

Faculty of physics

Particle dark matter

Notes of lectures by Prof. Alejandro Ibarra, Ph.D.

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Disclaimer: these notes are not official and were not endorsed by professor. The content is almost entirely based on the lectures, but occasionally the notes were modified or extended slightly to improve clarity or add extra information. In particular, the mistakes are almost surely mine. If you find any, please let me know at simon.bukovsek@tum.de or Simon.Bukovsek@campus.lmu.de.

Chapter 1

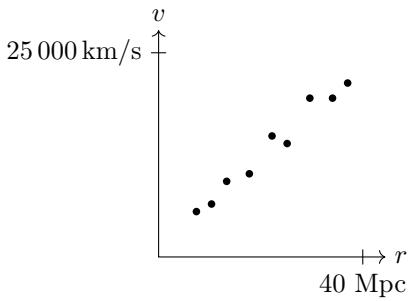
Evidence for dark matter

There are evidence on multiple scales:

- Galaxy clusters (scale of megaparsecs - Mpc),
- Galaxies (scale of 100 pc - 100 kpc),
- Cosmological scales (scale of gigaparsecs - Gpc),
- Maybe also our Solar system.

1.1 The virial theorem

The first cosmological evidence for dark matter came from the observation of galaxy clusters. Zwicky (1933) measured the velocity dispersion of galaxies in the Coma cluster.



Using the *virial theorem*, he estimated the total mass of the cluster. The second Newton's law for a galaxy i says

$$\mathbf{F}_i = m_i \frac{\mathrm{d}^2 \mathbf{r}_i}{\mathrm{d}t^2}.$$

Multiplying by $\mathbf{r}_i$ and summing over all galaxies, we get

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = \sum_i m_i \mathbf{r}_i \cdot \frac{\mathrm{d}^2 \mathbf{r}_i}{\mathrm{d}t^2} = \sum_i m_i \left(\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} |\mathbf{r}_i|^2 - \left| \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} \right|^2 \right).$$

In this expression we can identify several physical quantities:

$$\theta = \sum_i m_i |\mathbf{r}_i|^2, \qquad \text{moment of inertia;}$$

$$T_{\mathrm{T}} = \frac{1}{2}\sum_i m_i \left|\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right|^2, \qquad \text{total kinetic energy;}$$

$$\mathrm{Vir} = \sum_i \mathbf{r}_i \cdot \mathbf{F}_i \qquad \text{virial.}$$

So we can rewrite the previous equation as

$$\boxed{\mathrm{Vir} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\theta - 2T_{\mathrm{T}}.}$$

Another assumption Zwicky made is that the cluster is stationary so that the time average of the moment of inertia is constant. With this assumption, we get the virial theorem

$$\overline{\mathrm{Vir}} = -2\overline{T_{\mathrm{T}}}.$$

To get the value of the virial, we need to know the forces acting on each galaxy. Using internal gravitational forces only, we get

$$\mathbf{F}_i = -\sum_{j\neq i} G\frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}(\mathbf{r}_i - \mathbf{r}_j).$$

For a galaxy cluster with N galaxies, we have

$$\mathrm{Vir} = -G\sum_{i=1}^{N}\sum_{j\neq i}\frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}\mathbf{r}_i \cdot (\mathbf{r}_i - \mathbf{r}_j) = -\sum_{i=1}^{N}\sum_{j=1}^{i-1}\frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} = U,$$

where U is the total gravitational potential energy of the cluster. The expression for the potential energy can be independently calculated for a continuous mass distribution:

$$\mathrm{d}U = -\frac{GM(r)\,\mathrm{d}m}{r}$$

with $M(r)$ being the mass enclosed in a sphere of radius r. For $\rho(r) = \rho_0$ we get

$$-U = 4\pi \in_0^R G\frac{M(r)}{r}\rho_0 r^2\,\mathrm{d}r = 4\pi G\int_0^R \frac{4\pi}{3}\rho_0^2 r^4\,\mathrm{d}r = \frac{3}{5}\frac{GM^2}{R}.$$

In general, the potential energy is

$$U = -\alpha\frac{GM^2}{R},$$

where α depends on the distribution of matter. For example, if there are two equal point masses, we get $\alpha = 5/2$ and for three equal point masses, $\alpha = 10/3$. Zwicky assumed $\alpha < 5$. The other part of the virial theorem is the average of the total kinetic energy:

$$\overline{T_{\mathrm{T}}} = \frac{1}{2}\sum_{i=1}^{N} m_i \overline{v_i^2}.$$

Actually, we will consider the time average, averaged over all galaxies:

$$\underline{\overline{v^2}} = \sum_{i=1}^{N} \frac{m_i \overline{v_i^2}}{M}.$$

The overline means time average, the underline means average over all galaxies. Using the newly defined quantities, the virial theorem reads

$$M = \frac{R\underline{\overline{v^2}}}{G\alpha}.$$

If $\alpha < 5$ then we expect the mass of the cluster to be $M > R\underline{\overline{v^2}}/(5G)$. For experimental reasons, we should actually consider the projection of the velocity on the line of sight $v_\mathrm{s} = (\mathbf{v}\cdot\mathbf{e}_r) = v\cos\theta$. The average of the square of this quantity is

$$\underline{\overline{v_\mathrm{s}^2}} = \frac{\int_{-1}^{1} (\underline{\overline{\mathbf{v}}}\cdot\mathbf{e}_r)^2 \,\mathrm{d}\cos\theta}{\int_{-1}^{1} \mathrm{d}\cos\theta} = \frac{1}{3}\underline{\overline{v^2}}.$$

Using the quantities $\underline{\overline{v_\mathrm{s}^2}} \approx 5\cdot10^{15}\,\mathrm{cm}^2/\mathrm{s}^2$ and $R \approx 2\cdot10^6\,\mathrm{ly}$, we get the lower bound for the mass of the Coma cluster $M > 4.5\cdot10^{13}\,M_\odot$. Assuming there are around 1000 galaxies in the cluster, we get the average mass of a galaxy $M_\mathrm{gal} > 4.5\cdot10^{10}\,M_\odot$. This was in conflict with the calculations of the masses for nearby galaxies, which were around $M_\mathrm{gal} \approx 8\cdot10^7\,M_\odot$. This would indicate that the galaxies in the cluster are much more massive than nearby galaxies. Zwicky concluded that there must be some invisible matter in the cluster. However, this was very controversial at the time, as the derivation used many assumptions and simplifications (steady state, only internal gravitational forces, spherical symmetry, etc.). Today we know, that the mass of the Coma cluster is around $M \approx 1\cdot10^{15}\,M_\odot$ and more recent X-ray measurements confirmed that the most of the cluster's mass is in the form of hot gas ($3\cdot10^{14}\,M_\odot$) and the galaxies ($1\cdot10^{13}\,M_\odot$). However, the total mass is still much larger than the mass of visible matter, for a factor of 10.

