factor. Sadly, this expression is horribly divergent, so we introduce the hard cut-off regulator:

$$\begin{aligned} I(m_{\text{eff}}^2) &= \frac{2\pi^2}{2(2\pi)^4} \hbar \int_0^\Lambda \mathrm{d}k_{\text{E}} \, k_{\text{E}}^2 \ln\left(1 + \frac{m_{\text{eff}}^2}{k_{\text{E}}^2}\right) \\ &= \frac{1}{128\pi^2} \left(2m_{\text{eff}}^2(\phi_0)\Lambda^2 + 2m_{\text{eff}}^4(\phi_0) \ln\left(\frac{m_{\text{eff}}^2}{\Lambda^2}\right) + c\Lambda^4 + \text{const.}\right). \end{aligned}$$

Hence,

$$V_{\text{eff}}(\phi) = V(\phi) + \underbrace{A + B\phi^2 + C\phi^4}_{\text{counterterms}} + \frac{\Lambda^2}{32\pi^2} V''(\phi) - \frac{1}{64\pi^2} (V''(\phi))^2 \ln\left(\frac{e^{1/2}\Lambda^2}{V''(\phi)}\right).$$

The effective potential is renormalizable if the original potential doesn't contain any irrelevant operators, i.e., fields of power more than 4. If, for example, the potential included the term $c\phi^6$, the effective action would get the divergent term $V''(\phi)^2 \sim \phi^8$, which would require an infinite

number of counter-terms. Using the effective action, we can write the renormalized parameters as

$$\begin{aligned} m_{\mathrm{R}}^2 =& m^2 + \lambda c_2 + 2\lambda c_3, \\ \lambda_{\mathrm{R}} =& \lambda + 6\lambda^2 c_3, \\ \Lambda_{\mathrm{R}} =& c_1 + c_2 m^2 + c_3 m^4, \end{aligned}$$

where the coefficients $c_i$ come from the effective action

$$V_{\mathrm{eff}}(\phi) = V(\phi) + c_1 + c_2 m_{\mathrm{eff}}^2(\phi) + \frac{1}{64\pi^2} m_{\mathrm{eff}}^4 \ln(m_{\mathrm{eff}}^2) + c_3 m_{\mathrm{eff}}^4.$$

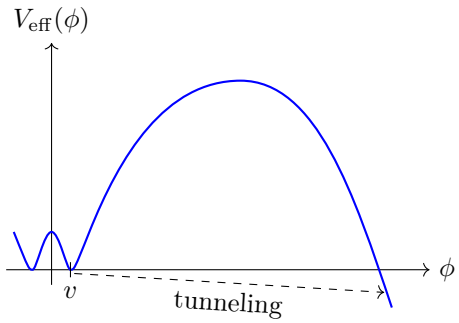
The renormalized potential is then

$$V_{\mathrm{eff}}^{\mathrm{R}}(\phi) = \frac{1}{2} m_{\mathrm{R}}^2 \phi^2 + \frac{\lambda_{\mathrm{R}}}{4!}\phi^4 + \hbar \frac{1}{64\pi^2} \left( m_{\mathrm{R}}^2 + \frac{\lambda_{\mathrm{R}}\phi^2}{2} \right)^2 \ln\left( m_{\mathrm{R}}^2 + \frac{\lambda_{\mathrm{R}}\phi^2}{2} \right).$$

This is called the *Coleman-Weinberg potential* (not Steven Weinberg but Eric Weinberg). We can add fermion functionals to this potential, $\mathcal{L}_{\mathrm{Yukava}} = g\phi\overline{\Psi}\Psi$. With this, we get a $\phi$-dependent fermion mass $m_\Psi(\phi) = m_\Psi^{(0)} + g\phi$. The contribution to the potential is

$$V_{\mathrm{eff}}^{\mathrm{R},\Psi}(\phi) = -\frac{(m_\Psi^{(0)} + g\phi)^4}{32\pi^2} \ln((m_\Psi^{(0)} + g\phi)^2).$$

The minus sign is from the fermion loop. What this could mean, is that the shape of the potential turns around at some very high energy ($\sim 10^{11}\,\mathrm{GeV}$) when this term starts to dominate. This means the local minimum our universe sits in right now might not be the global minimum. This is called the *metastable vacuum* since the expected time for quantum tunnelling to the global minimum is much greater than the age of the universe. So we can sleep well.



# Chapter 3

# The strong interaction

## 3.1 Quantum chromodynamics

We will start the discussion a bit more generally and first consider a SU($N_c$) gauge theory with $N_f$ quarks and the coupling constant $g_\mathrm{s}$. The coupling of such theory runs according to the beta function

$$\beta(g_\mathrm{s}) = \mu \frac{\mathrm{d}g_\mathrm{s}}{\mathrm{d}\mu} = -b_0 \frac{g_\mathrm{s}^3}{(4\pi)^3}.$$

The coefficient $b_0$ is given by

$$b_0 = \frac{11N_c - 2N_f}{3}.$$

The first term comes from the gauge field loops and the second part comes from fermion loops (in QED we only had the second term). Calculating this coefficient was a great triumph of the QCD theory. The solution to the renormalization group equation is

$$\frac{1}{g_\mathrm{s}^2(\mu)} = \frac{1}{g_0^2} - \frac{b_0}{(4\pi)^3} \ln\left(\frac{\Lambda_\mathrm{UV}^2}{\mu^2}\right)$$

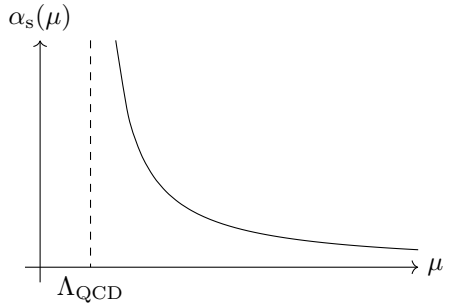
where $g_0 = g_\mathrm{s}(\Lambda_\mathrm{UV})$. In QCD, we have $N_c = 3$ and $N_f = 6$, so $b_0 = 7$ and $\beta(g) < 0$. This means that the coupling constant decreases with energy, i.e., the interaction becomes weaker at higher energies. This is called *asymptotic freedom* and is a key feature of QCD. Comparing this with QED (or any Abelian gauge theory), the $\beta$-function is positive, so the coupling constant increases with energy:

$$\alpha_\mathrm{QED}(m_Z) = \frac{1}{129}, \quad \alpha_\mathrm{QED}(m_e) = \frac{1}{137}.$$

The reference scale for the QCD coupling is at the mass of the $Z$-boson, $m_Z \approx 91\,\mathrm{GeV}$ and the value of the coupling constant is

$$\alpha_\mathrm{s}(m_Z) = 0.1181 \pm 0.0009.$$

The theory of weakly interacting particles (at high energies, e.g. LHC) can be done using the Feynman calculus. However, the gauge coupling increases severely at low energies, so the perturbation theory breaks down.




To determine exactly when the coupling constant has a pole, we can approximate it roughly by just using the first term in the beta function

$$\Lambda_{\text{QCD}} \approx \mu \exp\left(-\frac{8\pi^2}{b_0 g_s^2(\mu)}\right) \approx 200\,\text{MeV}.$$

For more accuracy, we would need to add all the higher order corrections to the beta function. Using the state-of-the-art fourth order corrections, the QCD scale is around $\Lambda_{\text{QCD}} \approx 213\,\text{MeV}$. In the low-energy limit, the perturbation theory breaks down, and we observe a lot of interesting physics: bound states and colour confinement.

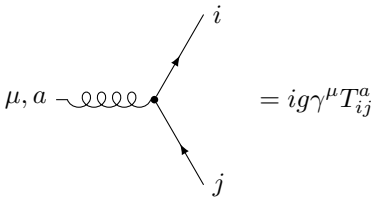
Let's consider the high energy regime for now. The action is

$$S_{\text{QCD}} = \int \mathrm{d}^4x \left(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + i\overline{\Psi}_i i\not{\partial}_{ij}\Psi_j - m_i\overline{\Psi}_i\Psi_i\right),$$

where the indices $i, j$ run from 1 to $N_f = 3$. The gauge particles are called *gluons* and the fermions *quarks*. In $\xi = 1$ gauge, the gluon propagator is

$$\nu, b \sim\sim\sim \mu, a \quad = i\frac{-\eta^{\mu\nu} + (1-\xi)p^\mu p^\nu/p^2}{p^2 + i\varepsilon}\delta^{ab}$$

and the gluon-quark vertex is




where $T^a = \lambda^a/2$ are the generators of SU(3) in the fundamental representation. The matrices $\lambda^a \in \mathbb{C}^{3\times3}$ are the *Gell-Mann matrices* and are defined as

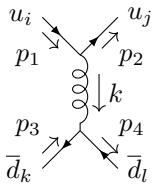
$$\lambda^1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \quad \lambda^2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}, \quad \lambda^3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$
$$\lambda^4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \quad \lambda^5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}, \quad \lambda^6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$
$$\lambda^7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}, \quad \lambda^5 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.$$

All matrices are rank 2, except $\lambda^8$, which is diagonal. The irreducible representations of SU(3) are classified by the Casimir $T^aT^a$ and $T^3$ and $T^8$ (those two are diagonal, so they trivially commute with all other generators). It also holds that $\mathrm{tr}(T^aT^a) = \delta^{ab}/2$. The masses of the quarks are

$$\begin{aligned} m_d &\approx 5\,\mathrm{MeV}, & m_u &\approx 2\,\mathrm{MeV}, \\ m_s &\approx 95\,\mathrm{MeV}, & m_c &\approx 1.3\,\mathrm{GeV}, \\ m_b &\approx 4.2\,\mathrm{GeV}, & m_t &\approx 173\,\mathrm{GeV}. \end{aligned}$$

Only the first three quarks (up, down, strange) are below the QCD scale. Below the QCD scale, the quarks are confined into composite particles called *hadrons*. Since the perturbation theory breaks down, we need to use the *lattice QCD* to calculate the properties of hadrons. If one switches the quarks off and only works with gluons (pure Yang-Mills theory) a very interesting phenomenon appears: the *mass gap* – there are no states with the energy below the mass of the *glueball* ($0^{++}$) which has the mass of around $\Lambda_{\mathrm{QCD}}$. This phenomenon is not understood yet analytically and is a part of the Millennium problems with a prize of 1 million dollars for a solution.

Let's consider a tree-level QCD amplitude for the process $u\overline{d} \to u\overline{d}$. The process is very similar to the QED process $e^+\mu^+ \to e^+\mu^+$. The scattering amplitude is




$$\begin{aligned} &= (ig_{\mathrm{s}}^2) T^a_{ji} T^b_{kl} \eta_{\mu\nu} \frac{-i}{k^2} \delta^{ab} \overline{u}_j(p_2)\gamma^\mu u_i(p_1) \cdot \overline{v}_k(p_3)\gamma^\nu v_l(p_4) \\ &= (ig_{\mathrm{s}})^2 T^a_{ij} T^a_{kl} \underbrace{\frac{1}{k^2} (\overline{u}_j \gamma_\mu u_i)(\overline{v}_k \gamma^\mu v_k)}_{\mathrm{QED}} . \end{aligned}$$

We see that the amplitude is almost exactly the same as in QED, except that we have to make the substitution

$$e^2 \to g_{\mathrm{s}}^2 T^a T^a.$$

To calculate the factor $T^aT^a$, we can use the *Fierz identity* for the fundamental representation of SU(3)

$$T^a_{ji} T^a_{kl} = \frac{1}{2} \left( \delta_{jl}\delta_{ik} - \frac{1}{N_c} \delta_{ij}\delta_{kl} \right),$$

where $N_c = 3$. Let's choose for example that $u$ is red ($i = 1$) and $d$ is anti-green ($k = 2$). We find

$$T^a_{j1} T^a_{2l} = -\frac{1}{2N_c} \delta_{1j}\delta_{2l}.$$

This means that the final state will have the same colour as the initial state, i.e., the colour is conserved. One important feature is also the negative sign in the Fierz identity. Since the interaction between electron an anti-muon is attractive, the interaction between quarks is *repulsive* (as the amplitude differs by a minus sign). The difference is also the factor 1/6 which is called the *colour factor*. Another choice of the initial state is $u$ being red and $d$ being anti-red. In this case, we find

$$T^a_{j1} T^a_{1l} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

This part of the amplitude gives *attractive* potential. The colour factor is the trace of this matrix, which is 4/3. In general, the product state of the initial particles (fundamental times anti-fundamental) can be split into the direct sum of irreducible representations

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8},$$

a singlet and an octet. The colour singlet is expressed as

$$|\mathbf{1}\rangle = \frac{1}{\sqrt{3}}(|r\bar{r}\rangle + |g\bar{g}\rangle + |b\bar{b}\rangle).$$

In the example where the colours of the quarks are the same, the final state can have any colour combination (as long as the outgoing quarks are anti-coloured). The tree-level potentials are

$$V^{\mathrm{QED}}(r) = -\frac{e^2}{4\pi r}, \qquad V^{\mathrm{QCD}}_{\mathrm{singlet}}(r) = -\frac{4}{3}\frac{g_s^2}{4\pi r}, \qquad V^{\mathrm{QCD}}_{\mathrm{octet}}(r) = +\frac{1}{6}\frac{g_s^2}{4\pi r}.$$

Heuristically, when the energy goes towards the QCD scale, we expect the colour singlet bound states (at least for two quark states) because the singlet potential is attractive (no bound colour octets). The colour singlet states of two quarks are called *mesons* ($q_i\bar{q}_i$).

## 3.2 Global symmetries of QCD

The QCD Lagrangian is

$$\mathcal{L}_{\mathrm{QCD}} = \overline{\Psi}_i \not{D} \Psi_i - (m_{ij}\overline{\Psi}_i\Psi_j + \mathrm{h.c.}).$$

What are the global symmetries of $\mathcal{L}_{\mathrm{QCD}}$? There is U(6) symmetry among the fields. The U(6) symmetry can be decomposed into $\mathrm{U}(6) = \mathrm{U}(1) \times \mathrm{SU}(6)$. The number of elements in the SU(6) is $n(\mathrm{SU}(6)) = N^2 - 1 = 35$

We use U(6) rotation to diagonalize and remove phases in the matrix $m_{ij} = \delta_{ij}m_j$, $m_j \in \mathbb{R}$. Initially the matrix $m_{ij}$ is complex symmetric, so there are

$$(\mathrm{angles}, \mathrm{phases}) = (N(N+1)/2, N(N+1)/2) = (21, 21)$$

parameters. By U(6) rotation, we can remove $N(N+1)/2 = 21$ phases and $N(N-1)/2 = 15$ real angles, so we are left with $21 - 15 = 6$ real parameters. These are the masses of the quarks. The Lagrangian is

$$\mathcal{L}_{\mathrm{QCD}} = i\overline{\Psi}_i \not{D} \Psi_i - \sum_i m_i \overline{\Psi}_i \Psi_i.$$

The global symmetry of this Lagrangian is

$$\mathrm{U}(1)^6 = \mathrm{U}(1)_u \times \mathrm{U}(1)_d \times \mathrm{U}(1)_s \times \mathrm{U}(1)_c \times \mathrm{U}(1)_t \times \mathrm{U}(1)_b.$$

This symmetry implies flavour conservation $\overline{u}u \to \overline{c}c$ (forbids reactions such as $\overline{u}c \to \overline{c}u$). However, when we include the weak interaction, this symmetry is broken and only the baryon number is conserved:

$$\mathrm{U}(1)^6 \to \mathrm{U}(1)_B.$$

Previously, we studied the interaction between two quarks. We found the potential behaves as $V(r) \sim r^{-1}$ for $r \ll \Lambda^{-1}_{\mathrm{QCD}}$. We have also seen that for singlet configurations of quarks, the

potential is attractive. However, when we increase the distance $r \gtrsim \Lambda_{\mathrm{QCD}}^2$ the potential starts to grow linearly $V(r) = \sigma r$. In the string theory, parameter $\sigma$ corresponds to the surface tension of the string. The colour is confined in so-called *flux tube*. If the distance between the quarks is large enough ($\sigma r \sim 2m_q$) the energy of the flux tube is converted into the mass of a new quark-antiquark pair, and the tube is split in two. These phenomena can be described by the lattice QCD.




Can we use symmetries to learn about low-energy QCD diagrams? The masses of the quarks are very hierarchically distributed:

| quark | up | down | strange | charm | bottom | top |
|---|---|---|---|---|---|---|
| mass [MeV] | 2.3 | 4.8 | 95 | 1275 | 4180 | 172000 |

We can split the theory into light quarks ($m \lesssim \Lambda_{\mathrm{QCD}}$, up, down, strange) and heavy quarks (charm, bottom, top). If we work in perturbation theory, we must be over the energy of $\Lambda_{\mathrm{QCD}}$, so we can assume that the masses of the light quarks are zero. This is clearly wrong, but it works surprisingly well. We can later add the masses as the perturbations to the theory. Since the lightest three quarks have the same(ish) mass, we can describe them by the U(3) symmetry. We will consider left- and right-handed spinors separately:

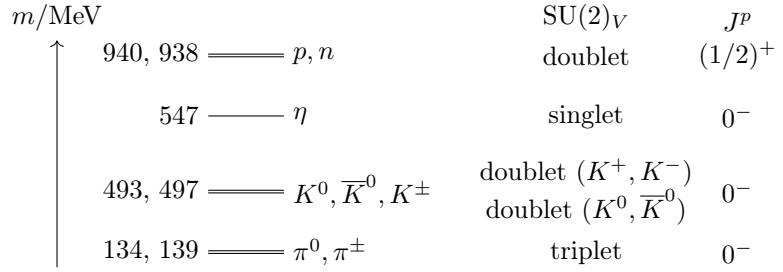
$$q_f^{\mathrm{L}} \to L_{ff'} q_{f'}^{\mathrm{L}}, \qquad q_f^{\mathrm{R}} \to R_{ff'} q_{f'}^{\mathrm{R}}$$

where $L, R \in \mathrm{U}(3)$. The Cartesian product of left- and right-handed symmetry is

$$\mathrm{U}(3)_{\mathrm{L}} \times \mathrm{U}(3)_{\mathrm{R}} = \mathrm{U}(1)_B \otimes \mathrm{U}(1)_A \otimes \mathrm{SU}(3)_V \otimes \mathrm{SU}(3)_A.$$

Let's discuss every symmetry separately:

- $\mathrm{U}(1)_B$: *baryon symmetry* $q_f \to e^{i\beta} q_f$. This symmetry remains even when we add masses and weak interaction. It implies that the lightest baryon (proton) is stable.
- $\mathrm{U}(1)_A$: *axial symmetry* $q_f \to e^{i\alpha\gamma^5} q_f$. The QCD action is invariant under this symmetry if we assume zero masses, however the corresponding current (and the effective action) is not conserved $\partial_\mu j_A^\mu = g_{\mathrm{s}}^2 F_{\mu\nu}^a \tilde{F}^{\mu\nu,a}$. This is called the chiral anomaly.
- $\mathrm{SU}(3)_V$: vector SU(3) symmetry. If $m = m_f$, this symmetry would hold. However, quark mass differences break this symmetry. Indeed, this is the case as $m_u, m_d \ll m_s$.
- $\mathrm{SU}(3)_A$: axial SU(3) symmetry. This symmetry gives the QCD resonance spectrum. In this spectrum, the $\mathrm{SU}(2)_V$ representation is manifest. Namely, we could do the same analysis for only the first two quarks $(u, d)$ and while breaking the $\mathrm{SU}(2)_A$ symmetry, the states can be sorted into different representations of the remaining $\mathrm{SU}(2)_V$ symmetry (which is not very broken as the masses of $u$ and $d$ are comparable). This can explain the lightness of the pions as the (pseudo-)Goldstone bosons (triplet representation of the unbroken representation $\mathrm{SU}(2)_V$). If the $\mathrm{SU}(3)_A$ symmetry actually existed, the Goldstones should be massless, but since it is only an approximate symmetry, pions still have some (relatively small) mass. They have $J^p = 0^-$ (spin 0, negative parity), which is another confirmation that they come from the spontaneously broken *axial* symmetry with the current $j_\mu^A = \overline{\Psi} T^a \gamma^\mu \gamma^5 \Psi$, where $T^a \in \mathrm{SU}(2)_A$ and $\Psi = (u, d)$. Furthermore, even heavier mesons (kaons, $\eta$) can be sorted into the representations of the unbroken $\mathrm{SU}(2)_V$ symmetry (as seen on the spectrum below).