1.2 Weak gravitational lensing

In order to discuss the weak *gravitational lensing*, we first need to understand the metric tensor in the presence of a weak potential. In the Minkowski space-time, the line element is

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$$

and in general Riemannian space-time, the line element is

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$$

and the *Einstein's field equations* are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}.$$

For weak gravitational fields, we can write the metric tensor as a small perturbation around the Minkowski metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1.$$

We could insert this back in the Einstein's field equations, and we would get a complicated expression. However, we can simplify them by using the trace reverse of the metric tensor

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\sigma}{}_{\sigma}.$$

The Einstein's field equations in the linearized form are

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \quad \partial_\mu\bar{h}^{\mu\nu} = 0.$$

These equations are very familiar from the electrodynamics. The solution is given by

$$\bar{h}^{\mu\nu}(ct,\mathbf{x}) = -\frac{4G}{c^4}\int \mathrm{d}^3\mathbf{y}\,\frac{T^{\mu\nu}(ct_\mathrm{r},\mathbf{y})}{|\mathbf{y}-\mathbf{x}|}, \quad ct_\mathrm{r} = ct - |\mathbf{x}-\mathbf{y}|.$$

Quantity ct_r is the retarded time, which takes into account the fact that changes in the energy-momentum tensor propagate with the speed of light. To further simplify the expression, we will assume that the source is stationary, so that $T^{\mu\nu}$ does not depend on time. In this case, we can write the tensor as

$$T^{\mu\nu}(\mathbf{x}) = \rho(\mathbf{x})u^\mu(\mathbf{x})u^\nu(\mathbf{x}).$$

For the stationary source, we have $u^\mu = (c,\mathbf{0})$. With this, the trace-reversed metric perturbation becomes

$$h^{00}(\mathbf{x}) = -\frac{4G}{c^2}\int \mathrm{d}^3\mathbf{y}\,\frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} = \frac{4\Phi(\mathbf{x})}{c^2}.$$

For any other combinations of indices, we get $h^{0i} = h^{i0} = h^{ij} = 0$. From this, we can obtain the non-trace-reversed metric perturbation:

$$h^{00} = \bar{h}^{00} - \frac{1}{2}\eta^{00}\bar{h} = \frac{2\Phi}{c^2}, \quad h^{ii} = \bar{h}^{ii} - \frac{1}{2}\eta^{ii}\bar{h} = -\frac{2\Phi}{c^2}.$$

The line element in the presence of a distribution of stationary non-relativistic dust is

$$\mathrm{d}s^2 = c^2\left(1 + \frac{2\Phi(\mathbf{x})}{c^2}\right)\mathrm{d}t^2 - \left(1 - \frac{2\Phi(\mathbf{x})}{c^2}\right)\mathrm{d}\mathbf{x}^2.$$

Setting $\mathrm{d}s^2 = 0$ for light, we get

$$\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right| \approx c\left(1 + \frac{2\Phi(\mathbf{x})}{c^2}\right) = c' < c.$$

Hence, we can pretend that the vacuum actually has some optical thickness, with the index of refraction $n = c/c'$. The light travels along the path that minimizes the travel time

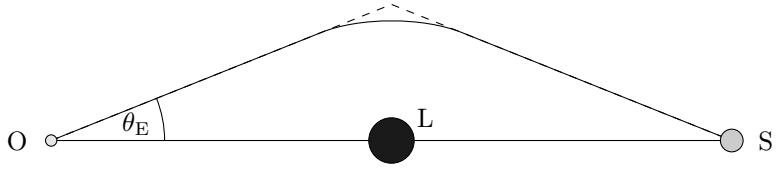
$$T = \int_A^B \mathrm{d}t = \int_A^B \frac{\mathrm{d}\ell}{c'(\mathbf{x}(\boldsymbol{\ell}))} = \frac{1}{c}\int_A^B n(\mathbf{x}(\ell))\,\mathrm{d}\ell = \frac{1}{c}\int_{\lambda_A}^{\lambda_B} \underbrace{n(\mathbf{x}(\lambda))\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\lambda}\right|}_{L}\,\mathrm{d}\lambda.$$

Varying this expression ($\delta T = 0$) and using the Euler-Lagrange equations (interpreting the integrand as a Lagrangian $L = n(\mathbf{x})|\dot{\mathbf{x}}|$)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0$$

we get the equation for the trajectory of light in a weak gravitational field:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(n(\mathbf{x})\frac{\dot{x}^i}{|\dot{\mathbf{x}}|}\right) = \frac{\partial n(\mathbf{x})}{\partial x^i}|\dot{\mathbf{x}}|.$$



However, this expression can be simplified using the normal coordinates, where $|\dot{\mathbf{x}}| = 1$ and $e^i = \dot{x}^i$ is the tangent vector to the trajectory:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(ne^i) = \frac{\partial n}{\partial x^i} = \dot{n}e^i + n\dot{e}^i = \frac{\partial n}{\partial x^j}e^j e^i + ne^i.$$

Moreover, the gradient of the index of refraction is

$$\nabla n = n\dot{\mathbf{e}} + (\nabla n \cdot \mathbf{e})\mathbf{e}.$$

From this, we see that

$$n\dot{\mathbf{e}} = \nabla n - (\nabla n \cdot \mathbf{e})\mathbf{e} = \nabla_\perp n$$

is the component of the gradient perpendicular to the trajectory. This can be simplified even further:

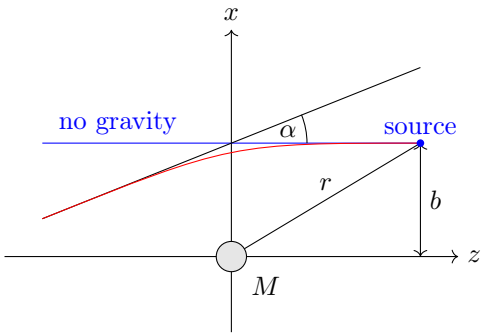
$$\dot{\mathbf{e}} = \frac{1}{n}\nabla_\perp n = \nabla_\perp \log n \approx -\frac{2}{c^2}\nabla_\perp \Phi.$$

The angle of deflection α is given by the integral of the change of the tangent vector between the source $\mathbf{e}_\mathrm{source}$ and the observer $\mathbf{e}_\mathrm{obs}$:

$$\alpha = \int_A^B \mathrm{d}\mathbf{e} = -\frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \nabla_\perp \Phi(\mathbf{x}(\lambda))\,\mathrm{d}\lambda.$$

In principle, this equation is useless, because we do not know the trajectory of light $\mathbf{x}(\lambda)$. However, if the deflection is small, we can solve the problem perturbatively. In the zeroth order, it is called the Born approximation, where we assume that the light travels in a straight line. We will assume that the light travels along the z-axis, so that $\mathbf{x}(\lambda) = (0, 0, \lambda)$ and $\mathrm{d}\lambda = \mathrm{d}z$. The angle of deflection becomes

$$\alpha = \frac{2}{c^2}\int_{-\infty}^{\infty} \nabla_\perp \Phi(0, 0, \lambda)\,\mathrm{d}\lambda$$



In an easy example of a deflection of a point source by a point mass, we have a point mass M at the origin and a source at $(b, 0, -\infty)$. The straight line trajectory of light is $\mathbf{x}(\lambda) = (b, 0, \lambda)$ and the distance of the photon to the source is $r = \sqrt{b^2 + \lambda^2}$. The gravitational potential is

$$\Phi = -\frac{GM}{r} = -\frac{GM}{\sqrt{x^2 + y^2 + z^2}}, \quad \nabla_\perp \Phi = \frac{GM}{r^3} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Finally, we get the angle of deflection

$$\alpha = \frac{2}{c^2} \int_{-\infty}^{\infty} \nabla_\perp \Phi(b, 0, \lambda)\, \mathrm{d}\lambda = \frac{2}{c^2} \int_{-\infty}^{\infty} \frac{GM}{(b^2 + \lambda^2)^{3/2}} \begin{pmatrix} b \\ 0 \end{pmatrix} \mathrm{d}\lambda = \frac{4GM}{bc^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

1.3 Rotational curves

The first evidence for dark matter on the scale of galaxies came from the observation of rotational curves. In 1939, Babcock measured the rotational curve of the Andromeda Galaxy (M31). He already proposed that there should be some invisible matter in the outer parts of the galaxy, but his results were largely ignored. In the 1970s, Vera Rubin and collaborators measured the rotational curves of many spiral galaxies and confirmed that the rotational curves remain flat at large distances from the galactic centre. This contradicts the assumption that most of the mass is concentrated in the centre of the galaxy, where most of the visible matter is. If the latter is the case, we would expect the rotational curves to fall off as $v(r) \propto 1/\sqrt{r}$ at large distances. However, the measurements show that the rotational curves remain flat, $v(r) \approx \text{const}$. This indicates that the matter distribution goes as $M(r) \propto r$ and $\rho(r) \propto r^{-2}$.

There is also another explanation for the flat rotational curves, which does not require dark matter. It is called *MOdified Newtonian Dynamics* (MOND) and was proposed by Milgrom in 1983. The idea is that Newton's second law is modified for very small accelerations $a < a_0$, where $a_0 \approx 10^{-10}\,\mathrm{m/s^2}$. The second Newton's law is modified as

$$\mathbf{F} = m\mathbf{a}\mu(a/a_0)$$

where the function $\mu(x)$ has the properties

$$\mu(x) \approx \begin{cases} 1 & x \gg 1 \\ x & x \ll 1 \end{cases}.$$

A possible example of such a function is $\mu(x) = x/(x+1)$. The force equilibrium for a star orbiting the galaxy at a distance r is

$$\frac{GMm}{r^2} = m\frac{a^2}{a_0} = \frac{m}{a_0}\frac{v^4}{r^2}, \quad v(r) \approx (GMa_0)^{1/4}.$$

This gives a constant rotational velocity at large distances. However, this theory has problems explaining other evidence for dark matter, such as the Bullet cluster or the Cosmic Microwave Background (CMB).

From the power spectrum of the CMB, we know that the composition of the universe is around 5% baryonic matter, 25% dark matter and 70% dark energy.

What do we know about dark matter particles so far?

- It has no electric charge. If it has positive charge, it would form a bound state with electrons X^+e^-, an anomalous heavy hydrogen atom. As a consequence we would be able to detect anomalously heavy water HXO. People have been searching for such water, and haven't found any. If it has negative charge, it would bind with nuclei, forming anomalous heavy isotopes. If there is a charged dark matter component, the process $e^+e^- \to X^+X^-$ should be possible at colliders. No such events have been observed, so the mass of such particles should be $m_X > 100\,\mathrm{GeV}$ (which is also unlikely by the absence of anomalously heavy water). A clever way out of this problem is to assume that dark matter has *micro-charge*.
- It is not made of baryons (or particles that behave like baryons). This can be concluded from the relative abundances of light elements (H, He, Li) formed during the Big Bang Nucleosynthesis (BBN). This is the conclusion from the cosmologist's point of view. From the particle physicist's point of view, we know that baryonic number is globally conserved $\mathrm{U}(1)_\mathrm{B}$ symmetry, where proton transforms as $p \to e^{i\alpha}p$, quarks as $q \to e^{i\alpha/3}q$ and leptons as $\ell \to \ell$.
- It was slow at the time of formation of the first structures (galaxies, galaxy clusters). If the dark matter was hot (relativistic) at that time, the simulations show much more clustered universe than we observe. This also excludes neutrinos as dark matter candidates, because their masses are too small to be non-relativistic at that time. There is also a middle notion of warm dark matter.
- It exists today. If dark matter particles were unstable, they should have a lifetime longer than the age of the universe. This implies a certain symmetry that stabilizes dark matter particles. For example, neutrinos are stable because of lepton number conservation and also the Lorentz symmetry (spin conservation, $\nu \to \gamma\gamma$ lifetime is estimated to be more than 10^{21} years) and electrons don't decay like $e^- \to \gamma\nu$ due to $\mathrm{U}(1)_\mathrm{em}$ symmetry (the decay rate is estimated to be more than 10^{26} years). Protons are also very stable, with estimated lifetime more than 10^{34} years, because of baryon number conservation $\mathrm{U}(1)_\mathrm{B}$. The proposed symmetries that stabilize dark matter are $\mathrm{U}(1)_\mathrm{dark}$ or $\mathbb{Z}_2$ which transforms dark matter particles as $X \to -X$. There is also a possibility that dark matter decays into another dark sector particles, $X \to \Psi\Psi$. However, if the mass of Ψ is significantly smaller than half of the mass of X, we would be able to observe the change in the dark matter density parameter.