Light mesons $\pi^0, \pi^\pm, K^0, \overline{K}^0, K^\pm, \eta$ all have $J^p = 0^-$. They have 8 degrees of freedom which suggests they might be (pseudo-)Goldstones for spontaneously broken $\mathrm{SU}(3)_A$ symmetry with the current $\overline{\Psi}\lambda^a\gamma^\mu\gamma^5\Psi$ for $\lambda^a \in \mathrm{SU}(3)$ and $\Psi = (u, d, s)$. This is the *octet* of mesons. There is one more meson, $\eta'$, which comes from the breaking of the $\mathrm{U}(1)_A$ symmetry. These nine particles together form the $\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \otimes \mathbf{8}$ representation of the unbroken $\mathrm{SU}(3)_V$ symmetry.

However, we do not observe $\mathrm{SU}(3)_A$ symmetry in the hadron spectrum (for once, they don't all have negative parity). This implies partners of SU(3) generators with opposite parity for each. The current $\overline{q}_f\gamma^\mu\gamma^5\lambda^a_{ff'}q_{f'}$ has negative parity (due to $\gamma^5$) and so has the conserved current $Q^a_A$. Take a proton $|p\rangle$ and assume it is $\mathrm{SU}(3)_A$ symmetric. Then $Q^a_A|p\rangle$ is also an eigenstate of Hamiltonian $H$. Applying the parity operator $U_\mathcal{P}$ to this state gives

$$U_\mathcal{P}(Q^a_A|p\rangle) = \underbrace{U_\mathcal{P}Q^a_AU^{-1}_\mathcal{P}}_{-Q^a_A}\underbrace{U_\mathcal{P}|p\rangle}_{+|p\rangle} = -Q^a_A|p\rangle\,.$$

This would predict an odd-parity state with the same mass as the proton, which is not observed. This means that the $\mathrm{SU}(3)_A$ symmetry is broken and $\delta_{\mathrm{SU}(3)_A}\Gamma \neq 0$. In other words, vacuum doesn't respect the $\mathrm{SU}(3)_A$ symmetry (again if $m_f = 0$) and it has spontaneous breaking of (approximate) $\mathrm{SU}(3)_A$ symmetry. Where does this breaking come from?

Recall the non-linear sigma model, where

$$\phi = (v + \sigma(x))e^{i\pi(x)/v}$$

with $m_\sigma = 2\mu$ and $m_\pi = 0$. Assume that we don't have enough energy to produce $\sigma$ particle ($E \ll m_\sigma$), which means we are only dealing with Goldstone bosons. How to set up most general Goldstone boson theory in IR?

Let's start with a familiar example, $\mathrm{SO}(3) \to \mathrm{SO}(2)$. The field is described as a three-tuple of real fields $\phi^i \in \mathbb{R}^3$ and the vacuum manifold satisfies $\phi^i\phi^i = v^2$ ($\mathcal{M}_0 = \mathrm{SO}(3)/\mathrm{SO}(2) \cong S^2$). In non-linear sigma model, the parametrization of states around the vacuum is

$$\boldsymbol{\phi} = (1 + \sigma(x)/v)\exp\left(i\mathcal{J}_1\pi_1(x) + \mathcal{J}_2\pi_2(x)\right)\boldsymbol{\phi}_0,$$

where $\mathcal{J}_1$ and $\mathcal{J}_2$ are broken generators and belong to SO(3). At low energies, we can ignore $\sigma(x)$ and the field is

$$\boldsymbol{\phi}(x) \approx \exp\left(i\mathcal{J}_1\pi_1(x) + \mathcal{J}_2\pi_2(x)\right)\boldsymbol{\phi}_0.$$

We want to parametrize the Goldstone bosons as

$$\boldsymbol{\phi}(x) = \Sigma(x)\boldsymbol{\phi}_0 = e^{iT^{\hat{a}}\pi^{\hat{a}}(x)}\boldsymbol{\phi}_0$$

where $T^{\hat{a}}$ are the broken generators. We still have to consider that the full theory must be $G = \mathrm{SO}(3)$ invariant, i.e., $\phi(x) \to g\phi(x)$ must produce the same physics. Sadly, $g\Sigma(x)$ cannot be written in the same form as $\Sigma(x)$.

We can find a new set of fields $\boldsymbol{\pi}'$, such that

$$g\Sigma(\boldsymbol{\pi}) = \Sigma(\boldsymbol{\pi}')h$$

where $g \in G$ (the initial symmetry) and $h \in H$ (the unbroken subgroup). This means that the general action of the group $G$ on the Goldstone bosons reparametrizes the Goldstone bosons into a new set of fields $\boldsymbol{\pi}'$ and also rotates the vacuum state $\boldsymbol{\phi}_0$ into a new vacuum state $h\boldsymbol{\phi}_0$.

Now consider the general case where the symmetry group $G$ spontaneously breaks to the subgroup $H$. We can decompose the generators of $G$ into unbroken and broken generators

$$\{T^a\} = \{T^i, T^{\hat{a}}\}$$

where $a \in \{1, \dots, \dim G\}$, $i \in \{1, \dots, \dim H\}$ and $\hat{a} \in \{1, \dots, \dim G - \dim H\}$. For any vacuum state $\boldsymbol{\phi}_0$ it holds $T^i\boldsymbol{\phi}_0 = 0$ and $T^{\hat{a}}\boldsymbol{\phi}_0 \neq 0$. Let $G$ have a compact semisimple Lie algebra $\mathfrak{g}$. In the full group, the commutation relation is

$$[T^a, T^b] = if^{ab}{}_c T^c.$$

If we only have the unbroken generators, the commutation relation is

$$[T^i, T^j] = if^{ij}{}_k T^k$$

since the unbroken subgroup is still a group and thus its algebra must be closed. The commutator between a broken and unbroken generator is

$$[T^i, T^{\hat{b}}] = if^{i\hat{b}}{}_{\hat{c}} T^{\hat{c}} + if^{i\hat{b}}{}_j T^j = if^{i\hat{b}}{}_{\hat{c}} T^{\hat{c}}.$$

The last term is zero as we can always cyclically permute the indices of the structure constant (in compact, semisimple Lie algebra) and obtain $f^{ji}{}_{\hat{b}}$ (which must be zero by the previous example of the commutator of two unbroken generators). The last case is

$$[T^{\hat{a}}, T^{\hat{b}}] = if^{\hat{a}\hat{b}}{}_{\hat{c}} T^{\hat{c}} + if^{\hat{a}\hat{b}}{}_j T^j.$$

In general, this cannot be simplified further. However, in a special case when $H$ has symmetric cosets it holds $f^{\hat{a}\hat{b}}{}_{\hat{c}} = 0$. (This happens when there exists an outer group automorphisms that leaves the unbroken subgroup invariant, but changes the broken generators.)

An example in QCD is the breaking of $\mathrm{SU}(2)_A$. In the chiral symmetry, the number of left- and right-handed particles is conserved (following from $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$). This can be thought of as the sum and the difference of the left- and right-handed particles is conserved ($\mathrm{SU}(2)_V$ and $\mathrm{SU}(2)_A$). When the axial symmetry breaks only the vectorial one remains:

$$\mathrm{SU}(2)_\mathrm{R} \times \mathrm{SU}(2)_\mathrm{L} = \mathrm{SU}(2)_V \otimes \mathrm{SU}(2)_A \to \mathrm{SU}(2)_V.$$

Since QCD is parity invariant (an outer automorphism), it holds

$$T^{\hat{a}}_A = T^{\hat{a}}_\mathrm{L} - T^{\hat{a}}_\mathrm{R}, \qquad T^i_V = T^i_\mathrm{L} + T^i_\mathrm{R}.$$

The pion coset is a symmetric coset, and since $\mathcal{P} : T^{\hat{a}}_A \to -T^{\hat{a}}_A$ it must hold $f^{\hat{a}\hat{b}}{}_{\hat{c}} = 0$.

## 3.3 Effective chiral Lagrangian

In the limit of $m_u, m_d, m_s \to 0$, QCD is $\mathrm{SU}(3)_\mathrm{R} \times \mathrm{SU}(3)_\mathrm{L}$ symmetric. Previously, we have seen that there are 8 Goldstone bosons of $\mathrm{SU}(3)_A$ in the spectrum. The symmetry $\mathrm{SU}(3)_V$ is "unbroken" (if we squint our eyes). Ideally, the breaking of the symmetry would be mediated by a scalar boson. But since we don't have a scalar boson in the QCD, we have to make a dynamical assumption - a chiral combination of quarks a non-zero vacuum expectation value:

$$\langle\Omega|\overline{q}^i_\mathrm{L} q^j_\mathrm{R} + \mathrm{h.c.}|\Omega\rangle = -v^3_\mathrm{QCD}\delta_{ij}$$

where $i, j = 1, 2, 3$ denote the flavour index and $v_\mathrm{QCD} \sim \Lambda_\mathrm{QCD}$. This is called the *quark condensate* and is exactly the same thing that happens in the BCS theory of superconductors. Technically, we could also choose

$$\langle\Omega|\,\mathrm{tr}(F^a_{\mu\nu}F^{\mu\nu,a})|\Omega\rangle \neq 0,$$

but it is not charged under flavour and wouldn't break the desired symmetry.

Our construction is $\mathrm{SU}(3)_V$ invariant, but it breaks the $\mathrm{SU}(3)_A$ symmetry (exactly what we need!). To see this, consider $q_\mathrm{L} \to Lq_\mathrm{L}$ and $q_\mathrm{R} \to Rq_\mathrm{R}$. If $L = R$, then we have $\mathrm{SU}(3)_V$ symmetry and

$$\overline{q}^i_\mathrm{L}\delta_{ij}q^j_\mathrm{R} \to \overline{q}_\mathrm{L}L^\dagger Rq_\mathrm{R} = \overline{q}_\mathrm{L}q_\mathrm{R}$$

so $\mathrm{SU}(3)_V$ is unbroken. If, however, $L = R^\dagger$, then we have $\mathrm{SU}(3)_A$ symmetry and

$$\overline{q}^i_\mathrm{L}\delta_{ij}q^j_\mathrm{R} \to \overline{q}_\mathrm{L}L^\dagger Rq_\mathrm{R} = \overline{q}_\mathrm{L}R^2q_\mathrm{R} \neq \overline{q}_\mathrm{L}q_\mathrm{R}.$$

So we see that $\mathrm{SU}(3)_A$ is broken.

We parametrize the vacuum state of the quark condensate as

$$\langle\phi\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} v_\mathrm{QCD}.$$

The field $\phi$ transforms in bi-fundamental representation

$$\phi \to L\phi R^\dagger$$

where $L \in \mathrm{SU}(3)_\mathrm{L}$ and $R \in \mathrm{SU}(3)_\mathrm{R}$. In $\mathrm{SU}(3)_V$, $L = R$ and $\phi$ transforms as

$$\langle\phi\rangle \to L\,\langle\phi\rangle\,L^\dagger = \langle\phi\rangle\,L^\dagger L = \langle\phi\rangle\,.$$

In $\mathrm{SU}(3)_A$, $L = R^\dagger$ and $\phi$ transforms as

$$\langle\phi\rangle \to L\,\langle\phi\rangle\,L = \langle\phi\rangle\,L^2 \neq \langle\phi\rangle\,.$$

The Goldstone bosons are parametrized as

$$\Sigma(x) = L(x)\,\langle\phi\rangle\,L(x) = \exp\left(\frac{2i}{f_\pi}\boldsymbol{\pi}(x)\right)$$

where $\boldsymbol{\pi}(x) = \lambda^a\pi^a(x)$ in SU(3) and $\boldsymbol{\pi}(x) = \tau^a\pi^a(x)$ in SU(2). Note that $\mathrm{tr}(\lambda^a\lambda^b) = 2\delta^{ij}$. Importantly, we didn't choose $f_\pi = v_\mathrm{QCD}$. In regular non-linear sigma model it has to be that the factor in the exponent is the same as the vacuum expectation value in order for the kinetic

term to have a factor of exactly 1/2. We will also require this, but only in the effective case on the other side of the QCD phase transition. This allows us to have two unrelated energy scales. The factor $f_\pi$ is the energy scale of the Goldstone boson being produced from the vacuum by the current:

$$\langle \pi(q)|J^5_\mu(y)|\Omega\rangle = iq_\mu f_\pi e^{iqy}.$$

The constant $f_\pi$ can be determined experimentally from the decay of the pion, hence it is called the *pion decay constant*. Denote $\mathbf{w}_\mathrm{L} = w^a_\mathrm{L}\lambda^a_\mathrm{L}$ and $\mathbf{w}_\mathrm{R} = w^a_\mathrm{R}\lambda^a_\mathrm{R}$. The symmetry operators can be parametrized as $L = e^{i\mathbf{w}_\mathrm{L}/2}$ and $R = e^{i\mathbf{w}_\mathrm{R}/2}$. The Goldstone bosons transform

$$\Sigma \to \Sigma' = L\Sigma R^{-1}.$$

In the unbroken subgroup $(\mathrm{SU}(3)_V)$, it holds $\mathbf{w}_\mathrm{L} = \mathbf{w}_\mathrm{R} = \mathbf{w}_V$ (since $L = R$) and consequently

$$\delta\boldsymbol{\pi} = \frac{i}{2}[\mathbf{w}_V, \boldsymbol{\pi}(x)] + \dots$$

In the broken subgroup $(\mathrm{SU}(3)_A)$, it holds $\mathbf{w}_\mathrm{L} = -\mathbf{w}_\mathrm{R} = \mathbf{w}_A$ and

$$\delta\boldsymbol{\pi} = \frac{v}{2}\mathbf{w}_A - \frac{1}{f_\pi}[\boldsymbol{\pi}, [\boldsymbol{\pi}, \mathbf{w}_A]] + \dots$$

The first term describes a shift symmetry $e^{i\pi/f_\pi} \to e^{i(\pi+\alpha f_\pi)/f_\pi}$. This shift symmetry forbids polynomial terms of $\boldsymbol{\pi}$ in $\mathcal{L}$. Let's calculate the leading term in $\mathcal{L}_\mathrm{EFT}$. The candidates are

$$[\mathrm{tr}(\Sigma^\dagger\partial_\mu\Sigma)]^2, \quad \mathrm{tr}(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma), \quad \mathrm{tr}((\Sigma^\dagger\partial_\mu\Sigma)^2).$$

The first candidate is

$$\mathrm{tr}(\Sigma^\dagger\partial_\mu\Sigma) = \mathrm{tr}\left((\mathbb{1} - 2i\pi^a\lambda^a/v)\partial_\mu(\mathbb{1} + 2i\pi^a\lambda^a/v)\right) = \mathrm{tr}(2i\partial_\mu\pi^a\lambda^a/v + \dots) \propto \mathrm{tr}(\lambda^a) = 0.$$

We claim that the second and the third term are the same. Using $\partial_\mu(\Sigma^\dagger\Sigma) = 0$ we get $(\partial_\mu\Sigma^\dagger)\Sigma = -\Sigma^\dagger(\partial_\mu\Sigma)$. With this, we can rewrite the third term to match the second term. The kinetic term and the leading interaction in the chiral Lagrangian is

$$\mathcal{L}_\mathrm{GB} = \frac{f_\pi^2}{16}\,\mathrm{tr}(\partial_\mu\Sigma\partial^\mu\Sigma^{-1}) = \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a.$$

The constant in front of the term was chosen such that the effective kinetic term has exactly a factor of 1/2. At low energies, the theory is completely fixed by unique leading term in the expansion of the Lagrangian. If we consider two flavours (SU(2)), we get the following field parametrization

$$\boldsymbol{\pi} = \sqrt{2}\begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}.$$

For three flavours (SU(3)), we have

$$\boldsymbol{\pi} = \sqrt{2}\begin{pmatrix} \pi_3/\sqrt{2} + \pi_8/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi_3/\sqrt{2} + \pi_8/\sqrt{6} & K^0 \\ K^- & \overline{K}^0 & -\sqrt{2/3}\pi_8 \end{pmatrix}.$$

In the chiral limit and at low energies, these are 8 massless, non-interacting scalar fields. The physical states of particles $\pi_3$ and $\pi_8$ are rotated to $\pi^0$ and $\eta$ mesons. If we want to consider the interaction between particles, we have to include higher order terms in the Lagrangian. It is all done using the effective field theory (EFT). The next order is

$$\mathcal{L}_{\text{int}} = -\frac{2}{3f_\pi^2}(\pi^a\pi^a\partial_\mu\pi^b\partial^\mu\pi^b - \pi^b\partial_\mu\pi^b\pi^a\partial^\mu\pi^a) + \dots$$

An interaction vertex would look like




The effective theory description is perturbative only if $E \ll f_\pi$. This is called the *chiral perturbation theory* ($\chi$PT) and it is a low-energy effective field theory of QCD.

Let's briefly return to the $\mathrm{U}(1)_B$ symmetry. Previously, we said that it is conserved, but in the perspective of the effective field theory, it is only *accidental*. This is because we cannot write down a $B$-violating term with 4 or fewer dimensions in the Standard model fields. The first term that would violate the baryon conservation would be $(qqq\ell)/\Lambda_B^2$ ($q$ for any quark and $\ell$ for any lepton). As long as we are well below $\Lambda_B$ (the hierarchy $E \ll \Lambda_B$), the $\mathrm{U}(1)_B$ symmetry is a good symmetry. Since we were not able to detect any violation of $\mathrm{U}(1)_B$ symmetry at all, we have a very large lower bound for that scale, $\Lambda_B > 10^{15}$ GeV.

## 3.4 Explicit breaking of chiral symmetry

So far, we have considered all the quarks to be massless. In turn, we got 3 or 8 *massless* Goldstone bosons. Clearly, though, mesons are definitely not massless. We want to construct a theory that would also describe the meson masses (and hopefully have some predictive power). The first thing we can do is to include the mass term to the Lagrangian:

$$\mathcal{L}_m = \overline{q}^i_{\mathrm{L}} M_{ij} q^j_{\mathrm{R}} + \text{h.c.}$$

where $M_{ij}$ is a diagonal matrix $M_{ij} = \mathrm{diag}(m_u, m_d, m_s)$. The full QCD Lagrangian is $\mathcal{L}_{\text{QCD}} = \mathcal{L}_{q,g} + \mathcal{L}_m$ where the first part enjoys $G = \mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{R}}$ symmetry and the right part has $\mathrm{U}(1)_u \times \mathrm{U}(1)_d \times \mathrm{U}(1)_s$ symmetry. How do we include the massive term in our $\chi$PT? The answer is, as always, symmetry. We will make $\mathcal{L}_m$ $G$-symmetrc by formally promoting the mass matrix to a field $M_{ij} \to M_{ij}(x)$. This field is called a *spurion*. We postulate the following transformation law (and we will later justify it makes sense):

$$M_{ij}(x) \to L M_{ij}(x) R^\dagger.$$

Combined with quark transformations $q_{\mathrm{L}} \to L q_{\mathrm{L}}$ and $q_{\mathrm{R}} \to R q_{\mathrm{R}}$ the mass term is indeed invariant under the desired chiral symmetry. The original theory is obtained when $M_{ij}(x)$ is frozen to its vacuum expectation value $M_{ij}$.

The strategy is to include spurions in $\chi$PT such that $\mathcal{L}_{\text{GB}}$ is still $G$-invariant ($G = \mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{R}}$) under $\Sigma$ and $M$ transformations. To the lowest order, we can write

$$\mathcal{L}_m = a\,\mathrm{tr}(M^\dagger\Sigma + \Sigma^\dagger M).$$

The constant $a$ can be determined by taking the vacuum limit $\boldsymbol{\pi} = 0$ or equivalently $\Sigma = \mathbb{1}$. In that case

$$\langle\Omega|\bar{q}_{\mathrm{L}}^{i} M_{ij} q_{\mathrm{R}}^{j} + \text{h.c.}|\Omega\rangle = -M_{ij} v_{\mathrm{QCD}}^{3} \delta_{ij} = -v_{\mathrm{QCD}}^{3} \operatorname{tr}(M).$$

Hence, the factor $a$ has to be proportional to $v_{\mathrm{QCD}}^{3}$. Our convention is

$$\boxed{\mathcal{L}_m = \frac{v_{\mathrm{QCD}}^{3}}{8} \operatorname{tr}(M^\dagger \Sigma + \Sigma^\dagger M).}$$

Let's first do a calculation in the easier case of $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$. The field $\Sigma$ is parametrized as

$$\Sigma = \exp\left(\frac{2i}{f_\pi} \pi^a \tau^a\right) = \mathbb{1} + \frac{2i}{f_\pi} \pi^a \tau^a - \frac{4}{f_\pi^2} \pi^a \tau^a \pi^b \tau^b.$$

With this, the massive part of the Lagrangian is

$$\mathcal{L}_m = \frac{v_{\mathrm{QCD}}^{3}}{8} \operatorname{tr}(M^\dagger \Sigma + \Sigma^\dagger M) \approx -\frac{v_{\mathrm{QCD}}^{3}}{2 f_\pi^2} \operatorname{tr}(M \tau^a \tau^b) \pi^a \pi^b = -\frac{v_{\mathrm{QCD}}}{2 f_\pi^2} \operatorname{Tr}(M) \pi^a \pi^a.$$

In the second step we used $\tau^a = \sigma^a/2$, then we symmetrized and used $\{\sigma^a, \sigma^b\} = 2\delta^{ab}$. Since $\operatorname{tr}(M)$ is a singlet in the representation, we expect that all three pions will have the same mass. Experimentally, we find $m_{\pi^0} = 134\,\mathrm{MeV}$ and $m_{\pi^\pm} = 139\,\mathrm{MeV}$. The pion mass term should look like $\mathcal{L}_m = m_\pi^2 (\pi^a \pi^a)/2$, so we can identify

$$m_{\pi^\pm}^2 = (m_u + m_d) \frac{v_{\mathrm{QCD}}^{3}}{f_\pi^2} \equiv m_0 (m_u + m_d).$$

At SU(2), the masses of pions are the same. Repeating the same steps for SU(3), we can express the masses of the kaons:

$$m_{K^\pm}^2 = m_0 (m_u + m_s), \quad m_{K^0}^2 = (m_s + m_d) m_0.$$

The mixing term between $\pi_3$ and $\pi_8$ is

$$\mathcal{L}_{\mathrm{mix}}/m_0 = \frac{1}{2}(m_u + m_d)\pi_3\pi_3 + \frac{1}{6}(m_u + m_d + 4m_s)\pi_8\pi_8 + \frac{1}{\sqrt{3}}(m_u - m_d)\pi_3\pi_8.$$

We can diagonalize this mass matrix $\pi_3, \pi_8 \to \pi_0, \eta$ and for simplicity we will assume that $m_u = m_d = \hat{m}$ (both are a few MeV in comparison to $m_s \sim 100\,\mathrm{MeV}$). Actually, after assuming that we don't even need to diagonalize. The masses are

$$m_{\pi^0}^2 = 2\hat{m} m_0, \quad m_\eta^2 = m_0 (2\hat{m} + 4m_s)/3.$$

Finally, the combination of all these formulas that eliminates the dependence on the individual quark masses (and also $m_0$) is

$$m_\eta^2 + m_{\pi^0}^2 = \frac{2}{3}(m_{K^0}^2 + m_{K^\pm}^2 + m_{\pi^\pm}^2).$$

This is an astounding result of the theory of chiral symmetry breaking, and it agrees very well with the experimental data (of course not perfectly, as $m_u \neq m_d$). The formula is called *Gell-Mann–Oaks–Renner relation*. If we don't assume $m_u = m_d$, the mass of $\pi^0$ becomes different

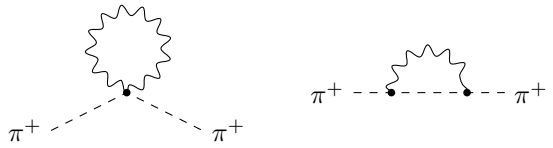
from the mass of $\pi^{\pm}$. Working out the details, we find

$$m_{\pi^0}^2 = m_0 \left( m_u + m_d - \frac{1}{4} \frac{(m_u - m_d)^2}{m_s} + \mathcal{O}(m_s^{-2}) \right),$$
$$m_\eta^2 = m_0 \left( \frac{m_u + m_d + 4m_s}{3} - \frac{1}{4} \frac{(m_u - m_d)^2}{m_s} + \mathcal{O}(m_s^{-2}) \right).$$

However, the theoretical prediction of the difference between the masses of $\pi^0$ and $\pi^{\pm}$

$$m_{\pi^\pm}^2 - m_{\pi^0}^2 \approx \frac{m_0}{4} \frac{(m_u - m_d)^2}{m_s}$$

does not agree well with the experimental data (the observed difference is much bigger). One aspect we didn't consider yet is that charged pions are charged, so they can interact with electromagnetic field.




The electromagnetic field can couple the fermion current

$$\mathcal{L}_m = Q_f \overline{\Psi}_f \gamma_\mu \Psi_f A^\mu = J_\mu A^\mu$$

where

$$Q = \begin{pmatrix} Q_u & 0 & 0 \\ 0 & Q_d & 0 \\ 0 & 0 & Q_s \end{pmatrix} = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} = \frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8$$

and $J_\mu$ is the vector current. Thus, we can embed it into the $\mathrm{SU}(3)_V$ symmetry which Goldstone bosons satisfy. Next, we promote this global symmetry to gauge redundancy

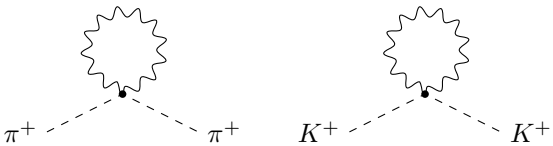
$$\Sigma \to e^{i\alpha(x)Q} \Sigma e^{-i\alpha(x)Q}, \quad \delta\Sigma = i\alpha(x)[Q, \Sigma].$$

The derivative is promoted to a covariant derivative $\mathrm{D}_\mu = \partial_\mu + i\alpha_\mu[Q, \bullet]$. The new interaction term of the Goldstone bosons is

$$\mathcal{L} = \frac{f_\pi^2}{16} \operatorname{tr}((\mathrm{D}_\mu \Sigma)^\dagger \mathrm{D}_\mu \Sigma).$$

Does the new version of Lagrangian respect the same symmetries as the old one? Even though we promoted a global $\mathrm{SU}(3)_V$ symmetry to a gauge symmetry, the gauging is only *partial*. This is explicit breaking of $\mathrm{SU}(3)_V$ since it is distinguishing Goldstone bosons by their charge.

Let's calculate the contributions from the electromagnetic interactions. The leading terms are



The corrections to the masses of $K^+$ and $\pi^+$ are

$$m_{\pi^\pm}^2 = m_0(m_u + m_d) + \Delta_{\mathrm{em},\pi}^2,$$
$$m_{\pi^\pm}^2 = m_0(m_u + m_s) + \Delta_{\mathrm{em},K}^2.$$

Since $Q_{\pi^+} = Q_{K^+}$ we expect $\Delta_{\mathrm{em},\pi}^2 = \Delta_{\mathrm{em},K}^2$. We use the measurements of Goldstone masses to extract the quark masses. We get $m_u/m_d \approx 1/2$ and $m_d/m_s \approx 1/20$. With this, we can predict the mass of the $\eta$ meson. The predicted value is $m_\eta^{\mathrm{th}} = 564\,\mathrm{MeV}$ and the measured value is $m_\eta^{\mathrm{exp}} = 547.86\,\mathrm{MeV}$. This is a rather decent prediction!

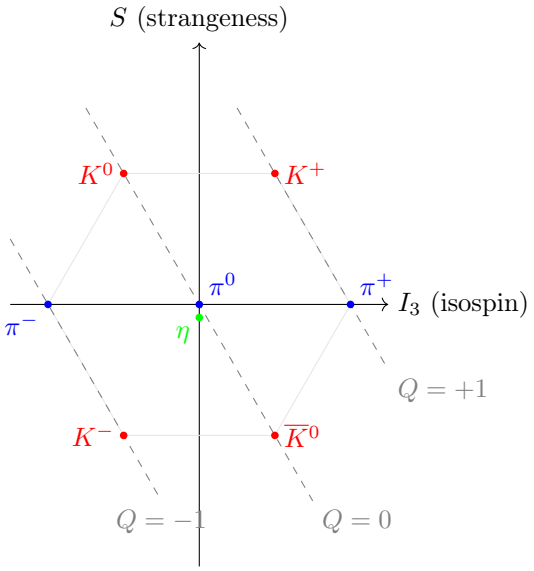
Remember that colour singlets are in the representation $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$. This means that mesons are in the $\mathbf{1}$ of $\mathrm{SU}(3)_\mathrm{c}$. We can do the same steps with flavours,

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \otimes \begin{pmatrix} \overline{u} & \overline{d} & \overline{s} \end{pmatrix} = \mathbf{1} \otimes \mathbf{8}.$$

Mesons can be in either octet or the singlet of the flavour. To understand this better, we have to understand the representation theory. The representation theory of SU(3) is a generalization of SU(2). In SU(2) we found a maximal set of commuting operators (Casimirs) $J^a J^a$ and $J^3$ and used the ladder operators $J_\pm = J_1 \pm i J_2$. We found

$$J_+ \left|J, m\right\rangle \propto \left|J, m+1\right\rangle.$$

In the language of the representation theory, the Casimirs are called *weights* and the ladder operators are called *root vectors*. In SU(3), we have two roots and three weights ($\lambda^a\lambda^a$, $\lambda^3$, $\lambda^8$).




This diagram is called the *eightfold way*. In addition to these mesons, we also have the *vector mesons* (spin 1).

| Symbol | composition | Mass [MeV] | lifetime [s] |
|---|---|---|---|
| $\rho^+$ | $u\overline{d}$ | 770 | $10^{-24}$ |
| $\rho^0$ | $(u\overline{u} - d\overline{d})/\sqrt{2}$ | 770 | $10^{-21}$ |
| $\omega$ | $(u\overline{u} + d\overline{d})/\sqrt{2}$ | 780 | $10^{-22}$ |
| $\phi$ | $s\overline{s}$ | 1020 | $10^{-22}$ |
| $(K^+)^*$ | $u\overline{s}$ | 890 | $10^{-24}$ |
| $(K^0)^*$ | $d\overline{s}$ | 890 | $10^{-24}$ |

The lifetimes correspond nicely to the mass. As a rule of thumb, 100 MeV corresponds to $10^{-23}$ s if the particle decays via the strong interaction. Let's make the same table for the pseudoscalar mesons (there is also $\eta'$ as a flavour singlet):

| Symbol | composition | Mass [MeV] | lifetime [s] |
|---|---|---|---|
| $\pi^+$ | $u\overline{d}$ | 139 | $10^{-8}$ |
| $\pi^0$ | $(u\overline{u} - d\overline{d})/\sqrt{2}$ | 134 | $10^{-16}$ |
| $\eta$ | $(u\overline{u} + d\overline{d} - 2s\overline{s})/\sqrt{6}$ | 548 | $10^{-19}$ |
| $\eta'$ | $(u\overline{u} + d\overline{d} + s\overline{s})/\sqrt{3}$ | 958 | $10^{-21}$ |
| $K^+$ | $u\overline{s}$ | 494 | $10^{-8}$ |
| $K^0$ | $d\overline{s}$ | 498 | $10^{-8}$–$10^{-11}$ |

The masses and lifetimes of the pseudo-scalar mesons do not correspond so nicely. This means that they probably decay via different interaction, $\pi^+ \to \mu^+ + \nu_\mu$ and $\pi^0 \to \gamma\gamma$. Another rule of thumb is the following:

| | |
|---|---|
| strong decay | $10^{-22}$–$10^{-24}$ s |
| electromagnetic decay | $10^{-16}$–$10^{-21}$ s |
| weak decay | $10^{-7}$–$10^{-13}$ s |

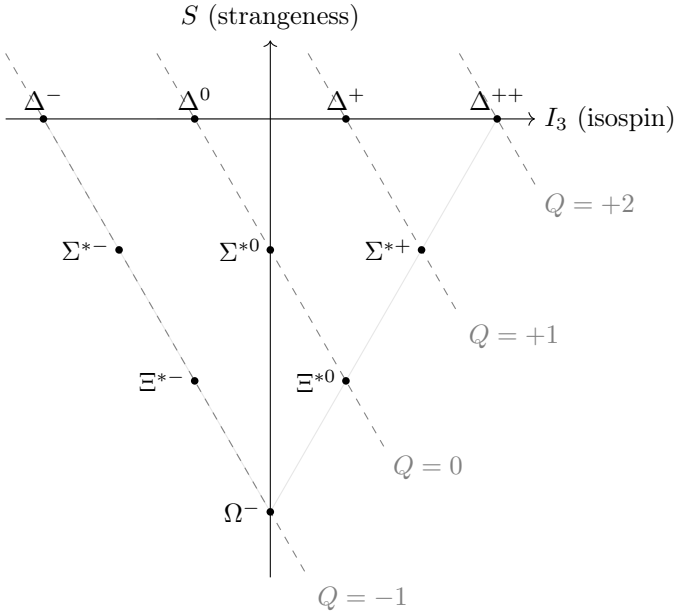
Lastly, let's discuss *baryons*. They are in the singlet colour representation of $\mathrm{SU}(3)_c$

$$B = \varepsilon^{abc} q_a q_b q_c.$$

What are the $\mathrm{SU}(3)_V$ representations? For a combination of two quarks, we have $\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6}$. We are not particularly interested in this, since we want to know the decomposition of three quarks. We have

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\overline{\mathbf{3}} \otimes \mathbf{3}) \oplus (\mathbf{6} \otimes \mathbf{3}) = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}.$$

This gives rise to many baryon resonances. The proton and neutron are part of the octet. The strong force at low energies can be approximated by pion exchange, giving rise to Yukawa potential $V(r) \sim e^{-m_\pi r}/r$.




Using the theory that we have just developed, let's now try to calculate the $\pi\pi \to \pi\pi$ scattering amplitude. The relevant part of the Lagrangian is

$$\mathcal{L}_{\text{int}} = \frac{f_\pi^2}{16}\,\text{tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) = -\frac{1}{6f_\pi^2}(\pi^a\pi^a\partial_\mu\pi^b\partial^\mu\pi^b - \pi^a\partial_\mu\pi^a\,\pi^b\partial^\mu\pi^b).$$

At tree level, the scattering amplitude is

$$\begin{aligned} i\mathcal{A}^{abcd}(\pi^a\pi^b \to \pi^c\pi^d) &= \frac{i}{6f_\pi^2}\left[\delta^{ab}\delta^{cd}(4(p_a p_b + p_c p_d) + 2(p_a p_c + p_a p_d + p_b p_c + p_b p_d))\right. \\ &\qquad \left. +(b \leftrightarrow c) + (b \leftrightarrow d)\right] \\ &= \frac{i}{6f_\pi^2}\left(\delta^{ab}\delta^{cd}s + \delta^{ac}\delta^{bd}t + \delta^{ad}\delta^{bc}u\right), \end{aligned}$$

where in the last step we used $m_\pi = 0$ and $s+t+u = 0$. As $s \to \infty$ the amplitude isn't bounded which violates tree-level unitarity. However, the full unitarity is not violated (Hamiltonian is still Hermitian), as we also need to sum all loop orders. The main issue is that perturbation theory breaks down at high enough energies. To see this, we will discuss higher order terms in the $\chi$PT.