Evidently, none of the Standard Model particles satisfy all these conditions. This is the main motivation for physics beyond the Standard Model.

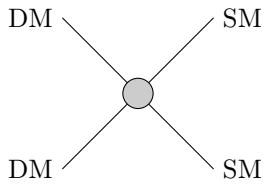
Chapter 2

Dark matter particles as relics from the early universe

2.1 WIMP history in a nutshell

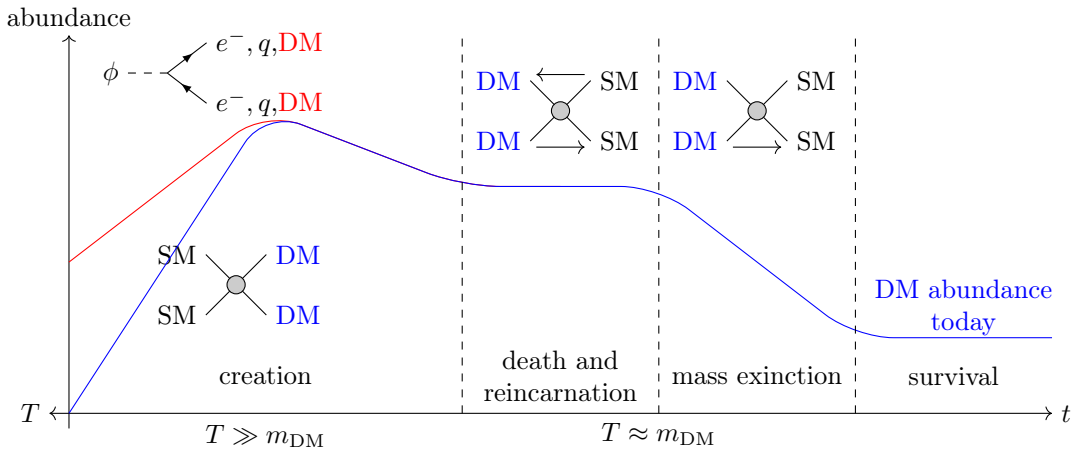
WIMPs are weakly interacting massive particles. We will be working with the following assumptions.

- They are stable on cosmological time-scales.
- They interact in pairs with the Standard Model particles



- The WIMP interaction strength is strong enough to keep the DM in chemical equilibrium with the SM particles.
- The WIMP interaction strength is weak enough to allow the dark matter to chemically decouple from the SM particles at some point in the early universe.

A brief history of the dark matter can be seen in the figure below. First, the universe was so hot that the dark matter particles were being created from other Standard model particles. It is also assumed that the inflaton field ϕ decayed into both SM and DM particles after the inflation. When the temperature dropped but was still larger than the mass of the dark particle, the density of dark matter particles remained roughly constant, as the creation and annihilation processes were in equilibrium. After the temperature dropped below the mass of the dark matter particle, the creation processes were suppressed, while the annihilation processes were still active. This led to a rapid decrease in the dark matter abundance. However, the dark matter particles probably didn't annihilate completely, as the universe kept expanding and the density of dark matter particles dropped (and with this also the annihilation rate). The dark matter is still being diluted today, but only due to the expansion of the universe.



2.2 Express review of the cosmology

At large scales, the universe is homogeneous and isotropic. The Friedmann-Lemaître-Robertson-Walker (FLRW) metric describes such a universe:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right],$$

where t is the *cosmic time* (a proper time for the fundamental observer), r, ϕ, θ are *comoving coordinates* (fixed for the fundamental observer), k is the *curvature parameter*

$$k = \begin{cases} +1 & \text{spherical geometry, closed universe,} \\ 0 & \text{flat geometry,} \\ -1 & \text{hyperbolic geometry, open universe} \end{cases}$$

and $R(t)$ is the *scale factor* (describes the expansion of the universe). We are interested in the solutions of the *Einstein's field equation* for the FLRW metric:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}.$$

The energy-momentum tensor $T_{\mu\nu}$ represents a perfect fluid

$$T_{\mu\nu}(x) = (\rho(x) + p(x))u_\mu(x)u_\nu(x) - p(x)g_{\mu\nu}(x).$$

The homogeneity and isotropy of the universe imply that $\rho(x) = \rho(t)$ and $p(x) = p(t)$. By construction of synchronous comoving coordinates, the fluid is at rest, so that $u^\mu = (1, 0, 0, 0)$. The simplified energy-momentum tensor is

$$T_{\mu\nu} = (\rho(t) + p(t))\delta^0{}_\mu\delta^0{}_\nu - p(t)g_{\mu\nu}(x).$$

Furthermore, we assume an elementary equation of state $p = w\rho$, where

$$w = \begin{cases} 1/3 & \text{radiation,} \\ 0 & \text{matter,} \\ -1 & \text{dark energy.} \end{cases}$$

The energy-momentum tensor is conserved:

$$\nabla_\mu T^{\mu\nu} = 0$$

which gives the fluid equation

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{R}}{R}.$$

Often times, it is useful to define the dimensionless scale factor $a(t) = R(t)/R_0$ where R_0 is the scale factor today. The solution of the fluid equation is

$$\rho \propto a^{-3(1+w)} = \begin{cases} a^{-4} & \text{radiation,} \\ a^{-3} & \text{matter,} \\ \text{const} & \text{dark energy.} \end{cases}$$

Now, we are ready to write down the Einstein's field equations for the FLRW metric. The time-time component gives the second *Friedmann equation*

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p)$$

and the space-space component gives

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} = 4\pi G(\rho - p) + \Lambda.$$

From these two equations, we can derive the first Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}.$$

To simplify the notation, we define the *Hubble parameter*

$$H(t) = \frac{\dot{a}(t)}{a(t)},$$

the *critical density* (the density required for a flat universe)

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}$$

and the density parameters

$$\Omega_i(t) = \frac{\rho_i(t)}{\rho_c(t)}, \quad \Omega_k(t) = -\frac{k^2}{H^2(t)a^2(t)}.$$

With these definitions, the first Friedmann equation becomes

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1.$$

Usually, we denote $\Omega = \Omega_m + \Omega_r + \Omega_\Lambda$ and its sign determines the geometry of the universe:

$$\Omega = 1 - \Omega_k = \begin{cases} > 1 & \text{spherical geometry, closed universe,} \\ = 1 & \text{flat geometry,} \\ < 1 & \text{hyperbolic geometry, open universe.} \end{cases}$$

Using $H_0 \approx 70\,\mathrm{km\,s^{-1}\,kpc^{-1}}$, we get the critical density today $\rho_{\mathrm{c},0} = 9.2 \cdot 10^{-27}\,\mathrm{kg\,m^{-3}} = 5.5\,m_{\mathrm{prot}}/\mathrm{m}^3$. There is no general analytical solution for the scale factor $a(t)$, but they exist for some special cases. For the matter dominated flat universe ($k = 0$, $w = 0$), we have

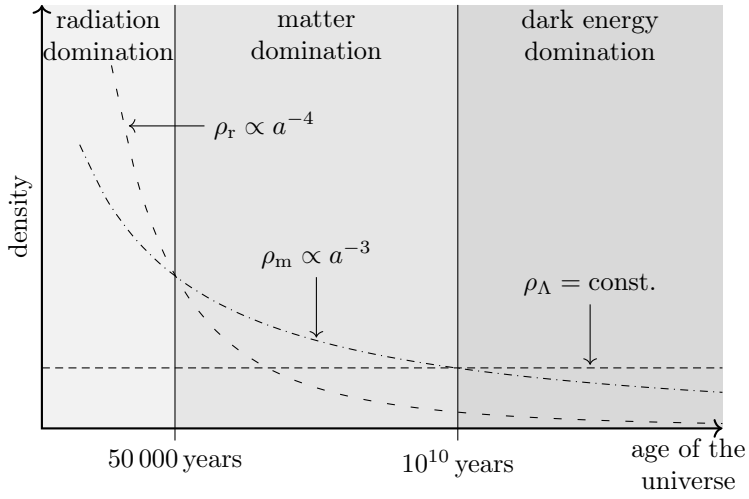
$$a(t) = (3H_0 t/2)^{2/3}$$

and for the radiation dominated flat universe ($k = 0$, $w = 1/3$), we have

$$a(t) = (2H_0 t)^{1/2}.$$

If there is only dark energy ($w = -1$), we have an exponential expansion

$$a(t) = e^{H_0 t}.$$



2.3 Thermodynamics of the early universe

Consider a gas of particles with mass m and g internal degrees of freedom in thermal equilibrium at temperature T and phase distribution $f(\mathbf{k})$, where $k^0 = \sqrt{m^2 + |\mathbf{k}|^2}$ and

$$f(\mathbf{k}) = \left(\exp\left(\frac{E(\mathbf{k}) - \mu}{k_{\mathrm{B}} T}\right) \pm 1\right)^{-1}.$$

The symbol μ represents the chemical potential. For the $+$ sign, it is called the *Fermi-Dirac distribution* (fermions) and for the $-$ sign, it is called the *Bose-Einstein distribution* (bosons). In the following, we will set $k_{\mathrm{B}} = 1$. The number density of particles is

$$n = g \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} f(\mathbf{k}) = \frac{g}{2\pi^2} \int_m^\infty \mathrm{d}E\, \frac{E(E^2 - m^2)^{1/2}}{\exp((E-\mu)/T) \pm 1},$$

the energy density is

$$\rho = g \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} E(\mathbf{k}) f(\mathbf{k}) = \frac{g}{2\pi^2} \int_m^\infty \mathrm{d}E\, \frac{E^2(E^2 - m^2)^{1/2}}{\exp((E-\mu)/T) \pm 1}$$

and the pressure is

$$p = g \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{|\mathbf{k}|^2}{3E(\mathbf{k})} f(\mathbf{k}) = \frac{g}{6\pi^2} \int_m^\infty \mathrm{d}E \, \frac{(E^2 - m^2)^{3/2}}{\exp((E-\mu)/T) \pm 1}.$$

To derive the last equation, we consider a box of size L and a particle with momentum $\mathbf{k}$ bouncing elastically between two walls. If it hits the wall at the angle θ, the force exerted on the wall is

$$F = \frac{2|\mathbf{k}| \cos\theta}{\Delta t}$$

where

$$\Delta t = \frac{2d}{v} = \frac{2L}{v \cos\theta}$$

is the average time between two hits and $v = |\mathbf{k}|/E$ is the velocity of the particle. The pressure is the force per unit area:

$$p = \frac{2|\mathbf{k}|^2 v \, \langle \cos^2\theta \rangle}{2LS} = \frac{|\mathbf{k}|^2}{EV} \frac{\int_{-1}^1 \cos^2\theta \, \mathrm{d}(\cos\theta)}{\int_{-1}^1 \mathrm{d}(\cos\theta)} = \frac{|\mathbf{k}|^2}{3EV}.$$

The integrals for n, ρ and p are very ugly, but they can be solved exactly in some limits. In the relativistic limit ($T \gg m$) and for $\mu = 0$, we have

$$n = \begin{cases} \frac{\zeta(3)}{\pi^2} g T^3 & \text{bosons,} \\ \frac{3\zeta(3)}{4\pi^2} g T^3 & \text{fermions,} \end{cases}$$

and

$$\rho = \begin{cases} \frac{\pi^2}{30} g T^4 & \text{bosons,} \\ \frac{7\pi^2}{240} g T^4 & \text{fermions,} \end{cases} \qquad p = \frac{\rho}{3}.$$

In calculating these integrals, we use the following very useful relations:

$$\int_0^\infty \frac{x^{n-1}\,\mathrm{d}x}{e^x - 1} = \Gamma(n)\zeta(n), \quad \int_0^\infty \frac{x^{n-1}\,\mathrm{d}x}{e^x + 1} = (1 - 2^{1-n})\Gamma(n)\zeta(n)$$

where $\Gamma(n)$ is the Gamma function and $\zeta(n)$ is the Riemann zeta function. For whole numbers, we have $\Gamma(n) = (n-1)!$ and $\zeta(3) \approx 1.202$, $\zeta(4) = \pi^4/90$. In the non-relativistic limit ($T \ll m$) the distributions reduce to

$$f(k) = \exp\left(-\frac{E-\mu}{T}\right), \quad E = m + \frac{|\mathbf{k}|^2}{2m}.$$

We get

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m-\mu}{T}\right), \quad \rho = mn + \frac{3}{2}nT, \quad p = nT.$$

The total energy density of the plasma is dominated by relativistic particles, so that

$$\rho = \sum_{\text{bosons}} \rho_i + \sum_{\text{fermions}} \rho_i = \sum_{\text{b}} \frac{\pi^2}{30} g_i T_i^4 + \sum_{\text{f}} \frac{\pi^2}{30}\frac{7}{8} g_i T_i^4 = \frac{\pi^2}{30} g_* T^4.$$

We introduced the effective number of degrees of freedom

$$g_* = \sum_{\mathrm{b}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{\mathrm{f}} g_i \left(\frac{T_i}{T}\right)^4.$$

At high enough temperatures ($T \gg 300\,\mathrm{GeV}$), all species are in the thermal equilibrium, so they all have the same temperature $T_i = T$. Let's count the effective degrees of freedom at that temperature:

- quarks: 6 quarks, each with 4 components (Dirac spinor) and 3 colours, in total 72 degrees of freedom,
- charged leptons: 3 leptons, each with 4 components (Dirac spinor), in total 12 degrees of freedom,
- neutrinos: 3 neutrinos, each with 2 components (Weyl spinor), in total 6 degrees of freedom,
- gluons: 8 gluons, each with 2 polarizations, in total 16 degrees of freedom,
- electroweak gauge bosons (before electroweak symmetry breaking): $W^{1,2,3}, B$, each with 2 polarizations, in total 8 degrees of freedom,
- Higgs boson: 1 complex scalar doublet, in total 4 degrees of freedom.