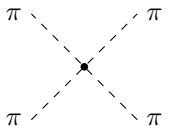
First, we start with the massive Lagrangian,

$$\mathcal{L} = \frac{f_\pi^2}{16}\,\text{tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) + \frac{m_0^2 f_\pi^2}{8}\,\text{tr}(\Sigma^\dagger M + M^\dagger \Sigma).$$

For adding higher order terms, we have two options:

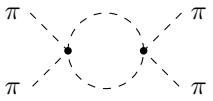
$$\begin{aligned} \text{tr}(\mathrm{D}^\mu \mathrm{D}_\mu \Sigma^\dagger \mathrm{D}_\mu \mathrm{D}^\mu \Sigma) &\sim \frac{f_\pi^2}{\Lambda^2}, \\ \text{tr}(\mathrm{D}^\mu \Sigma^\dagger \mathrm{D}^\mu \Sigma m_0 M) &\sim \frac{f_\pi^2}{\Lambda^2}. \end{aligned}$$

We see that every pair of derivatives and every mass term adds a factor of $\Lambda^{-2}$. We can see this as an EFT expansion in terms of $p^2/\Lambda^2$ and $m_0 M/\Lambda^2$, which is valid when $p^2, m_0 M \ll \Lambda^2$. To see how large $\Lambda$ is, we consider the $\pi\pi \to \pi\pi$ scattering. At tree level, we have



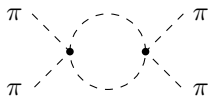

$$\sim \frac{p^2}{f_\pi^2}\pi\pi\pi\pi.$$

The one loop integral is




$$\sim \int \mathrm{d}^4 p \, \frac{1}{p^4}\frac{p^4}{f_\pi^4} \sim \frac{\Lambda^4}{f_\pi^4}.$$

This cannot be correct, as it would correspond to a Lagrangian term of form $\mathcal{L} \sim (\Lambda^4/f_\pi^4)\pi\pi\pi\pi$ (which violates the shift symmetry). In fact, it should correspond to a Lagrangian with at least two derivatives. With a Lagrangian $\mathcal{L} = \pi^3\Box\pi$ the one loop correction is




$$\sim p_{\mathrm{ext}}^2 \frac{\Lambda^2}{(4\pi)^2 f_\pi^2}\pi^4(x).$$

In order for this statement to make sense, the tree level amplitude should be generally larger than the one loop correction:

$$\frac{p_{\mathrm{ext}}^2}{f_\pi^2}\pi^4(x) \gtrsim p_{\mathrm{ext}}^2 \frac{\Lambda^2}{(4\pi)^2 f_\pi^2}\pi^4(x) \quad \Longrightarrow \quad \Lambda \lesssim 4\pi f_\pi.$$

Since $f_\pi \approx 200\,\mathrm{MeV}$, the broad estimate for the $\Lambda$ is $1-2$ GeV. Hence, the EFT of pions is valid in the range $0 < E < (1-2)$ GeV.

The effective field theory is a very general approach to treat field theories and is not confined to the low-energy QCD only. An important example is the treatment of gravity at small energies. With the expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/M_{\mathrm{Pl}}$, one can write the Einstein-Hilbert action as an effective theory:

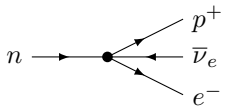
$$\mathcal{L}_{\mathrm{EH}} = M_{\mathrm{Pl}}^2 \int \mathrm{d}^4 x \, \sqrt{-g}R = \int \mathrm{d}^4 x \, \left(\partial_\mu h \partial^\mu h + \frac{h^2 \Box h}{M_{\mathrm{Pl}}} + \dots\right).$$

# Chapter 4

# Leptonic standard model

## 4.1 Fermi theory

The starting point of this chapter is the $\beta$-decay and Fermi theory. The $\beta$-decay $n \to p^+ e^- \overline{\nu}_e$ was historically not known in detail for a long time (neutrino was a big mystery), but they knew it resembled a three body interaction:




Fermi proposed a four-fermion interaction to describe this decay – the *Fermi theory*. It was known that this interaction has to be weak in comparison with other interactions known at that time. The decay times were orders of magnitude higher than expected by electromagnetism. This is why it is called *weak interaction*. A Lagrangian that would describe this interaction in the most general Lorentz-invariant form is

$$\mathcal{L}_{\mathrm{I}} = C_{ij}(\overline{\psi}_u \mathcal{O}_i \psi_p)(\overline{\psi}_{\overline{\nu}_e} \mathcal{O}_j \psi_e)$$

where $\mathcal{O}_i$ and $\mathcal{O}_j$ are some combinations of Dirac matrices. The cross-sections of the interactions depend on the choice of these matrices. These cross-sections were measured, and it was a big triumph of 1950s when they figured there is only one possible option

$$O_i = O_j = \gamma^\mu(1-\gamma_5)/2.$$

In other words, only left-handed particles and right-handed antiparticles interact. The matrix structure between the two pairs of fermions turned out to be trivial $C_{ij} = 4G_{\mathrm{F}}\delta^{ij}/\sqrt{2}$. This type of interaction is called the *$V-A$ interaction*. This is because vector ($\gamma_\mu$) and axial ($\gamma_\mu\gamma_5$) currents never occur individually, but always in the $V-A$ combination. Using a single-dimensional coupling $G_{\mathrm{F}}$ we get the following Lagrangian:

$$\mathcal{L}_{\mathrm{I}} = \frac{G_{\mathrm{F}}}{\sqrt{2}}(\overline{\psi}_u \gamma_\mu(1-\gamma^5)\psi_p)(\overline{\psi}_{\overline{\nu}_e}\gamma^\mu(1-\gamma^5)\psi_e).$$

The value of the *Fermi constant* is $G_{\mathrm{F}} \approx 10^{-5}/m_p^2$.

Since the weak force treats left- and right-handed particles differently, it cannot be parity invariant. A Dirac spinor consists of a 2-component left-handed Weyl spinor and a 2-compnent right-handed Weyl spinor,

$$\Psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix}.$$

In the chiral basis, the $\gamma^0$ matrix is block-off-diagonal identity matrix

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

The parity transformation maps $\mathcal{P} : \mathbf{x} \to -\mathbf{x}$ and

$$\mathcal{P} : \Psi(x_0, \mathbf{x}) \to \Psi(x, -\mathbf{x}) = \pm\gamma^0 \Psi(x_0, \mathbf{x})$$

$$\mathcal{P} : \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix} \to \pm \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix} = \pm \begin{pmatrix} \psi_{\mathrm{R}} \\ \psi_{\mathrm{L}} \end{pmatrix},$$

where the sign depends on the internal parity of the particle. Since the weak interaction only couples to one handedness, we say it violates $\mathcal{P}$-symmetry *maximally*. This was a great historical surprise, because everything they knew at the time was parity symmetric, and they didn't expect parity to be violated at all. In fact, the only reason we use the Dirac spinors is that QED and QCD are parity invariant. In the weak interaction, left- and right-handed particles have different quantum numbers. We can keep using the Dirac spinors, but we will need to project out the right- or left-handed part of the Dirac spinor. The projection operators are (in the chiral representation)

$$P_{\mathrm{L}} = \frac{1}{2}(1 - \gamma^5) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{\mathrm{R}} = \frac{1}{2}(1 + \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

Because of the asymmetry it is sometimes more convenient to write Weyl spinors. For example, the Dirac Lagrangian written with Weyl spinors is

$$\mathcal{L}_{\mathrm{Dirac}} = \psi_{\mathrm{L}}^\dagger i\overline{\sigma}^\mu \partial_\mu \psi_{\mathrm{L}} + \psi_{\mathrm{R}}^\dagger i\sigma^\mu \partial_\mu \psi_{\mathrm{LR}} - m(\psi_{\mathrm{R}}^\dagger \psi_{\mathrm{L}} + \psi_{\mathrm{L}}^\dagger \psi_{\mathrm{R}}).$$

From this notation it is obvious that left- and right-handed components propagate independently and are joined together only by the mass term. For a massless fermion, the two chiralities are independent degrees of freedom. Thus, it is not much of a stretch to consider that left- and right-handed particles have different couplings to some interaction. In particular, weak gauge bosons have exactly zero coupling to right-handed particles (and non-zero to left-handed ones).

However, we will, for the time being, stick with Dirac spinors and use the projector operators, because we are more familiar with gamma matrix and trace formalism.

We can think of the Fermi Lagrangian as a contraction of two fermion currents:

$$\mathcal{L}_{\mathrm{Fermi}} = \frac{G_{\mathrm{F}}}{\sqrt{2}} \underbrace{\left(\overline{\Psi}_n \gamma_\mu (1 - \gamma_5) \Psi_p\right)}_{J_\mu^{\mathrm{had}}} \underbrace{\left(\overline{\Psi}_{\overline{\nu}_\mu} \gamma^\mu (1 - \gamma_5) \Psi_\mu\right)}_{J_{\mathrm{lep}}^\mu} = \frac{G_{\mathrm{F}}}{\sqrt{2}} J_\mu^{\mathrm{had}} J_{\mathrm{lep}}^\mu.$$

Compared to QED (with $m_\gamma \neq 0$), where the interaction term is

$$\mathcal{L}_{\mathrm{QED}}^{\mathrm{int}} = eA_\mu J_{\mathrm{QED}}^\mu = eA_\mu(\overline{\Psi}\gamma^\mu \Psi),$$

the tree level $2 \to 2$ scattering amplitude is

$$\mathcal{A}_{\text{QED}} \sim e^2 J_\mu \frac{i\eta^{\mu\nu}}{q^2 - m_\gamma^2} J_\nu.$$

It has a familiar structure of (current) $\cdot$ (propagator) $\cdot$ (current). This could be a six-dimensional term in the QED effective Lagrangian, very similar to the one we have in Fermi theory. In fact, if we assume $q^2 \ll m_\gamma^2$, the structure becomes exactly the same. If we assume that the 4-point weak interaction is due to a heavy gauge boson $m_W^2 \gg q^2$ the scattering amplitude and the term in the effective Lagrangian is

$$g^2 J_\mu^{\text{had}} \frac{\eta^{\mu\nu}}{q^2 - m_W^2 + i\varepsilon} J_\mu^{\text{lep}} \approx \frac{g^2}{m_W^2} J_\mu^{\text{had}} J_{\text{lep}}^\nu,$$

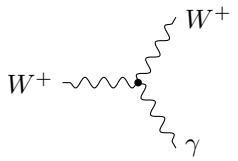
which is exactly the same as in massive QED case. Identifying $g^2/m_W^2 = G_{\text{F}}/\sqrt{2}$ and assuming $e \approx g$, we can broadly estimate the mass of the $W$-boson, $m_W \approx 100\,\text{GeV}$.

The term in the effective Lagrangian is non-renormalizable since the fields carry mass dimension of 6 and the coupling a mass dimension of $-2$. The break-down of perturbation theory was predicted at the scale of 100 GeV, and it was eventually confirmed when $W$ boson was discovered at CERN. If we want to describe processes above this scale we have to complete the EFT with new degrees of freedom.

## 4.2 Gauge theory of weak interaction

We want a theory with

- massive boson $W$,
- massless photon $\gamma$,
- different left and right representations of weak interaction.
- From the $\beta$-decay, we know that $W$-boson has a QED charge, which means it couples to the photon:




Given these requirements, we can try to construct a gauge theory. We clearly need something non-Abelian, since we have seen that the gauge boson $W$ has to interact with itself. We will start with the simplest possible option, and we will fail. Then we will try the next simplest one, and that one will work.

### 4.2.1 Georgi-Glashow model

The easiest non-Abelian candidate for the weak interaction is SU(2):

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \to \underbrace{e^{i\tau^a\alpha^a}}_{A} \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad \det A = 1, \quad A^\dagger A = \mathbb{1}.$$

The generators are Pauli matrices $\tau^a = \sigma^a/2$ that obey the commutation relation

$$[\tau^a, \tau^b] = i\varepsilon^{abc}\tau^c.$$

The covariant derivative and a field strength tensor are

$$\mathrm{D}_\mu = \partial_\mu - igA^a_\mu(x)T^a, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu,$$

where $f^{abc}$ are the structure constants and $T$ is some representation. Since gauge invariance forbids a massive gauge field, we have to add a spontaneous symmetry breaking. So, we include the Higgs field with a potential

$$V(\phi) = -\mu^2|\phi|^2 + \lambda|\phi|^4.$$

The Lagrangian without fermions is

$$\mathcal{L} = -\frac{1}{4}\mathbf{F}^2_{\mu\nu} + |\mathrm{D}_\mu\phi|^2 - V(\phi).$$

We expand the field around the vacuum expectation value $\langle\phi\rangle = v/\sqrt{2}$. The gauge boson acquires mass $M_A^2 = g^2v^2$. From experiments, we know that we also need

- a left-handed pair of fermions in some representation of SU(2),
- a right-handed electron in the trivial representation,
- and a massless photon and massive $W$-boson.

The simplest option to satisfy these requirements is the breaking of SU(2) symmetry to U(1) symmetry. This is called the *Georgi-Glashow model*. The generator $Q$ (the charge) of the unbroken U(1) must be one of the generators $T^i$ of SU(2), so we choose $Q = T^3$. The two broken generators correspond to $W^\pm$ bosons and the unbroken one to the photon. A minimal representation of the fermions is spin-1/2 (fundamental) representation where the charge of electrons is $Q = -1$ and the charge of neutrinos is $Q = +1$. This does not agree with the experimental data, so we try the next easiest representation – the spin-1 triplet representation:

$$L_\mathrm{L} = \begin{pmatrix} X_\mathrm{L} \\ \nu_\mathrm{L} \\ e_\mathrm{L} \end{pmatrix}, \quad L_R = \begin{pmatrix} Y_\mathrm{R} \\ Z_\mathrm{R} \\ e_\mathrm{R} \end{pmatrix}.$$

This representation brings three new fermions, two charged ones ($X_\mathrm{L}$, $Y_\mathrm{R}$) and one neutral one ($Z_\mathrm{R}$). These particles have never been detected, so we can disregard this representation. Any other representation would only bring more new (unseen) particles, so we might want to search for some other type of symmetry breaking.

### 4.2.2 Glashow-Salam-Weinberg model

The issue with the previous approach was that the electric charge was tied directly to SU(2) generator. Maybe, we could try to include another (Abelian) gauge boson $B^\mu$ and enlarge the symmetry group to

$$\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y.$$

The added Abelian group is distinct from QED U(1) group, though we will see that they are somehow related. We have seen that setting the electric charge to $Q = T^3$ doesn't work well. Moreover, setting $Q = Y$ also doesn't work. What works is

$$Q = T_3 + Y\mathbb{1}.$$

In the fundamental (doublet) representation, the $T_3$ generator has eigenvalue $+1/2$ on the upper component of the doublet $L_\mathrm{L} = (\nu_e, e)^T$ and $-1/2$ on the lower component (and obviously 0 on the singlet)

$$T_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

If we set $Y = -1/2$ on the doublet and $Y = -1$ on the singlet, the electrical charges of electrons and neutrino are exactly what we measure experimentally.

| Particle | $\begin{pmatrix}\nu\\e\end{pmatrix}_\mathrm{L}$ | $e_\mathrm{R}$ | $\begin{pmatrix}u\\d\end{pmatrix}_\mathrm{L}$ | $u_\mathrm{R}$ | $d_\mathrm{R}$ |
|---|---|---|---|---|---|
| $\mathrm{SU}(2)_\mathrm{L}$ representation | 2 | 1 | 2 | 1 | 1 |
| $\mathrm{U}(1)_Y$ representation | $-1/2$ | $-1$ | $1/6$ | $2/3$ | $-1/3$ |

Now that we have constructed the particles with charges, we have to spontaneously break the symmetry by introducing a scalar field. It must be charged under both symmetries in order to break them. The simplest possibility is a doublet in $\mathrm{SU}(2)_\mathrm{L}$. The field is a tuple of two complex scalar fields

$$H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \quad \text{with charges} \quad Q = T_3 + Y = \begin{pmatrix} 1/2 + Y \\ -1/2 + Y \end{pmatrix}.$$

We need $Y = \pm 1/2$ in order for the generator $Q$ to not be broken. We pick $Q = +1/2$ and the vacuum expectation value is

$$\langle H \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

which comes from the Mexican hat potential $V(H) = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2$. The gauge boson masses come from the kinetic term of Lagrangian $(\mathrm{D}_\mu H)^\dagger (\mathrm{D}^\mu H)$. The derivative is

$$\mathrm{D}_\mu = \partial_\mu - ig\frac{\tau^a}{2} W_\mu^a - i\frac{g'}{2} B_\mu \mathbb{1} = \partial_\mu - \frac{i}{2} \begin{pmatrix} gW_\mu^3 + g'B_\mu & g(W^1\mu - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^3) & -gW_\mu^3 + g'B_\mu \end{pmatrix}.$$

Acting with this on $\langle H \rangle$ gives

$$(\mathrm{D}_\mu \langle H \rangle)^\dagger (\mathrm{D}^\mu \langle H \rangle) = \frac{g^2 v^2}{8}((W_\mu^1)^2 + (W_\mu^2)^2) + \frac{v^2}{8}(-gW_\mu^3 + g'B_\mu)^2.$$

We don't like the mixing terms between $W^3$ and $B$, so we move to some other basis $(Z, A)$ where the two fields don't mix. After diagonalizing the mixing matrix, we get

$$\begin{aligned} Z_\mu &= \cos\theta_\mathrm{W}\, W_\mu^3 - \sin\theta_\mathrm{W}\, B_\mu, \\ A_\mu &= \sin\theta_\mathrm{W}\, W_\mu^3 + \cos\theta_\mathrm{W}\, B_\mu. \end{aligned}$$

The angle $\theta_\mathrm{W}$ is called the *weak mixing angle* or *Weinberg angle* (both start with the letter W). If we actually diagonalized the matrix, we would find

$$Z_\mu = \frac{gW_\mu^3 - g'B_\mu}{\sqrt{g^2 + g'^2}}, \qquad \Longrightarrow \qquad \cos\theta_\mathrm{W} = \frac{g}{\sqrt{g^2 + g'^2}}.$$

Finally, we also mix $W^1$ and $W^2$ because we want them to be charged and conjugate of each other.

$$W^\pm = \frac{1}{\sqrt{2}}(W^1 \pm iW^2).$$

When we plug these into the kinetic term, we get the following Lagrangian terms

$$\mathcal{L}_M = \frac{-g^2v^2}{4} W_\mu^+ W_\mu^- + \frac{1}{2}\frac{g^2+g'^2}{4} v^2 Z_\mu Z^\mu + 0.$$

Notice the absence of the mass term for the field $A$. Collecting the mass terms gives

$$M_W^2 = \frac{g^2v^2}{4}, \qquad M_Z^2 = \frac{g^2+g'^2}{4}v^2.$$

The masses and the Weinberg angle can be combined in a single factor, called the *ρ parameter*

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_\mathrm{W}} = 1.$$

Other models give different values for this parameter and since it has been measured very accurately to be close to one it gives a good confirmation for the GWS theory.

### 4.2.3 Coupling to fermions

The particles that interact with the weak gauge bosons will be characterized by two numbers, the *weak isospin* $T$ (a half-integer) and the *weak hypercharge* $Y$ (any real number). The weak isospin is the $\mathrm{SU(2)_L}$ representation (so it will be denoted by a bold whole number $2T+1$). The third component of the weak isospin is $T_3$. A particle with weak isospin $T = 1/2$ and weak hypercharge $Y = -1/2$ would be denoted as $\mathbf{2}_{-1/2}$. Higgs boson has the numbers $\mathbf{2}_{1/2}$.