So we have 90 fermionic degrees of freedom and 28 bosonic degrees of freedom. The effective number of degrees of freedom at high temperatures is

$$g_* = \underbrace{28}_{\text{bosons}} + \underbrace{\frac{7}{8}\cdot 90}_{\text{fermions}} = 106.75.$$

The number g_* changes dramatically with decreasing temperature because some particles become non-relativistic and annihilate and at 200 MeV the QCD phase transition occurs, where quarks and gluons become confined into hadrons. At very low temperatures ($T \ll 1\,\mathrm{MeV}$), the only relativistic particles left are photons and neutrinos, so that

$$g_* = 2\left(\frac{T_\gamma}{T}\right)^4 + 6\frac{7}{8}\left(\frac{T_\nu}{T}\right)^4 = 2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3} \approx 3.36.$$

We have assumed that $T = T_\gamma$ and

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}.$$

The latter relation comes from the fact that neutrinos decouple before electron-positron annihilation, which heats up the photons but not the neutrinos (see homework).

Now, we can combine the first Friedmann equation with the total energy density of the plasma to get the Hubble parameter in the radiation dominated era:

$$H^2 = \frac{8\pi G}{3}\rho = \frac{4\pi^3 G}{45} g_* T^4.$$

Plugging in the numbers, we get

$$H(T) \approx 1.66 g_*(T)^{1/2} \frac{T^2}{M_{\mathrm{Pl}}}.$$

In the radiation dominated era, the scale factor grows as $a(t) \sim t^{1/2}$ and hence $H = 1/(2t)$

$$t \approx 0.3 g_*(T)^{-1/2} \frac{M_{\rm Pl}}{T^2} \sim \left(\frac{T}{\rm MeV}\right)^{-2} \text{s}.$$

An important aspect of the early universe thermodynamics is the conservation of entropy. It's value in the equilibrium is given by

$$T\mathrm{d}S = \mathrm{d}U + p\mathrm{d}V = V\mathrm{d}\rho + (p+\rho)\mathrm{d}V = V\frac{\mathrm{d}\rho}{\mathrm{d}T}\mathrm{d}T + (p+\rho)\mathrm{d}V.$$

The last expression can be seen as only a function of T and V. The two prefactors before $\mathrm{d}T$ and $\mathrm{d}V$ are actually related by a *Maxwell relation*. The latter states that for any function $f(x, y)$, we have

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)_y = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)_x.$$

Applying this to $\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V$, we get

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_p.$$

The differential of the entropy is

$$\mathrm{d}S(T, V) = \frac{(\mathrm{d}\rho/\mathrm{d}T)V}{T}\mathrm{d}T + \frac{p+\rho}{T}\mathrm{d}V.$$

The two possible second derivatives are

$$\frac{\partial}{\partial V}\left(\frac{\partial S}{\partial T}\right)_V = \frac{\mathrm{d}\rho/\mathrm{d}t}{T}, \quad \frac{\partial}{\partial T}\left(\frac{\partial S}{\partial V}\right)_T = \frac{\mathrm{d}(p+\rho)/\mathrm{d}T}{T} - \frac{p+\rho}{T^2}.$$

The Schwarz theorem states that these two derivatives are equal, which gives the relation

$$\frac{\mathrm{d}p}{\mathrm{d}T} = \frac{p+\rho}{T}.$$

We can also write the differential of entropy as

$$\mathrm{d}S = \frac{1}{T}\mathrm{d}(\rho V) + \frac{p}{T}\mathrm{d}V = \frac{1}{T}\mathrm{d}((\rho+p)V) - \frac{V}{T^2}(\rho+p)\mathrm{d}T = \mathrm{d}\left(\frac{\rho+p}{T}V\right).$$

Integrating this expression gives

$$S = \frac{\rho+p}{T}V + \text{const.}$$

The only assumptions for this derivation were thermal equilibrium and the first law of thermodynamics. The energy-momentum conservation is

$$\dot{\rho} = -3(p+\rho)\frac{\dot{a}}{a} = -(p+\rho)\frac{\dot{V}}{V}.$$

This implies

$$\mathrm{d}((\rho+p)V) = V\mathrm{d}p = \frac{\rho+p}{T}V\mathrm{d}T, \quad \mathrm{d}\left(\frac{p+\rho}{T}V\right) = 0.$$

This means that the total entropy is conserved. In the following it will be very useful to operate with the *entropy density*

$$s = \frac{S}{V} = \frac{\rho + p}{T}.$$

It is dominated by the relativistic degrees of freedom ($\rho = p/3$):

$$s = \frac{2\pi^2}{45} g_{*S}(T) T^3,$$

where g_{*S} is the effective number of degrees of freedom for entropy

$$g_{*S}(T) = \sum_{\mathrm{b}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\mathrm{f}} g_i \left(\frac{T_i}{T}\right)^3.$$

2.4 The WIMP miracle

Using the knowledge of the last two sections, we would like to estimate the relic abundance of WIMPs. When the reactions DM+DM $\rightleftharpoons$ SM+SM were in equilibrium ($\Gamma \gg H$), the number density of WIMPs followed the equilibrium distribution n_{eq}. When the temperature dropped below the mass of the WIMP ($T < m_{\mathrm{DM}}$), the number density dropped by the annihilation of WIMPs into SM particles. However, at some point the annihilation rate became smaller than the Hubble expansion rate ($\Gamma < H$) and the WIMPs chemically decoupled from the SM particles. Recall the relation between the Hubble rate and the temperature in the radiation dominated era

$$H \simeq 1.66 g_*(T)^{1/2} \frac{T^2}{M_{\mathrm{Pl}}}$$

and the expression for the reaction rate is $\Gamma = \sigma v n$. Considering only the *cold dark matter* (CDM) case, we have

$$n_{\mathrm{eq}} = g \left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m}{T}\right)$$