To find the couplings between weak gauge bosons and fermions, we first write down the covariant derivative

$$\mathrm{D}_\mu = \partial_\mu - ig\tau^a W_\mu^a - ig'YB_\mu.$$

Since the physical fields are $W^\pm, Z$ and $A$ we have to modify representation matrices. For the $W$ bosons, we define

$$\tau^\pm = \tau^1 \pm i\tau^2 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

With these, the part in the covariant derivative is

$$\tau^1 W_\mu^1 + \tau^2 W_\mu^2 = \frac{1}{\sqrt{2}}(\tau^+ W_\mu^+ \tau^- W_\mu^-).$$

For the photon and $Z$ boson, we unmix the $W^3$ and $B$ fields

$$\begin{pmatrix}B\\ W^3\end{pmatrix} = \begin{pmatrix}\cos\theta_\mathrm{W} & -\sin\theta_\mathrm{W}\\ \sin\theta_\mathrm{W} & \cos\theta_\mathrm{W}\end{pmatrix}\begin{pmatrix}A\\ Z\end{pmatrix},$$

where

$$\sin\theta_\mathrm{W} = \frac{g'}{\sqrt{g^2+g'^2}}, \quad \cos\theta_\mathrm{W} = \frac{g}{\sqrt{g^2+g'^2}}.$$

The full covariant derivative becomes

$$\mathrm{D}_\mu = \partial_\mu - \frac{ig}{\sqrt{2}}(\tau^+ W_\mu^+ + \tau^- W_\mu^-) - ig\tau^3(\sin\theta_\mathrm{W}\, A_\mu + \cos\theta_\mathrm{W}\, Z_\mu) - ig'Y(\cos\theta_\mathrm{W}\, A_\mu - \sin\theta_\mathrm{W}\, Z_\mu).$$

Furthermore, we will express the third component of the weak isospin $T_3$ with the hypercharge $Y$ and the electromagnetic charge $Q$ as $T_3 = Q - Y$. The goal is to express the derivative only using $A$, $Z$, $Q$ and $\tau_3$. We get

$$\begin{aligned}
\mathrm{D}_\mu &= \partial_\mu - i\frac{g}{\sqrt{2}}(\tau^+ W_\mu^+ + \tau^- W_\mu^-) - i\sqrt{g^2+g'^2}\cos\theta_\mathrm{W}\,(Q-Y)(\sin\theta_\mathrm{W}\,A_\mu + \cos\theta_\mathrm{W}\,Z_\mu) + \\
&\quad - i\sqrt{g^2+g'^2}\sin\theta_\mathrm{W}\,Y(\cos\theta_\mathrm{W}\,A_\mu - \sin\theta_\mathrm{W}\,Z_\mu) \\
&= \partial_\mu - i\frac{g}{\sqrt{2}}(\tau^+ W_\mu^+ + \tau^- W_\mu^-) - i\sqrt{g^2+g'^2}\cos\theta_\mathrm{W}\sin\theta_\mathrm{W}\,QA_\mu + \\
&\quad - i\sqrt{g^2+g'^2}Z_\mu((Q-Y)\cos^2\theta_\mathrm{W} + Y\sin^2\theta_\mathrm{W}) \\
&= \partial_\mu - i\frac{g}{\sqrt{2}}(\tau^+ W_\mu^+ + \tau^- W_\mu^-) - ig\sin\theta_\mathrm{W}\,QA_\mu - \frac{ig}{\cos\theta_\mathrm{W}}Z_\mu(\tau_3 - Q\sin^2\theta_\mathrm{W}).
\end{aligned}$$

From the last line, we can read off the electromagnetic coupling

$$e = g\sin\theta_\mathrm{W}.$$

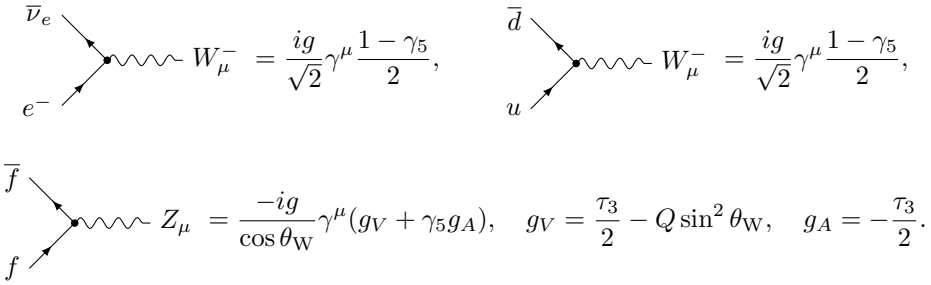
We are now far enough that we can write down the fermion interaction Lagrangian. As we have seen, fermions come in either singlet or doublet representations of $\mathrm{SU}(2)_\mathrm{L}$. The doublet of electron and neutrino will be denoted as $E_\mathrm{L}$, and the doublet of up and down quark as $Q_\mathrm{L}$. All right-handed fermions are singlets and are denoted by lower-case letters. We could generalize the notation to all three generations by adding an index to each of these fields, but we won't do that yet. The interaction Lagrangian is

$$\mathcal{L} = \overline{E}_\mathrm{L} i\partial\!\!\!/ E_\mathrm{L} + \overline{e}_\mathrm{R} i\partial\!\!\!/ e_\mathrm{R} + \overline{Q}_\mathrm{L} i\partial\!\!\!/ Q_\mathrm{L} + \overline{u}_\mathrm{R} i\partial\!\!\!/ u_\mathrm{R} + \overline{d}_\mathrm{R} i\partial\!\!\!/ d_\mathrm{R} + g(W_\mu^+ J_+^\mu + W_\mu^- J_-^\mu + Z_\mu J_Z^\mu) + eJ_Q^\mu A_\mu.$$

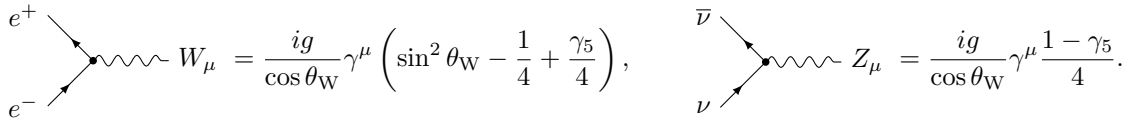
The currents that couple to the gauge bosons are

$$\begin{aligned}
J_Q^\mu &= (-1)\overline{e}\gamma^\mu e + (2/3)\overline{u}\gamma^\mu u + (-1/3)\overline{d}\gamma^\mu d, \quad \text{Dirac spinors} \\
J_+^\mu &= \frac{1}{\sqrt{2}}(\overline{\nu}_\mathrm{L}\gamma^\mu e_\mathrm{L} + \overline{u}_\mathrm{L}\gamma^\mu d_\mathrm{L}), \\
J_-^\mu &= \frac{1}{\sqrt{2}}(\overline{e}_\mathrm{L}\gamma^\mu \nu_\mathrm{L} + \overline{d}_\mathrm{L}\gamma^\mu u_\mathrm{L}), \\
J_Z^\mu &= \frac{1}{\cos\theta_\mathrm{W}}\sum_{i=\mathrm{L,R}} \overline{\Psi}_i\gamma^\mu(\tau^3 - Q\sin^2\theta_\mathrm{W})\Psi_i.
\end{aligned}$$

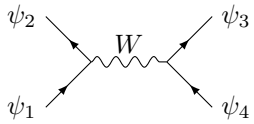
With this at hand, we can write down the Feynman rules for the weak interaction.



The diagrams for $W^+$ can be obtained by rotating the ones for $W^-$ (the vertex factors are the same). Two specific examples for neutral current are




Using the Feynman rules, we can try to match the Fermi coupling exactly. The process is shown on the following diagram.



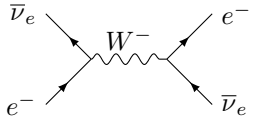

The effective vertex is

$$\mathcal{L}_{\text{Fermi}} = \frac{G_F}{\sqrt{2}}(\overline{\Psi}_2\gamma_\mu(1-\gamma_5)\tau^+\Psi_1)(\overline{\Psi}_4\gamma^\mu(1-\gamma_5)\tau^+\Psi_3).$$

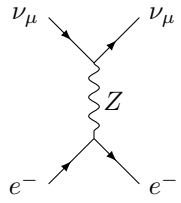
Matching all the factors, we find

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}.$$

As we have seen, there exists a neutral current which distinguishes the GSW theory from the Georgi-Glashow model. At low energies, the photon exchange dominates over the $Z$-boson exchange, which means it is very difficult to detect $Z$-boson (at least at low energies). Any process that can run via a photon can also run via a $Z$-boson. However, the converse is not always the case. Namely, the $Z$-boson can couple to neutrinos. In a typical electron-neutrino scattering, the charged current is usually present.




However, in some cases, we can convince the neutrino to interact via the neutral current. To do this, smart people in the 1970s tried to collide electrons with charged pions. Pions decay predominantly to muons ($\pi^+ \to \mu^+ + \nu_\mu$) since the production of $e^+ + \nu_e$ pair is suppressed by a factor of $m_e^2/m_\mu^2$ due to angular momentum. A muonic neutrino emitted by the pion can interact with the electron only via the neutral current.




This process has no QED contribution and no charged current at tree level. The detection of the interaction between the muonic neutrino and the electron was a great success of the GSW

theory when it was detected at Gargamelle experiment at CERN in 1973. Later, the masses of the $W$ and $Z$ bosons were measured very accurately

$$M_W = (80.369 \pm 0.013)\,\text{GeV}, \qquad M_Z = (91.1880 \pm 0.0020)\,\text{GeV}.$$

The value of the Weinberg angle is $\sin^2\theta_\text{W} = 0.231\,20 \pm 0.000\,15$.

Recap

| Group | $\begin{pmatrix}\nu\\ e\end{pmatrix}_\text{L}$ | $e_\text{R}$ | $\begin{pmatrix}u\\ d\end{pmatrix}_\text{L}$ | $u_\text{R}$ | $d_\text{R}$ | $H$ |
|---|---|---|---|---|---|---|
| $\text{SU}(2)_\text{L}$ | **2** | **1** | **2** | **1** | **1** | **2** |
| $\text{U}(1)_Y$ | $-1/2$ | $-1$ | $1/6$ | $2/3$ | $-1/3$ | $1/2$ |
| $\text{SU}(3)_c$ | **1** | **1** | **3** | **3** | **3** | **1** |

$$V(H) = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2, \quad \Longrightarrow \quad \langle H \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix},$$

$$M_W^2 = \frac{g^2 v^2}{4}, \qquad M_Z^2 = \frac{g^2 + g'^2}{4} v^2, \quad \frac{M_W^2}{M_Z^2} = \cos^2\theta_\text{W},$$

$$Z_\mu = \cos\theta_\text{W} W_\mu^3 - \sin\theta_\text{W} B_\mu, \quad A_\mu = \sin\theta_\text{W} W_\mu^3 + \cos\theta_\text{W} B_\mu,$$

$$W^\pm = \frac{W_\mu^1 \mp i W_\mu^2}{\sqrt{2}}, \quad e = g \sin\theta_\text{W}, \qquad \frac{G_\text{F}}{\sqrt{2}} = \frac{g^2}{8 M_W^2},$$

$$M_W = 80.4\,\text{GeV}, \quad M_Z = 90\,\text{GeV}, \quad M_H = 125\,\text{GeV},$$

$$v = 176\,\text{GeV}, \quad \cos\theta_\text{W} \approx 0.23.$$

## 4.3 Masses for the matter fields and flavour mixing

The Dirac mass term is

$$\overline{\Psi} m \Psi = m(\psi_\text{L}^\dagger \psi_\text{R} + \psi_\text{R}^\dagger \psi_\text{L}).$$

It is not allowed by the $\text{SU}(2)_\text{L}$ symmetry. An option to solve this problem is to use the Higgs field. A term that we could add is

$$\underset{\overline{\mathbf{2}}_{1/2}}{\overline{(\nu,\, e)}_\text{L}} \underset{\mathbf{2}_{1/2}}{H} \underset{\mathbf{1}_{-1}}{e_\text{R}} + \text{h.c.}$$

The numbers under the terms are the $\text{SU}(2)_\text{L}$ representation in bold (this number is $2T+1$ where $T$ is the weak isospin) and the hypercharge in subscript. This is a special version of a *Yukawa term*

$$\mathcal{L}_\text{Yuk} = Y_{ij}^e \overline{E}_\text{L}^i e_\text{R}^j H + \text{h.c.}$$

where $i$ and $j$ are generation indices and $Y_{ij}^e$ is a complex $3 \times 3$ matrix. After the electro-weak symmetry breaking we get a mass term

$$Y_{ij}^e \begin{pmatrix} \overline{\nu}_\text{L} & \overline{e}_\text{L} \end{pmatrix}^i \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} e_\text{R}^j + \text{h.c.} = Y_{ij}^e \frac{v}{\sqrt{2}} \overline{e}_\text{L}^i e_\text{R}^j + \text{h.c.}.$$

This looks as a mass term for charged leptons

$$m_{ij}^e = \frac{v}{\sqrt{2}} Y_{ij}^e.$$

In order to preserve the symmetry, we have to treat the Yukawa coefficients as spurions. Since the fields transform as

$$E_\mathrm{L} \to U_{E_\mathrm{L}} E_\mathrm{L}, \quad e_\mathrm{R} \to U_{e_\mathrm{R}} e_\mathrm{R}$$

and the kinetic term in the Lagrangian is invariant under $\mathrm{SU}(3)_{E_\mathrm{L}} \times \mathrm{SU}(3)_{e_\mathrm{R}}$, the Yukawa term has to transform as

$$Y^e \sim (\mathbf{3}_\mathrm{E}, \overline{\mathbf{3}}_{e_\mathrm{R}}), \quad Y^e \to U_{E_\mathrm{L}} Y^e U_{e_\mathrm{R}}^\dagger$$

It is not easy to diagonalize $Y_{ij}^e$ as it is not Hermitian. However, we can manually construct two manifestly Hermitian matrices, $Y_e Y_e^\dagger$ and $Y_e^\dagger Y_e$. Their representations and transformations are

$$Y_e Y_e^\dagger \sim (\mathbf{8}_{E_\mathrm{L}}, \mathbf{1}_{e_\mathrm{R}}) \to U_{E_\mathrm{L}}^\dagger Y_e Y_e^\dagger U_{E_\mathrm{L}},$$

$$Y_e^\dagger Y_e \sim (\mathbf{1}_{E_\mathrm{L}}, \mathbf{8}_{e_\mathrm{R}}) \to U_{e_\mathrm{R}}^\dagger Y_e^\dagger Y_e U_{e_\mathrm{R}}.$$

We can diagonalize them such that

$$Y_e Y_e^\dagger = U_e^\dagger D_e^2 U_e,$$

$$Y_e^\dagger Y_e = W_e^\dagger D_e^2 W_e,$$

where

$$D_e = \frac{\sqrt{2}}{v} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}$$

and $U_e, W_e \in \mathrm{U}(3)$. Then, we can perform the *singular value decomposition*, so that $Y_e$ transforms in the bi-unitary way using the same transformation matrices as for the previously constructed Hermitian matrices

$$Y_e = U_e^\dagger D_e W_e.$$

The singular value decomposition splits any complex matrix $M$ into a product of three matrices, $M = UDV^\dagger$, where $U$ and $V$ are unitary matrices and $D$ is a diagonal matrix with non-negative real numbers on the diagonal. The Yukawa term in the Lagrangian then becomes

$$\mathcal{L}_\mathrm{Yuk} = Y_e^{ij} \overline{E}_\mathrm{L}^i H e_\mathrm{R}^j = (U_e^\dagger)^{ik} D_e^{kk'} W_e^{k'j} \overline{E}_\mathrm{L}^i H e_\mathrm{R}^j = \overline{E}_\mathrm{L}^i (U_e^\dagger)^{ik} D_e^{kk'} H W_e^{k'j} e_\mathrm{R}^j = \overline{\tilde{E}}_\mathrm{L}^k D^{kk'} \tilde{e}_\mathrm{R}^{k'},$$

where in the last step we defined the rotated fields (flavour and mass eigenstates)

$$\tilde{E}_\mathrm{L}^k = U_e^{ik} E_\mathrm{L}^i, \quad \tilde{e}_\mathrm{R}^{k'} = W_e^{k'j} e_\mathrm{R}^j.$$

Using these rotated fields, the Lagrangian is diagonal and the rotations $U$ and $W$ disappear from the Lagrangian. For the leptons it works nicely, but for the quarks it does not! In this case, we have to consider the colours:

$$\mathcal{L}_\mathrm{Yuk} = Y_d^{ij} \underset{\overline{\mathbf{2}}_{-1/6}}{\overline{Q}_\mathrm{L}^i} \underset{\mathbf{2}_{1/2}}{H} \underset{\mathbf{1}_{-1/3}}{d_\mathrm{R}^j} + Y_u^{ij} \underset{\overline{\mathbf{2}}_{-1/6}}{\overline{Q}_\mathrm{L}^i} \underset{\mathbf{2}_{-1/2}}{(i\epsilon H^*)} \underset{\mathbf{1}_{2/3}}{u_\mathrm{R}^j},$$

where $\epsilon$ is $2 \times 2$ antisymmetric matrix. The indices $u$ and $d$ on Yukawa coefficients represent the up-type and down-type quark and the indices $i, j \in \{1, 2, 3\}$ represent the generation. Note

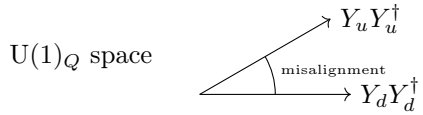
that the fundamental representation of SU(2) is pseudo-real, so $\overline{\mathbf{2}} \cong S\mathbf{2}S^{-1}$ and $S^T = -S$. The flavour symmetry of the quark kinetic term is $\mathrm{U}(3)_Q \times \mathrm{U}(3)_{d_\mathrm{R}} \times \mathrm{U}(3)_{u_\mathrm{R}}$. Do not confuse this for the symmetry $\mathrm{SU}(3)_f$ that we had in QCD. Hence, we follow the same steps as in the leptonic case. In the following table, all Hermitian combinations of the Yukawa terms are shown, together with their representations under the flavour symmetry $\mathrm{U}(3)_Q \times \mathrm{U}(3)_{d_\mathrm{R}} \times \mathrm{U}(3)_{u_\mathrm{R}}$.

| Yukawa term | Q | $u_\mathrm{R}$ | $d_\mathrm{R}$ | diagonalization |
|---|---|---|---|---|
| $Y_u$ | $\mathbf{3}$ | $\overline{\mathbf{3}}$ | $\mathbf{1}$ | $U_u^\dagger D_u V_u$ |
| $Y_d$ | $\mathbf{3}$ | $\mathbf{1}$ | $\overline{\mathbf{3}}$ | $U_d^\dagger D_d V_d$ |
| $Y_u Y_u^\dagger$ | $\mathbf{8}$ | $\mathbf{1}$ | $\mathbf{1}$ | $V_u^\dagger D_u^2 V_u$ |
| $Y_u^\dagger Y_u$ | $\mathbf{1}$ | $\mathbf{8}$ | $\mathbf{1}$ | $U_u^\dagger D_u^2 U_u$ |
| $Y_d Y_d^\dagger$ | $\mathbf{8}$ | $\mathbf{1}$ | $\mathbf{1}$ | $V_d^\dagger D_d^2 V_d$ |
| $Y_d^\dagger Y_d$ | $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{8}$ | $U_d^\dagger D_d^2 U_d$ |

We can rotate the right-handed quarks independently (as they are singlets) and remove $V_u$ and $V_d$ from the Lagrangian by defining

$$u_\mathrm{R}^i \to (V_u)^{ij} u_\mathrm{R}^j, \qquad d_\mathrm{R}^i \to (V_d)^{ij} d_\mathrm{R}^j.$$

Sadly, we cannot rotate $u_\mathrm{L}$ and $d_\mathrm{L}$ independently, since they form the doublet in $\mathrm{SU}(2)_\mathrm{L}$. As we can see, both $Y_u Y_u^\dagger$ and $Y_d Y_d^\dagger$ are in non-trivial representations of $\mathrm{U}(3)_Q$, which means that there can be a misalignment between them – specifically, the set of weak eigenstates and the mass eigenstates can be different. The flavour symmetry breaking geometry looks like this.