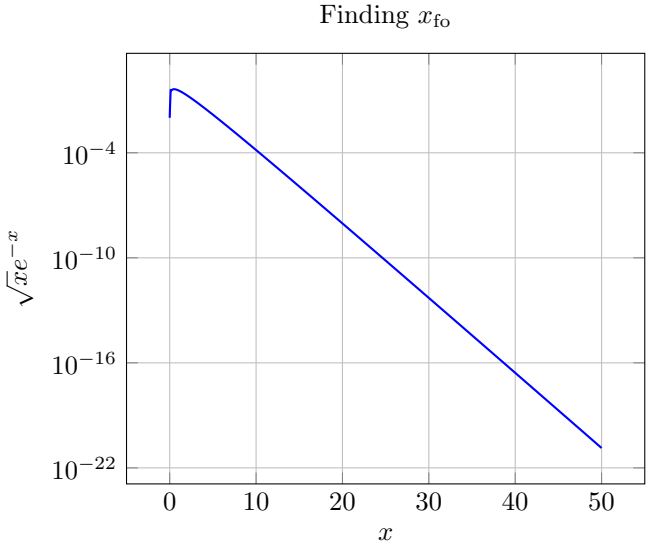
(we ignored the chemical potential). The time of freeze-out corresponds to $H \equiv H$:

$$\sigma v \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} = 10^{-66} g_*(T)^{1/2} \frac{T^2}{M_{\mathrm{Pl}}}.$$

In order to extract as much sense from this equation analytically, we define $x_{\mathrm{fo}} = m/T$ and assume that $g_*(T)$ is roughly constant during the freeze-out. The freeze-out condition becomes

$$\sqrt{x_{\mathrm{fo}}} e^{-x_{\mathrm{fo}}} = 1.66 g_*^{1/2} \frac{1}{\sigma v m M_{\mathrm{Pl}}}.$$

If we insert $\sigma v \sim G_{\mathrm{F}}^2 m^2$ and $m = 100\,\mathrm{GeV}$, we get $x_{\mathrm{fo}} \approx 34$. For $m = 1\,\mathrm{GeV}$.



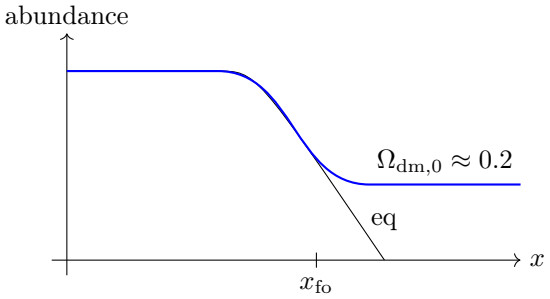
A sensible range for the right-hand side is 10^{-8} to 10^{-18}. As we can see from the graph, the corresponding range of x_{fo} is between 20 and 40. There seems to be another solution at very small values of x, but this is the ultra-relativistic regime, where these equations don't hold any more. The dark matter density parameter today is

$$\Omega_{\mathrm{dm},0} = \frac{\rho_{\mathrm{dm},0}}{\rho_{\mathrm{c}}} = \frac{m}{\rho_{\mathrm{c}}} n_{\mathrm{dm,fo}} \left(\frac{T_0}{T_{\mathrm{fo}}}\right)^3 = \frac{1.66\, x_{\mathrm{fo}}}{\sigma v} g_*^{1/2} \frac{T_0^3}{\rho_{\mathrm{c}} M_{\mathrm{Pl}}}.$$

The values of T_0, M_{Pl} and ρ_{c} are known and g_* can be approximated as ~ 100. The dark matter density parameter is thus

$$\Omega_{\mathrm{dm},0} \approx 0.2 \cdot \frac{x_{\mathrm{fo}}}{20} \cdot \frac{3 \cdot 10^{-25}\,\mathrm{cm^3\,s^{-1}}}{\sigma v}.$$

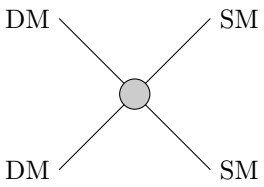
Assume that there is a stable particle, that reached thermal equilibrium and that it interacts with the Standard Model as $\mathrm{DM} + \mathrm{DM} \rightleftharpoons \mathrm{SM} + \mathrm{SM}$.



We can notice that

$$\sigma v \sim 3 \cdot 10^{-26}\,\mathrm{cm^3\,s^{-1}} = 1\,\mathrm{pb} \cdot 3 \cdot 10^{10}\,\mathrm{cm\,s^{-1}}.$$

If the interaction



had a cross-section of the order of a picobarn, it would give roughly the correct relic abundance today. This is called the *WIMP miracle*, because this is the approximate cross-section of the weak interactions. This is one of the main motivations for WIMPs as dark matter candidates.

2.5 Boltzmann's equation

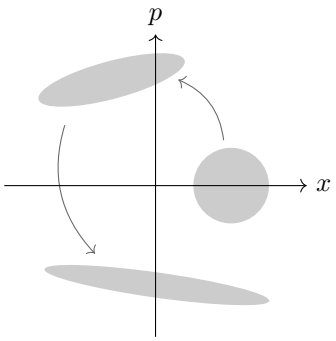
The *Boltzmann equation* describes the time evolution of the phase space distribution $f(\mathbf{x}, \mathbf{p}, t)$ of a particle species:

$$\hat{L}[f] = \hat{C}[f]$$

where $\hat{L}$ is the *Liouville operator* (reallocation of particles in the phase space) and $\hat{c}$ is the *collision operator* (creation/destruction of particles). In the non-relativistic limit, the Liouville operator is

$$\hat{L}[f] = \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} + \sum_i \frac{\partial f}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} = \frac{\partial f}{\partial t} + (\nabla_x f)\cdot\mathbf{x} + (\nabla_p f)\cdot\frac{\mathbf{F}}{m}.$$

If the collision operator is zero, the area of the phase space occupied by a set of particles is conserved (*Liouville's theorem*).