The weak interaction Lagrangian for the quarks is

$$\overline{Q}_\mathrm{L}^i \mathrm{D}_\mu Q_\mathrm{L}^i = \overline{Q}_\mathrm{L}^i \mathrm{D}_\mu \begin{pmatrix} u_\mathrm{L} \\ d_\mathrm{L} \end{pmatrix}^i = \overline{Q}_\mathrm{L}^i \left( \partial_\mu - ig \begin{pmatrix} Q_u^Z Z_\mu & W_\mu^+ \\ W_\mu^- & Q_d^Z Z_\mu \end{pmatrix} \right) \begin{pmatrix} u_\mathrm{L} \\ d_\mathrm{L} \end{pmatrix}^i.$$

Because the matrix is not diagonal, the up and down quarks can talk to each other. If we ignore the effect of $W$ bosons ($W^\pm = 0$), the kinetic term is invariant under the extended symmetry

$$\mathrm{U}(3)_{Q_u} \times \mathrm{U}(3)_{Q_d} \times \mathrm{U}(3)_{d_\mathrm{R}} \times \mathrm{U}(3)_{u_\mathrm{R}}.$$

This means that we can use $U_u, U_d, V_u, V_d$ simultaneously without changing the kinetic term – they disappear from $\mathcal{L}_\mathrm{kin}^{W=0}$ in the mass basis. It holds in the standard model that there are no *flavour changing neutral currents* at tree level.

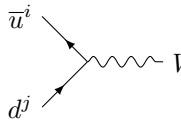
The $W$ field couples up-type and down-type quarks, and consequently the misalignment $U_u \neq U_d$ is a physically measurable value. In the mass basis, the coupling term is

$$\frac{g}{\sqrt{2}} \overline{u}_\mathrm{L}^i \gamma_\mu d_\mathrm{L}^i W_\mu^+ + \mathrm{h.c.} \to \frac{g}{\sqrt{2}} (\overline{u}_\mathrm{L} U_u^\dagger)^i \gamma_\mu W_\mu^+ (U_d d_\mathrm{L})^i = \frac{g}{\sqrt{2}} \overline{u}_\mathrm{L}^i \gamma_\mu d_\mathrm{L}^j V_\mathrm{CKM}^{ij} W_\mu^+ + \mathrm{h.c.}$$

The flavour changing matrix is called the *Cabibbo-Kobayashi-Maskawa matrix* (CKM matrix) and it is a $3 \times 3$ unitary matrix

$$V_\mathrm{CKM}^{ij} \equiv (U_u^\dagger)^{ik} U_d^{kj}.$$

This means that $W$ bosons can change generations and this is called the *flavour changing charged current* (FCCC):




$$= \frac{ig\gamma_\mu}{\sqrt{2}} V^{ij}_{\mathrm{CKM}}.$$

The first question on our minds is how many entries in $Y_\ell$ and $Y_u, Y_d$ are physical.

Let's first look at an analogy. Assume that a quantum system $H_0$ has rotational symmetry (for example hydrogen atom). We perturb a system slightly by, say, external magnetic field $H_p = -\mathbf{B}\cdot\boldsymbol{\mu}$. Since the external magnetic field has a preferred direction, the rotational symmetry is broken. We can use the rotational symmetry of the original Hamiltonian to rotate the coordinate system, so that the magnetic field points in the $z$ direction. This gives one physical parameter that describes the perturbed Hamiltonian $H_p$ – only the $z$-component of the magnetic field. In this analogy, the perturbing field had three real components (and zero complex phases), which we denote $(3,0)$. After the inclusion in the symmetric ground Hamiltonian, only one real parameter remained, denoted $(1,0)$.

We can now take this analogy to the Yukawa coupling. Initially, Yukava matrix is $3\times 3$ complex matrix, so in our notation the number of parameters is $(\text{angles}, \text{phases}) = (9,9)$. We can redefine fields $E_{\mathrm{L}}$ and $e_{\mathrm{R}}$ using $\mathrm{U}(3)_{\mathrm{L}}$ and $\mathrm{U}(3)_{\mathrm{R}}$ symmetry transformations. How many angles and phases are there in the $\mathrm{U}(N)$? The number of angles is the same as in $\mathrm{O}(N)$, which is $N(N-1)/2$. The number of phases must be such that the total number of parameters is $N^2$, that is $N(N+1)/2$. So we have

$$\mathrm{U}(N): \quad (\text{angles}, \text{phases}) = (N(N-1)/2, N(N+1)/2).$$

The leptonic Yukawa coefficient matrix respects one $\mathrm{U}(1)_\ell$ symmetry which corresponds to the conservation of the lepton number. The symmetries of the leptonic Yukawa term are (the underlined group is the one that is conserved)

$$\mathrm{U}(3)_{\mathrm{L}} \times \mathrm{U}(3)_{\mathrm{R}} = \underline{\mathrm{U}(1)_\ell} \times \mathrm{U}(1)_A \times \mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{R}}.$$

Counting the number of parameters, we have

$$\underset{Y^{ij}}{(9,9)} - (2\cdot \underset{U_e, W_e}{(3,6)} - \underset{\mathrm{U}(1)_\ell}{(0,1)}) = (3,0).$$

So we see there are three real parameters and no phases in the leptonic Yukawa matrix. Actually, the number of phases should be negative, but (how do you resolve this???) The three parameters are the masses of the charged leptons, which we can write as

$$Y^e_{ij} = \frac{\sqrt{2}}{v}\begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}.$$

Let's do the same counting for the quark Yukawa coefficients. Initially, we will do the counting for $N$ different generations of quarks, and then we will specialize to $N=3$ and $N=2$. The initial number of parameters is $2(N^2, N^2)$ for two complex square matrices $Y_u$ and $Y_d$. Each of the two singlets and the doublet can remove $N(N-1)/2$ angles and $N(N+1)/2$ phases (corresponding to $\mathrm{U}(N)$ symmetry). Finally, there is a conservation of baryon number which has to be subtracted. We get

$$2(N^2,N^2) - (3\cdot(N(N-1)/2, N(N+1)/2) - (0,1)) = (N(N+3)/2, (N-2)(N-1)/2)\,.$$

Of course, $2N$ of the real parameters are used for the quark masses, so the final result for the number of parameters in the CKM matrix is

$$(\text{mixing angles}, \text{CP-violating phases}) = \left(\frac{N(N-1)}{2}, \frac{(N-1)(N-2)}{2}\right).$$

Now, we focus on $N = 3$. For physical parameters, we have

$$2 \cdot \underset{Y_u, Y_d}{(9,9)} - (3 \cdot \underset{U_Q, V_u, W_d}{(3,6)} - \underset{\mathrm{U}(1)_B}{(0,1)}) = (9,1).$$

The real parameters are the six quark masses and three angles and there is also one complex phase called *Kobayashi-Maskawa phase* (or KM phase). The two bases we can choose are

$$Y_d = \frac{\sqrt{2}}{v}\begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \qquad Y_u = \frac{\sqrt{2}}{v} V_{\mathrm{CKM}}^{\dagger} \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}.$$

In the CKM matrix, there have to be three angles and one phase. If we repeated the same steps for only two generations of matter fields, we would find

$$2 \cdot (4,4) - (3 \cdot (1,3) - (0,1)) = (5,0).$$

In the five parameters, there are four masses and one real angle, called the *Cabibbo angle* $\theta_{\mathrm{C}}$. Importantly, there are no complex phases in the two-generation case. This means that there cannot be any CP violation in the two-generation case, and we will now explain why. A CP transformation maps a left-handed electron to a right-handed positron, and vice versa.

> *CP is a good symmetry of the Lagrangian if we can find a basis where all parameters are real.*

To show this, we first write down the Yukawa term

$$\mathcal{L}_{\mathrm{Yuk}} = Y_{ij}\overline{\psi_{\mathrm{L}}}^{i}\phi\psi_{\mathrm{R}}^{j} + Y_{ij}^{*}\overline{\psi_{\mathrm{R}}}^{j}\phi\psi_{\mathrm{L}}^{i}.$$

The CP transformation acts on the fermions as

$$\phi \to \phi^{*}, \quad \psi_{\mathrm{L}} \to \overline{\psi_{\mathrm{L}}}, \quad \psi_{\mathrm{R}} \to \overline{\psi_{\mathrm{R}}}$$

where we use the convention

$$\overline{\psi_{\mathrm{L}}} = \overline{P_{\mathrm{L}}\psi} = \overline{\psi}P_{\mathrm{R}}.$$

After the CP transformation, the Yukawa term becomes

$$\mathcal{L}_{\mathrm{Yuk}} \to \mathcal{L}_{\mathrm{Yuk}}^{\mathrm{CP}} = Y_{ij}^{*}\overline{\psi_{\mathrm{L}}}^{i}\phi\psi_{\mathrm{R}}^{j} + Y_{ij}\overline{\psi_{\mathrm{R}}}^{j}\phi\psi_{\mathrm{L}}^{i}.$$

This means that $\mathcal{L}_{\mathrm{Yuk}} = \mathcal{L}_{\mathrm{Yuk}}^{\mathrm{CP}}$ if and only if $Y_{ij} = Y_{ji}^{*}$. If all Yukawa couplings are real, the Lagrangian is CP symmetric, hence with two generations of fermions, there is no CP violation. In practice, the treatment is a bit more subtle, because we can define the CP transformation in a more general way. The correct criterion is that CP is violated only if, after exhausting all possible phase redefinitions of the fields, no basis exists in which every coupling is real.

| Particle | Yukawa term | Mass |
|---|---|---|
| top | $y^t \approx 1$ | $m_t \approx 173\,\mathrm{GeV}$ |
| bottom | $y^b \approx 0.025$ | $m_b \approx 4.2\,\mathrm{GeV}$ |
| charm | $y^c \approx 7.5 \cdot 10^{-3}$ | $m_c \approx 1.3\,\mathrm{GeV}$ |
| strange | $y^s \approx 5.5 \cdot 10^{-4}$ | $m_s \approx 93\,\mathrm{MeV}$ |
| up | $y^u \approx 1.3 \cdot 10^{-5}$ | $m_u \approx 2\,\mathrm{MeV}$ |
| down | $y^d \approx 2.7 \cdot 10^{-5}$ | $m_d \approx 5\,\mathrm{MeV}$ |
| electron | $y^e \approx 10^{-2}$ | $m_e \approx 0.511\,\mathrm{MeV}$ |
| muon | $y^\mu \approx 6.4 \cdot 10^{-4}$ | $m_\mu \approx 106\,\mathrm{MeV}$ |
| tau | $y^\tau \approx 2.9 \cdot 10^{-6}$ | $m_\tau \approx 1.8\,\mathrm{GeV}$ |

With two generations, the CKM matrix is

$$V_{\mathrm{CKM}} = \begin{pmatrix} \cos\theta_{\mathrm{C}} & \sin\theta_{\mathrm{C}} \\ -\sin\theta_{\mathrm{C}} & \cos\theta_{\mathrm{C}} \end{pmatrix}, \qquad \sin\theta_{\mathrm{C}} \approx 0.22.$$

Somewhat interestingly, Cabibbo angle is very similar to Weinberg angle. In three generations, we write the CKM matrix as

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix},$$

and we use the standard (PDG) parametrization

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

with the shorthand $s_{ij} \equiv \sin\theta_{ij}$ and $c_{ij} \equiv \cos\theta_{ij}$. The mixing angles are $\theta_{12}$, $\theta_{23}$, $\theta_{13}$ and the CP phase is $\delta$, with numerical values

$$\sin\theta_{12} = 0.225\,00 \pm 0.000\,67, \qquad \sin\theta_{23} = 0.041\,82^{+0.000\,85}_{-0.000\,74},$$
$$\sin\theta_{13} = 0.003\,69 \pm 0.000\,11, \qquad \delta = 1.144 \pm 0.027.$$

We can notice that the CKM matrix is almost diagonal, so people have come up with a simpler parametrization that sacrifices some precision. It is called the *Wolfenstein parametrization*, and it is defined as

$$V = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4),$$

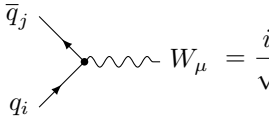
where

$$\lambda = \sin\theta_{12}, \quad A\lambda^2 = \sin\theta_{23}, \quad A\lambda^3(\rho - i\eta) = \sin\theta_{13}e^{-i\delta}.$$

Best fits for these parameters are

$$\rho = 0.159 \pm 0.010, \quad \eta = 0.348 \pm 0.010, \quad A = 0.826^{+0.018}_{-0.015}, \quad \lambda = 0.225\,00 \pm 0.000\,67.$$

This shows that CKM matrix is very close to the identity matrix and that there is a pronounced hierarchy between the quark masses. A generic vertex between two quarks and a $W$ boson is




$$= \frac{ig}{\sqrt{2}} \begin{pmatrix} \overline{u}_{\mathrm{L}} & \overline{c}_{\mathrm{L}} & \overline{t}_{\mathrm{L}} \end{pmatrix} V_{\mathrm{CKM}} \gamma^\mu \begin{pmatrix} d_{\mathrm{L}} \\ s_{\mathrm{L}} \\ b_{\mathrm{L}} \end{pmatrix}$$

Ideally, we would like to find a coordinate-free way of determining whether the CP is violated or not. One way is the *Jarlskog invariant* $J$, defined as

$$\mathrm{Im}\,(V_{ij}V_{kl}V_{il}^{*}V_{kj}^{*}) = J\sum_{m,n=1}^{3}\varepsilon_{ikm}\varepsilon_{jln}$$

with no implicit summation. In terms of PDG angles and Wolfenstein parameters, it is

$$J = c_{12}c_{23}^{2}c_{13}s_{12}s_{23}s_{13}\sin\delta \approx \lambda^{6}A^{2}\eta = (3.00^{+0.15}_{-0.09})\cdot 10^{-5}.$$

The theoretical maximal value for this invariant (the absolute unitary bound) is

$$|J| \leq \frac{1}{6\sqrt{3}} \approx 0.1.$$

Clearly, the actual value is way below the theoretical limit, but it is not zero. Thus, we say that the CP is violated *weakly*. Equipped with the Jarlskog invariant, we can state the necessary and sufficient condition for the CP violation:

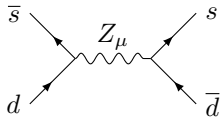
$$X_{\mathrm{CP}} \equiv \Delta_{tc}^{2}\Delta_{tu}^{2}\Delta_{cu}^{2}\Delta_{bs}^{2}\Delta_{bd}^{2}\delta_{sd}^{2}J \neq 0,$$

where $\Delta_{ij}^{2} = m_{i}^{2} - m_{j}^{2}$ is the mass difference between the $i$-th and $j$-th quark. The theory would conserve the CP symmetry if either Jarlskog determinant was zero or if any pair of like-charge quarks had the same mass. We can also rewrite the CP violating invariant in truly coordinate-free way as

$$X_{\mathrm{CP}} = \mathrm{Im}\,(\det([Y_{d}Y_{d}^{\dagger}, Y_{u}Y_{u}^{\dagger}])).$$

## 4.4 GIM mechanism

This mechanism explains why the strength of the flavour-changing neutral currents is so weak. It is named after Glashow, Iliopoulos and Maiani, hence it is called the *GIM mechanism*. The process with which this mechanism was first studied is the mixing between neutron kaons $K = (d\overline{s})$ and $\overline{K} = (\overline{d}s)$. At tree level, we could have a $Z$-boson exchange.




The contribution from this diagram would be

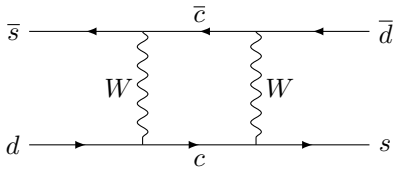
$$\frac{1}{M_{Z}^{2}} \approx \frac{1}{(100\,\mathrm{GeV})^{2}}.$$

What they actually measure is

$$\text{experiment}: \quad \frac{1}{\Lambda^{2}} \approx \frac{1}{(10^{3}\,\mathrm{TeV})^{2}}.$$

Clearly, the mixing is far too weak to be produced by the exchange of $Z$-boson. Moreover, we already know that $Z$ only couples diagonally in flavour, so it cannot change flavours. This is consistent with the statement that there is no FCNC at tree level. However, we can still have a

contribution from the charged current with one-loop contribution. The naive computation would be the following.




$$\approx \frac{g^4}{16\pi^2}\frac{1}{M_W^2} \approx \frac{1}{(3\,\mathrm{TeV})^2}.$$

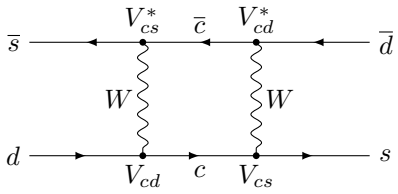
This value is still way off the experimental results. We still have to add the CKM matrix contribution to the calculation. We have to sum over all possible up-type propagators:

$$\mathcal{A} \propto \sum_{i=u,c,t} (V_{is}^* V_{id})(V_{js}^* V_{jd}) f(m_i, m_i)$$

where $f(m_i, m_j)$ is the loop function. This expression is very similar to the Jarlskog invariant. If $m_u = m_c$, this contribution would be zero exactly. No matter what the propagator is ($u$, $c$ or $t$) the change of quark generation has to happen at least at two vertices. Since off-diagonal terms in CKM matrix are very small, any generation change is significantly suppressed. This is called *Cabibbo suppression*. The coupling of quarks from first and third generation together is even more suppressed (so-called double Cabibbo suppression). If the propagators in the loop are $u$ or $c$ quark, there are Cabibbo suppressions at two vertices. If the propagator is $t$, then there are two singular and two double suppressions, which means that we can ignore this channel (even though the amplitude is multiplied by the top mass quark). From the other two options for the propagators ($u$ and $c$), the charm contributes more, since the loop factor contributes the mass of the quark. The main contribution to the CKM factor in FCNC loop is thus

$$\mathcal{A} \sim G_\mathrm{F}^2 (V_{cs} * V_{cd})^2 m_c^2 \approx \frac{m_c^2}{M_W^2} \sin^2\theta_\mathrm{C}.$$

The Fermi constant was added to make the expression dimensionless. The CKM corrected one-loop FCNC amplitude is




$$\approx \frac{g^4}{16\pi^2}\frac{m_c^2}{M_W^4}\sin^2\theta_\mathrm{C} \approx \frac{1}{(10^3\,\mathrm{TeV})^2}.$$

This result is consistent with observations.

# Chapter 5

# Beyond the Standard Model

## 5.1 Grand unified theories

In the electroweak theory, the low-energy regime splits into electromagnetic sector (unbroken $\mathrm{U}(1)_{\mathrm{em}}$) and the spontaneously broken weak sector with three massive gauge bosons. At high energies, there is $\mathrm{SU}(2) \times \mathrm{SU}(1)$ theory with four effectively massless gauge bosons and no difference between electromagnetic and weak interactions. How can we add the QCD into the picture? At mass of the $Z$ boson, the strong coupling is much stronger than the weak coupling ($\alpha_{\mathrm{s}} \gg \alpha_{\mathrm{w}}$). However, these two couplings run with different speeds, according to the $\beta$-function:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -\frac{\alpha^2}{2\pi}\left[\frac{11}{3}C_{\mathrm{adj}} - \frac{2}{3}N_f^{\mathrm{Weyl}}C_{\mathrm{f}} - \frac{1}{6}N_c\right], \qquad t = \ln\mu.$$

This is a very general expression; specifically for our case, we have

$$\frac{\mathrm{d}\alpha_{\mathrm{s}}}{\mathrm{d}t} = -\frac{\alpha_{\mathrm{s}}^2}{2\pi}\cdot 7, \quad \frac{\mathrm{d}\alpha_{\mathrm{w}}}{\mathrm{d}t} = -\frac{\alpha_{\mathrm{w}}^2}{2\pi}\frac{19}{6}.$$

Solving for $\alpha_{\mathrm{s}}(M_G) = \alpha_{\mathrm{w}}(M_G)$, we get $M_G \sim 10^{16}\,\mathrm{GeV}$. This is way above any detectable scales, however it is still smaller than Planck's energy, so it makes sense to talk about this.

To unify the theory, Georgi, Glashow and partly Salam (in 1974) proposed that there might be some gauge unified coupling at the energy of $M_G$. The gauge group $G$ should be greater than the Standard model group, $G \supset \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. If we count the number of commuting generators of these groups, we get $2+1+1=4$, so $\mathrm{rank}(G) > 0$. What turns out to work is the group $G = \mathrm{SU}(5)$, the $5 \times 5$ Hermitian traceless matrices, which are generated by 24 generators. This is one of the earliest *grand unified theories*(GUT). Some generators can be identified by considering SU(5) as block diagonal of SU(2) and SU(3):

$$T^{1,2,3} = \begin{pmatrix} \tau^{1,2,3}_{\mathrm{Pauli}} & 0 \\ 0 & 0 \end{pmatrix}, \quad T^{4-11} = \begin{pmatrix} 0 & 0 \\ 0 & T^{1-8}_{\mathrm{GM}} \end{pmatrix}, \quad T^{12} = \frac{1}{\sqrt{60}}\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} = Y/c.$$

These generators correspond to SU(2), SU(3) and U(1), respectively. The last generator could also be multiplied by any non-zero scalar $c$ to give the hypercharge $Y$. How do the Standard

model fermions fit into this discussion? We will work with Weyl spinors. In the Standard model, there are left- and right-handed spinors $\Psi_{\mathrm{L}}$ and $\Psi_{\mathrm{R}}$. However, in this context it is more convenient to work with $\Psi_{\mathrm{r}}^c = \varepsilon\Psi_{\mathrm{R}}^*$, which also transforms as a left-handed spinor.

| Gauge group | $L = \begin{pmatrix}\nu\\ e\end{pmatrix}_{\mathrm{L}}$ | $e_{\mathrm{R}}^c$ | $Q = \begin{pmatrix}u\\ d\end{pmatrix}_{\mathrm{L}}$ | $u_{\mathrm{R}}^c$ | $d_{\mathrm{R}}^c$ |
|---|---|---|---|---|---|
| $\mathrm{SU}(2)_{\mathrm{L}}$ | $\mathbf{2}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\mathbf{1}$ | $\mathbf{1}$ |
| $\mathrm{SU}(3)_{\mathrm{C}}$ | $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{3}$ | $\overline{\mathbf{3}}$ | $\overline{\mathbf{3}}$ |
| $\mathrm{U}(1)_{\mathrm{Y}}$ | $-1/2$ | $+1$ | $+1/6$ | $-2/3$ | $-1/3$ |

The fundamental representation of SU(5) is **5**, which can be decomposed into the Standard model representations as

$$\mathbf{5} \to (\mathbf{2},\mathbf{1})_{3\beta} + (\mathbf{1},\mathbf{3})_{-2\beta}$$
$$\overline{\mathbf{5}} \to (\mathbf{2},\mathbf{1})_{-3\beta} + (\mathbf{1},\overline{\mathbf{3}})_{2\beta}.$$

The notation $(\mathbf{a},\mathbf{b})_c$ means that the representation is $a$ dimensional in SU(2), $b$ dimensional in SU(3) and has charge $c$ in $\mathrm{U}(1)_Y$ (we used $\beta = c/\sqrt{60}$). Hence, we can pick one SU(2) doublet (which has to be colour singlet) and one SU(3) triplet (which has to be SU(2) singlet) and put them together into a **5** representation of SU(5):

$$\Psi_{\overline{5}} = \begin{pmatrix}\varepsilon L\\ d_{\mathrm{R}}^c\end{pmatrix}$$

If we want the hypercharge of electron to be $-1/2$, then see that $\beta = 1/6$ and $c = \sqrt{60}\beta = \sqrt{60}/6 = \sqrt{5/3}$. Hence, $Y = \sqrt{5/3}T^{12}$ and the charges fit perfectly:

$$Q = T^3 + Y = \begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&-1/3&0&0\\0&0&0&-1/3&0\\0&0&0&0&-1/3\end{pmatrix}, \qquad Y = \begin{pmatrix}1/2&0&0&0&0\\0&1/2&0&0&0\\0&0&-1/3&0&0\\0&0&0&-1/3&0\\0&0&0&0&-1/3\end{pmatrix}.$$

We still have 10 elements left to represent: 6 elements in $Q_{\mathrm{L}}$ (three colours times two isospins), 1 element in $e_{\mathrm{R}}^c$ and 3 elements in $u_{\mathrm{R}}^c$. Together, there are 10 elements, so maybe we can fit all fermions into **10** representation of SU(5)? Yes, they do! This gives an antisymmetric 2-tensor $\Psi^{ij}$. The particles fit into the matrix as follows:

$$\begin{pmatrix}0&e_{\mathrm{R}}^c&u_{\mathrm{L}}^1&u_{\mathrm{L}}^2&u_{\mathrm{L}}^3\\ -e_{\mathrm{R}}^c&0&d_{\mathrm{L}}^1&d_{\mathrm{L}}^2&d_{\mathrm{L}}^3\\ -u_{\mathrm{L}}^1&-d_{\mathrm{L}}^1&0&u_{\mathrm{R}}^{c1}&u_{\mathrm{R}}^{c2}\\ -u_{\mathrm{L}}^2&-d_{\mathrm{L}}^2&-u_{\mathrm{R}}^{c1}&0&u_{\mathrm{R}}^{c3}\\ -u_{\mathrm{L}}^3&-d_{\mathrm{L}}^3&-u_{\mathrm{R}}^{c2}&-u_{\mathrm{R}}^{c3}&0\end{pmatrix}$$

Such organization of fermions helps solve the problem of charge quantization. In the Standard model it holds $Q = T_3 + Y$, however, there is no reason for integer charges on $Y$ ($\mathrm{U}(1)_Y$ could also be any real number). In Grand Unified Theory, the element $Y$ is SU(5) generator (up to one global prefactor), so the only possible charges are 1 and 1/3. It follows immediately that $Q_q = Q_e/3$. This is called the *GUT miracle*.

Next, we will try to see how the gauging works in SU(5). There are 24 generators, so there must be 24 gauge bosons in adjoint representation. Since there are 12 gauge bosons in the

Standard model (8 gluons, $W^{\pm}$, $Z$ and $\gamma$), there must be additional 12 more gauge bosons in SU(5). We will denote them by $X_i$ and $Y_i$, where $i = 1, 2, 3$ (and we also allow the antiparticles):

$$A^a_\mu T^a = \begin{pmatrix} & & X_1 & X_2 & X_3 \\ & & Y_1 & Y_2 & Y_3 \\ \overline{X}_1 & \overline{Y}_1 & & & \\ \overline{X}_2 & \overline{Y}_2 & & & \\ \overline{X}_3 & \overline{Y}_3 & & & \end{pmatrix}$$

The empty blocks are filled with the Standard model gauge bosons (gluons in the lower block and SU(2) × U(1)). Their charges are $Y_X = +4/3$ and $Q_Y = +1/3$. These bosons mix leptons and quarks, so they violate the conservation of the baryon number and predict the proton decay. The gauge bosons couple to fermions as

$$\overline{\Psi}_5 \gamma_\mu g_5 A^{\mu,a} T^a \Psi_{\overline{5}}$$

and

$$\overline{\Psi}^{ij}_{10}(A^{\mu,a} T^a_{ik}\delta_{jl} + A^{\mu,a} T^a_{jl}\delta_{ik})\Psi^{kl}_{10}.$$

These fields couple together particles, that don't interact within the Standard model. For example, the $X$ boson couples to $\overline{d}^i_{\mathrm{L}} u^{cj}_{\mathrm{R}}$ and $\overline{e}^i_{\mathrm{R}} \nu^j_{\mathrm{L}}$.

Now that we have all particles, we want that SU(5) breaks down to SU(3) × SU(2) × U(1) at the energy scale $M_G$. This should give bosons $X$ and $Y$ with masses of the order $M_G$. At the GUT scale, the coupling constants for SU(2) and SU(3) are the same, but at lower energies they run differently, so they differ at energies that we can observe. In order to break GUT, we need some sort of Higgs mechanism. The simplest one that works is a scalar $\Sigma$ in the **24** (adjoint) representation of SU(5), which has the following vacuum expectation value

$$\langle \Sigma \rangle = v_5 \begin{pmatrix} -3 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The action of the adjoint representation is the commutator, hence we require $[\langle \Sigma \rangle, T^{1,\dots,12}] = 0$ in order for the Standard model symmetry to be unbroken. Fortunately, this is exactly what happens. The scale of the vacuum expectation value is $v_5 \sim M_G$. The standard model Higgs boson can be in **5** representation (at scale $m_H \sim 100\,\mathrm{GeV}$). Since the scalar masses are so far apart, this is called the *gauge hierarchy problem*.

Let's take a look at some predictions of the SU(5) GUT.

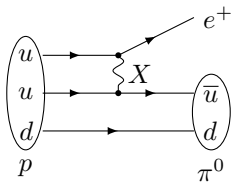
- It can predict the **weak mixing angle**. Previously, we showed $Y = \sqrt{5/3}T_{12}$, so $\mathrm{U}(1)_Y$. The hypercharge coupling $g'$ is related to $T_{12}$ generator via the equation $g'Y = g_5 T_{12}$. It follows $g' = \sqrt{3/5}g_5$ and of course $g = g_5$. The naive calculation of the weak angle gives

$$\sin^2\theta_{\mathrm{W}} = \frac{g^2}{g^2 + g'^2} = \frac{3}{8} = 0.325.$$

  The actual measurement is $\sin^2\theta_{\mathrm{W}} = 0.231$, but this is measured at $M_Z$ scale. We can try to solve, how the weak angle is running with the energy and try to infer its value at

$M_Z$ scale. We have to be careful doing this, as we actually shouldn't use all the $\beta$-function knowledge we have. If we try to evolve the three coupling constants, we just find that they never meet all together. What we do instead is that we postulate that they do meet at some scale $M_G$: $\alpha_s(M_G) = \alpha_W(M_G) = \alpha_Y(M_G)$. As the initial condition we use the values of $\alpha_S(M_Z)$, $\alpha_W(M_Z)$ and $\sin^2\theta_W|_{M_Z}$, and we solve for $\sin^2\theta_W$, $M_G$ and $\alpha_Y(M_Z)$ (for the letter we pretend is unknown). We find $\sin^2(\theta_W)|_{M_G} \approx 0.183$. This is a bit better prediction, but still not very good.

- GUT also predicts **proton decay**.




The momentum in the propagator is much smaller than the mass of the $X$ boson, so the amplitude is simply $\mathcal{A} \sim g_5^2/M_X^2$. The decay rate is

$$\Gamma \sim \frac{g_5^2}{M_x^4} m_p^5$$

which gives the lifetime $\tau_p \sim 10^{25}$ years. Doing the calculation a bit more precisely, we get even higher value $\tau_p \sum 10^{28}$ to $10^{30}$ years. The measured bound for $p \to \pi^0 e^+$ decay rate is around $\tau_p > 2.4 \cdot 10^{34} years$ which is not particularly consistent with the GUT prediction.

- **Fermion masses**: the Higgs boson in the **5** representation has the vacuum expectation value of $\langle H \rangle = \mathrm{diag}(0, v/\sqrt{2}, 0, 0, 0)$. It can couple to fermions either between **5** and **10**

$$\lambda \Psi_{10}^{ij} \Psi_{\overline{5}}^{i} (H^*)_j = \frac{\lambda v}{\sqrt{2}} (-\overline{e}_R e_L + \overline{e}_L e_R) + \text{h.c.}$$

or between two **10** representations

$$\lambda_i \Psi_{10}^{ij} \Psi_{10}^{kl} H^m \varepsilon_{ijklm} = \frac{\lambda_u v}{\sqrt{2}} \overline{u}_R u_L + \text{h.c.}$$

This predicts $m_e = m_d$, $m_\mu = m_s$ and $m_\tau = m_b$. After considering the running, the last generation matches the experimental results quite well, but the others don't. We can avoid this discrepancy by using a more complicated Higgs representation, for example **45**, but this doesn't sound particularly appealing.

- **Neutrino masses**: we can add $\nu_R$ as a singlet in SU(5) (and consequently a singlet in all the Standard model symmetry groups). The mass would be

$$\overline{\Psi}_5 H^* \nu_R = \frac{v}{\sqrt{2}} \overline{\nu}_L \nu_R + \text{h.c.}$$

If neutrino has a Majorana mass $M_G \nu_R^T \varepsilon \nu_R$, then this would explain the smallness of the left-handed neutrino masses

$$m_{\nu_R} = \frac{v^2}{M_G} \approx 0.1\,\text{eV to } 10^{-3}\,\text{eV}.$$

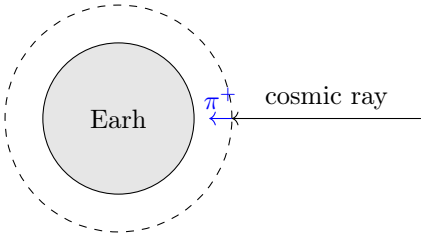
A more detailed description of possible sources of neutrino is provided by the *seesaw mechanism*.

## 5.2 Neutrinos

So far, we have discussed neutrinos only as a doublet partners of left-handed electrons. They have $T_3 = 1/2$ and $Y = -1/2$, so they don't carry any electric charge. They take part in one of the Yukawa terms with Higgs field, $Y_E^{ij}\overline{L}_{\mathrm{L}}^i e_{\mathrm{R}}^j H$. We can diagonalize the Yukawa matrix using rotations from $\mathrm{SU}(3)_L$ and $\mathrm{SU}(3)_{e_{\mathrm{R}}}$, so that only electron, muon and tau obtain masses, $vY_E^{ij}\overline{e}_{\mathrm{L}}^i e_{\mathrm{R}}^j + \mathrm{h.c.}$ One consequence of this is that the individual lepton flavour is conserved:

$$\mathrm{U}(1)_e \times \mathrm{U}(1)_\mu \times \mathrm{U}(1)_\tau.$$

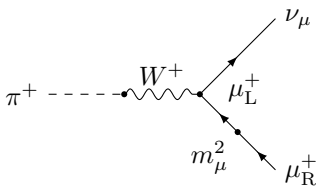
Additionally, the total lepton number is also conserved, $L = L_e + L_\mu + L_\tau$. What does the nature tell us about neutrinos? When high energy cosmic rays hit the atmosphere, a number of $\pi^+$ particles is created.




These pions decay predominantly to $\mu^+$ and $\nu_\mu$. The channel with electron is also allowed, but is suppressed by the square of the electron mass

$$\frac{\Gamma(\pi^+ \to \mu^+\nu_\mu)}{\Gamma(\pi^+ \to e^+\nu_e)} \approx \frac{m_\mu^2}{m_e^2} \approx 10^4.$$

This is because the interaction that mediates the decay is weak, hence both lepton and neutrino are produced in the same chirality state and hence opposite helicity states (the neutrino is an antiparticle, so it will be right-handed). However, the spin of pion is zero, hence the lepton has to interact with Higgs field to change the helicity state to the opposite spin. The interaction with Higgs field gives a factor of lepton mass squared to the amplitude.




The muon then also decays to $\mu^+ \to e^+\mu_e\overline{\nu}_\mu$. The overall reaction is thus

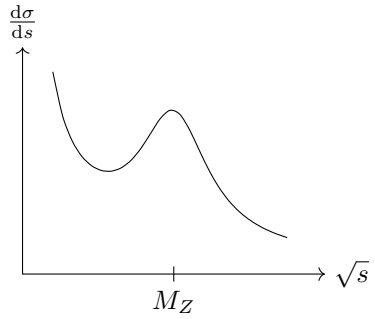
$$\pi^+ \to e^+\nu_\mu\nu_e\overline{\nu}_\mu.$$

What we expect then is that the ratio between electronic and muonic *atmospheric neutrinos* is $1:2$. We can observe this number experimentally! If we look up in the atmosphere, we get approximately the expected result. However, if we look into the Earth and measure neutrinos that were produced on the other side of the Earth (and travelled mostly uninterrupted through it), we don't find what we expect.

Even more problematic is the measurement of *solar neutrinos*. When people measured (very accurately, $5\sigma$) the abundance of electronic neutrinos coming from the Sun, they found that some

electron neutrinos get converted to muonic or tau neutrinos. This means that $\mathrm{U}(1)_e \times \mathrm{U}(1)_\mu \times \mathrm{U}(1)_\tau$ must be violated (just like with quarks).

How many generations of neutrinos are there? We can measure that by observing the $Z$-boson decay width. If we collide electrons together, we get a very distinct cross-section dependence on the centre-of-mass energy around the mass of $Z$-boson.




The cross-section features a very prominent Breit-Wigner peak at the mass of the $Z$-boson,which comes from the propagator of the $Z$-boson

$$\frac{i}{p^2 - M_Z^2 + iM_Z\Gamma_Z}.$$

By studying the width of the peak, we can measure the decay width $\Gamma_Z$. The $Z$-boson decays either in quark-antiquark pair, or in lepton-antilepton pair or in neutrino-antineutrino pair. The individual decay widths can be calculated and also measured by selectively only looking at certain decay products. Neutrinos cannot be detected directly, but we can measure all other decay channels and the total decay width. The difference between the total decay width and the sum of the other decay widths gives us the neutrino decay width. We can then compare this number with the theoretical prediction for a single neutrino species, and we get a relative factor of $2.984 \pm 0.008$. With this, we conclude that the number of neutrinos (with mass less than $M_Z/2$) is exactly 3.

How can we introduce masses of neutrinos? Let's try to do this by only using the Standard model fields. We use $L = (\nu, e)_\mathrm{L}$ and write down one possible Lagrangian (called *Weinberg operator*):

$$\mathcal{L}_\text{Weinberg} = C_{nm}\varepsilon^{\alpha\beta}L_\alpha^{i,n}L_\beta^{j,m}H^iH^j + \text{h.c.},$$

where $i, j$ are the $\mathrm{SU}(2)_\mathrm{L}$ indices, $\alpha, \beta$ are Weyl indices and $m, n$ are the generation indices. The matrix $C_{mn}$ is complex, $3 \times 3$ and symmetric. In fact, this is the only dimension-5 operator that can be constructed from the Standard model fields to describe neutrino masses. A cosmological bound for the sum of neutrino masses is

$$\sum_i m_{\nu_i} \leq 0.25\,\text{eV}.$$

The energy scale of the Weinberg operator (since it is an irrelevant operator, it needs to have some EFT scale) is $\Lambda$ and the mass of the neutrinos is approximately

$$m_\nu \approx C\frac{v^2}{\Lambda}.$$

The scale $\Lambda$ can be very roughly approximated as

$$\Lambda \sim \frac{Cv^2}{m_\nu} \approx 10^{15}\,\text{GeV}.$$

This is suspiciously close to the GUT scale. The bigger the scale, the smaller the neutrino mass. This is called the *seesaw mechanism* (as GUT energy and neutrino energy are in some sort of opposite balance – one goes up, the other goes down).

Let's see how many degrees of freedom we obtain by this mechanism. The Yukawa matrix has $(9,9)$ parameters (angles/masses, phases) and $C$ has $(6,6)$ parameters (because of the symmetry). The symmetry that is preserved at the end is still $\mathrm{U}(3)_L \times \mathrm{U}(3)_{e_\mathrm{R}}$. So we have:

$$\underset{Y_e}{(9,9)} + \underset{C}{(6,6)} - \underset{\mathrm{U}(3)_L}{(3,6)} - \underset{\mathrm{U}(3)_{e_\mathrm{R}}}{(3,6)} = (9,3).$$

The nine real parameters are the six masses (three for neutrinos and three for charged leptons) and three mixing angles $(\theta_{12}, \theta_{13}, \theta_{23})$. Additionally, we have three CP phases $(\delta_\mathrm{CP}, \alpha, \beta)$. This is a big difference from quarks. A typical parametrization of the neutrino mixing is

$$m_e = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad m_\nu = V_\mathrm{PMNS}^* \begin{pmatrix} m_1 e^{-2i\alpha} & 0 & 0 \\ 0 & m_2 e^{-2i\beta} & 0 \\ 0 & 0 & m_3 \end{pmatrix} V_\mathrm{PMNS},$$

where the *PMNS matrix* (Pontecorvo-Maki-Nakagawa-Sakata matrix) is a $3 \times 3$ unitary matrix that describes the mixing of neutrinos. It is parametrized as

$$V_\mathrm{PMNS} = R_{23}(\theta_{23}) R_{13}(\theta_{13}) \begin{pmatrix} e^{i\delta_\mathrm{CP}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} R_{12}(\theta_{12}).$$

The angles $\alpha$ and $\beta$ are the *Majorana phases* (they only exist if there is Majorana mass, which hasn't been detected yet), and $\delta_\mathrm{CP}$ is the *Dirac phase*. The values for the mixing angles are

$$\theta_{12} \approx 33.41^{+0.75\circ}_{-0.72}, \quad \theta_{23} \approx 49.1^{+1.0\circ}_{-1.3}, \quad \theta_{13} \approx 8.54^{+0.11\circ}_{-0.12}, \quad \delta_\mathrm{CP} = 197^{+42\circ}_{-25}.$$

This is very different from the CKM matrix, where a significant hierarchy of the angles exists.

How do we complete this theory in the UV regime? We postulate that there exists a right-handed neutrino $\nu_\mathrm{R}$, which is a singlet under $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{U}(1)_Y$. Hence, $Y = Q = 0$. It is called a *sterile neutrino*. With this, we can make a one-to-one copy of the quark Higgs sector:

$$\mathcal{L}_\mathrm{lep} = Y_e^{ij} \overline{L}_\mathrm{L}^i e_\mathrm{R}^j H + Y_\nu^{ij} \overline{L}_\mathrm{L}^j \nu_\mathrm{R}^j \tilde{H} + \text{h.c.}$$

However, the physics is not the same as for quarks, because the right-handed neutrino is a singlet, so there are additional terms allowed, namely the *Majorana mass term*:

$$\mathcal{L}_\mathrm{Maj} = \frac{1}{2} M^{ij} \nu_{\mathrm{R},\alpha}^i \nu_{\mathrm{R},\beta}^j \varepsilon^{\alpha\beta} + \text{h.c.},$$

where $M^{ij}$ is complex symmetric $3 \times 3$ matrix. The total mass is then

$$\mathcal{L}_\mathrm{mass} = \frac{1}{2} M^{ij} (\nu_\mathrm{R}^i \cdot \nu_\mathrm{R}^j) + \underbrace{v Y_\nu^{ij}}_{m_\mathrm{D}} (\nu_\mathrm{L}^i)^\dagger \nu_\mathrm{R}^j + \text{h.c.} = \frac{1}{2} \begin{pmatrix} \nu_\mathrm{R} & \nu_\mathrm{L}^\dagger \end{pmatrix} \begin{pmatrix} M & m_\mathrm{D}^T \\ m_\mathrm{D} & 0 \end{pmatrix} \begin{pmatrix} \nu_\mathrm{R} \\ \nu_\mathrm{L}^\dagger \end{pmatrix},$$

where we denoted $(\nu_{\mathrm{R}}^i \bullet \nu_{\mathrm{R}}^j) = \nu_{\mathrm{R},\alpha}^i \nu_{\mathrm{R},\beta}^j \varepsilon^{\alpha\beta}$. The eigenvalues of the mass matrix are

$$m = \frac{1}{2}\left(M \pm \sqrt{M^2 + m_{\mathrm{D}}^2}\right) \overset{M \gg m_{\mathrm{D}}}{\approx} \begin{cases} M, \\ m_{\mathrm{D}}^2/M = m_\nu. \end{cases}$$

If we take $m_{\mathrm{D}} \sim yv$, the mass of the right-handed neutrino is $M \sim y^2 \cdot 10^{15}\,\mathrm{GeV}$. This is called type-I seesaw mechanism. We could also put the Higgs boson in a different representation (for example, a triplet Higgs representation would give a type-II seesaw mechanism).

What is the low energy behaviour of this EFT when we integrate out $\nu_{\mathrm{R}}$ ($M \gg E$)? The easiest way to do this is to use EOM matching. We take $p_{\nu_{\mathrm{R}}}^\mu = (M, 0, 0, 0)$ and the equation of motion

$$M_{ij}(\nu_{\mathrm{R}}^i)_\alpha \varepsilon^{\alpha\beta} + m_{\mathrm{D}}^{ij}(\overline{\nu}_{\mathrm{L}}^i)^\beta = 0.$$

After solving for $\nu_{\mathrm{R}}$ and plugging into the $\mathcal{L}_{\mathrm{mass}}$, we get

$$\mathcal{L}_{\mathrm{mass}} = \frac{1}{2}(\nu_{\mathrm{L}}^\dagger)^i m_{\mathrm{D}}^{ij}(M^{-1})^{jk} m_{\mathrm{D}}^{lk}(\nu_{\mathrm{L}}^*)_\beta^k \varepsilon^{\alpha\beta}.$$

This is exactly the same as the Weinberg operator, with $m_{\mathrm{D}}^{ij} = Y_\nu^{ij}\langle\tilde{H}\rangle$ and $C_{nm} = \Lambda(M^{-1})^{jk}$.

A consequence of the neutrino mass is a phenomenon called *neutrino oscillations*. We know the hierarchy of the neutrino masses

$$|m_3^2 - m_2^2| = 2.5 \cdot 10^{-3}\,\mathrm{eV}, \quad |m_2^2 - m_1^2| = 7.4 \cdot 10^{-5}\,\mathrm{eV}.$$

It could also happen that one of the neutrinos is massless. Let's say that we have an initial state

$$|\nu(x=0)\rangle = |\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle\,.$$

Note that the initial state $|\nu_e\rangle$ is the weak eigenstate, and it is not the same as any single mass eigenstate $|\nu_1\rangle$. The evolution of the state is given by

$$|\nu(x)\rangle = \cos\theta e^{ip_1 x}\,|\nu_1\rangle + \sin\theta e^{ip_2 x}\,|\nu_2\rangle\,,$$

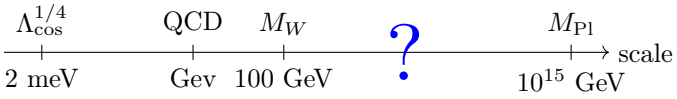
where $E^2 = m_i^2 + p_i^2$. We expand the spatial momentum as $p_i = E - m_i^2/2E + \ldots$ and after some algebra we get

$$P(\nu_e \to \nu_\mu) = \sin^2(2\theta)\sin^2\left(\frac{m_2^2 - m_1^2}{4E}L\right),$$

where $L$ is the distance travelled by the neutrino. If we take GeV for energy and eV for the mass, then the oscillation length is of order kilometres.

## 5.3 Outlook

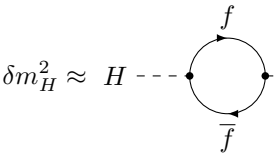
The Standard model explains all experiments performed in laboratories on Earth.



The scales we have tested go up to 10 TeV. We have a huge hierarchy in the energy scales:

$$\frac{G_{\mathrm{F}}}{G_{\mathrm{grav}}} \sim 1.7 \cdot 10^{33} \sim \frac{M_{\mathrm{Pl}}^2}{M_W^2}, \qquad \frac{m_p^2}{G_{\mathrm{grav}}} \sim 1.7 \cdot 10^{38} \sim \frac{M_{\mathrm{Pl}}^2}{\Lambda_{\mathrm{QCD}}^2}.$$

The second hierarchy can be explained: $\Lambda_{\mathrm{QCD}}$ can be explained dynamically by theory of dimensional transformation and $\beta$-function. The first hierarchy is much more problematic. There is a theoretical problem with EW scale, particularly with the Higgs mass term $\mu^2 H^\dagger H$:

$$\delta m_H^2 \approx$$ 

$$\approx g^2 \frac{m_f^2}{4\pi^2} \ln(\dots).$$

If $m_F \gg m_H^{\mathrm{exp}}$, it would require large cancellation between $\mu^2$ and $\delta m_H^2$. If a relevant operator is not protected by symmetries, it is suitable to UV mass thresholds. This requires *fine-tuning* to keep $\mu^2 \ll \Lambda_{\mathrm{UV}}^2$. There is a formalization of this "naturality" argument (*t' Hooft criterion*): if symmetry of the Lagrangian increases by removing a term, then this term is *natural*.

We could forbid $\mu^2 H^\dagger H$ if $H$ was a Goldstone boson (because of the *shift symmetry*). However, we know that Higgs has mass (and there are other non-shift-symmetric terms in the Lagrangian, such as $y_t \overline{Q} H t_{\mathrm{R}}$). There is a way to make the Higgs boson a Goldstone boson (called composite Higgs) by breaking SO(5) to SO(4) (the custodial symmetry), which produces $n_{\mathrm{G}} = 10 - 6 = 4$ parameters — exactly the number of fields in the Higgs doublet.

Another symmetry prohibiting the Higgs mass term is *scale symmetry*: $H(x) \to e^{-\alpha} H(e^\alpha x)$. The term $(H^\dagger H)^2$ is scale invariant but the term $H^\dagger H$ is not. Sadly, adding the Higgs mass doesn't break the scale invariance, since the Standard model already has one other dimensionfull parameter, the Planck mass. And even if we ignore the large masses, there still are the $\beta$-functions which are not scale invariant. Scale symmetry is anomalous in general, which means that the *dilation current* coming from the scale symmetry is not conserved:

$$\partial_\mu j_{\mathrm{D}}^\mu = \underbrace{T^\mu{}_\mu}_{0} + \beta(g(\mu))(\dots).$$

Can we make the Higgs mass like a fermion mass? Yes, using supersymmetry:

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} T_\mu, \quad \delta\phi = \epsilon_\alpha \psi^\alpha$$

where $\epsilon$ and $\psi$ are Grassmannians and $\alpha$ and $\dot\beta$ are Weyl spin indices. The chiral symmetry protecting fermion masses can now also protect the Higgs mass term. We add a superpartner to every Standard model field. Since we have new particles, the running of couplings should be affected. However, the particles of the Standard model are in complete multiplets of SU(5) ($\overline{\mathbf{5}}$ and $\mathbf{10}$) and if we want to include the right-handed neutrino, we can work in SO(10) where all particles fit in $\mathbf{16}$ representation. Using this, adding the new particles (Higgsino, Wino, gluino) which are still charged under the same symmetries as their regular partners actually makes the grand unification work.

There are other issues with the Standard model, which we will just list here due to the lack of time.

- Only 5 % of the energy budget are in the Standard model particles, the rest is dark matter and dark energy.
- Inflation.
- Baryon asymmetry: there is more matter than antimatter in the universe.

# Index


**A**
accidental symmetry ..... 61
adjoint representation ..... 18
anti-unitary operator ..... 23
asymptotic freedom ..... 52
atmospheric neutrinos ..... 88
axial symmetry ..... 56

**B**
Baker-Campbell-Hausdorff formula ..... 17
baryon ..... 65
baryon symmetry ..... 56
Bianchi identity ..... 15, 28

**C**
Cabibbo angle ..... 80
Cabibbo suppression ..... 83
Cabibbo-Kobayashi-Maskawa matrix .... 78
Casimir operator ..... 7
charge conjugation ..... 21
chiral perturbation theory ..... 61
chiral symmetry ..... 13
Coleman-Mandula theorem ..... 9
Coleman-Weinberg potential ..... 51
colour factor ..... 54
compact Lie group ..... 17
compensating factor ..... 25
conjugate representation ..... 18
connection ..... 25
continuous spin representations ..... 8
covariant derivative ..... 16, 25

**D**
dilation current ..... 92
Dirac equation ..... 13
Dirac phase ..... 90
Dirac spinor ..... 10
domain walls ..... 34
double-well potential ..... 32

**E**
effective action ..... 38
effective potential ..... 38, 49
eightfold way ..... 64

**F**
Faddeev-Popov procedure ..... 29
Fermi constant ..... 68
Fermi theory ..... 68
field strength tensor ..... 14
Fierz identity ..... 54
fine-tuning ..... 92
flavour changing charged current ..... 79
flavour changing neutral current ..... 78
flux tube ..... 56
fundamental representation ..... 18

**G**
gauge hierarchy problem ..... 86
Gell-Mann matrices ..... 53
Gell-Mann–Oaks–Renner relation ..... 62
generators ..... 16
Georgi-Glashow model ..... 71
GIM mechanism ..... 82
glueball ..... 54
gluons ..... 53
Goldstone boson ..... 35
Goldstone theorem ..... 36
gradient expansion ..... 38
grand unified theory ..... 84
Grassmann variable ..... 13
GUT miracle ..... 85

**H**
hadrons ..... 54

helicity ..... 8

**I**
ideal ..... 17

**J**
Jacobi identity ..... 15, 17
Jarlskog invariant ..... 82

**K**
Killing form ..... 19
Kobayashi-Maskawa phase ..... 80
Kronecker delta ..... 12

**L**
ladder algebra ..... 8
lattice QCD ..... 54
Lie algebra ..... 16
Lie group ..... 16
linear sigma model ..... 35
little group ..... 8
Lorentz algebra ..... 10
Lorentz transformation ..... 5

**M**
Majorana mass ..... 14
Majorana mass term ..... 90
Majorana phases ..... 90
mass gap ..... 54
mass matrix ..... 36
maximal symmetry violation ..... 69
Maxwell action ..... 14
mesons ..... 55
metastable vacuum ..... 51
Minkowski metric ..... 5

**N**
neutrino oscillations ..... 91

**O**
octet ..... 57

**P**
parallel transport matrix ..... 26
parity transformation ..... 19
Pauli matrices ..... 10
Pauli-Lubanski vector ..... 7
pion decay constant ..... 60
PMNS matrix ..... 90
Poincaré symmetry ..... 5
proper orthochronous Lorentz group ..... 5

**Q**
quark condensate ..... 59
quarks ..... 53

**R**
representation ..... 18
$\rho$ parameter ..... 73

**S**
seesaw mechanism ..... 87, 90
semisimple Lie algebra ..... 17
simple Lie group ..... 17
singlet representation ..... 18
singular value decomposition ..... 77
solar neutrinos ..... 88
spinors ..... 10
spurion ..... 13, 61
stabilizer subgroup ..... 43
sterile neutrino ..... 90
structure constants ..... 16

**T**
t' Hooft criterion ..... 92
trivial representation ..... 18

**U**
unitary gauge ..... 47

**V**
$V - A$ interaction ..... 68
vacuum expectation value ..... 36
vacuum manifold ..... 43
vector meson ..... 64

**W**
Ward-Takahashi identity ..... 40
weak hypercharge ..... 73
weak interaction ..... 68
weak isospin ..... 73
weak mixing angle ..... 72
Weinberg angle ..... 72
Weinberg operator ..... 89
Weyl basis ..... 10
Weyl spinors ..... 10
Wolfenstein parametrization ..... 81

**Y**
Yukawa term ..... 76