In the relativistic case, the distribution is a function of the four-momentum $p^\mu(\tau)$ and the four-position $x^\mu(\tau)$. The Liouville operator can be written in a covariant form:

$$\hat{L}[f] = \frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\partial f}{\partial x^\mu}\frac{\partial x^\mu}{\partial \tau} + \frac{\partial f}{\partial p^\mu}\frac{\partial p^\mu}{\partial \tau} = p^\mu\frac{\partial f}{\partial x^\mu} - \Gamma^\mu{}_{\alpha\beta}p^\alpha p^\beta\frac{\partial f}{\partial p^\mu}.$$

We used the geodesic equation and $p^\mu = \partial x^\mu/\partial\tau$ (ignoring the external forces). In the homogeneous and isotropic universe, the distribution function depends only on the magnitude of the momentum $|\mathbf{p}|$ and on time t (or equivalently on time and energy E). The Liouville operator reduces to

$$\hat{L}[f] = E\frac{\partial f}{\partial t} - \Gamma^0{}_{\rho\sigma}p^\mu p^\sigma\frac{\partial f}{\partial E}.$$

In FLRW metric, the only non-zero Christoffel symbols with a time index are

$$\Gamma^0{}_{rr} = \frac{a\dot{a}}{1-kr^2}, \quad \Gamma^0{}_{\theta\theta} = a\dot{a}r^2, \quad \Gamma^0{}_{\varphi\varphi} = a\dot{a}r^2\sin^2\theta$$

or in short $\Gamma^0{}_{ij} = Hg_{ij}$. Using this, we get

$$\Gamma^0{}_{ij}p^ip^j = -\frac{\dot{a}}{a}g_{ij}p^ip^j = H|\mathbf{p}|^2.$$

Hence, the Boltzmann equation in the FLRW metric is

$$\boxed{E\frac{\partial f}{\partial t} - H|\mathbf{p}|^2\frac{\partial f}{\partial E} = \hat{C}[f].}$$

The equation is still very difficult to solve, so we will first try to apply it to the number density n. We take the Boltzmann equation

$$\frac{\partial f}{\partial t} - H\frac{|\mathbf{p}|^2}{E}\frac{\partial f}{\partial E} = \frac{1}{E}\hat{C}[f]$$

and integrate it over the 3-momentum space:

$$\frac{g}{(2\pi)^3}\int \mathrm{d}^3\mathbf{p}\,\frac{\partial f}{\partial t} - \frac{g}{(2\pi)^3}\int \mathrm{d}^3\mathbf{p}\,H\frac{|\mathbf{p}|^2}{E}\frac{\partial f}{\partial E} = \frac{g}{(2\pi)^3}\int \mathrm{d}^3\mathbf{p}\,\frac{1}{E}\hat{C}[f].$$

The first term is simply the time derivative of the number density. The second term can be simplified using integration by parts:

$$\begin{aligned}
\int \mathrm{d}^3\mathbf{p}\,\frac{|p^2|}{E}\frac{\partial f}{\partial E} &= \int \mathrm{d}\Omega \int \mathrm{d}|\mathbf{p}|\frac{|\mathbf{p}|^4}{E}\frac{\partial f}{\partial E} \\
&= \int \mathrm{d}\Omega \int \mathrm{d}p\frac{p^4}{E}\frac{\partial f}{\partial E}\frac{\mathrm{d}p}{\mathrm{d}E} \\
&= 4\pi\int \mathrm{d}p\,p^3\frac{\partial f}{\partial p} \\
&= 4\pi\left(\int_0^\infty \mathrm{d}p\,\frac{\partial}{\partial p}(p^3 f) - \int \mathrm{d}p\,3p^2 f\right) \\
&= -3\int \mathrm{d}^3\mathbf{p}\,f.
\end{aligned}$$

The Boltzmann equation for the number density is thus

$$\frac{\mathrm{d}n}{\mathrm{d}t} + 3Hn = \frac{g}{(2\pi)^3}\int \mathrm{d}^3\mathbf{p}\frac{1}{E}\hat{C}[f].$$

The equation is very intuitive – if the collision term is zero, we retrieve the conservation of the comoving number density $N = na^3$:

$$0 = \dot{N} = \dot{n}a^3 + 3n\dot{a}a^2 = a^3(n + 3Hn) = 0.$$

Next, we focus on the collision term. Consider the reaction

$$a + b \longrightarrow i + j.$$

The knowledge of quantum field theory tells us that the reaction rate (at given momenta and spins of a and b) is

$$\Gamma_{ab\to ij}(p_a,p_b,s_a,s_b)=\frac{1}{(2\pi)^6}\frac{1}{2E_aE_b}\sum_{s_i,s_j}\int\frac{\mathrm{d}^3p_i}{2E_i}\frac{\mathrm{d}^3p_j}{2E_j}|M_{ab\to ij}|^2(2\pi)^4\delta^{(4)}(p_a+p_b-p_i-p_j).$$

The species a and b have momentum distributions $f_a(t,E_a)$ and $f_b(t,E_b)$. The total reaction rate is thus

$$\Gamma^{\mathrm{tot}}_{ab\to ij}=\sum_{s_a,s_b}\int\frac{\mathrm{d}^3p_a}{(2\pi)^3}f_a\,\frac{\mathrm{d}^3p_b}{(2pi)^3}f_b\,\Gamma_{ab\to ij}(p_a,p_b,s_a,s_b).$$

For the collision term, we have to consider both the creation and destruction of particles:

$$\Gamma_{ij\to ab}-\Gamma_{ab\to ij}=\sum_{\mathrm{spins}}\frac{1}{(2\pi)^{12}}\int\frac{\mathrm{d}^3p_a}{2E_{\mathbf{p}_a}}\frac{\mathrm{d}^3p_b}{2E_{\mathbf{p}_b}}\frac{\mathrm{d}^3p_i}{2E_{\mathbf{p}_i}}\frac{\mathrm{d}^3p_j}{2E_{\mathbf{p}_a}}(2\pi)^4\delta^{(4)}(p_a+p_b-p_i-p_j)\cdot$$
$$\cdot\left(-|M_{ab\to ij}|^2f_af_b+|M_{ij\to ab}|^2f_if_j\right).$$

We will now use three further assumptions:

(i) T invariance (or equivalently CP invariance): $|M_{ab\to ij}|^2=|M_{ij\to ab}|^2$;

(ii) the gas is diluted: $f^{\mathrm{eq}}\simeq e^{-E/T}$ (Maxwell-Boltzmann distribution);

(iii) the Standard model particles (i and j) are in thermal equilibrium: $f_i=f_{i,\mathrm{eq}}$. Moreover, it holds
$$f_if_j=f_i^{\mathrm{eq}}f_j^{\mathrm{eq}}=e^{(E_i+E_j)/T}=e^{(E_a+E_b)/T}=f_a^{\mathrm{eq}}f_b^{\mathrm{eq}}.$$

Using these assumptions, we can write

$$-|M_{ab\to ij}|^2f_af_b+|M_{ij\to ab}|^2f_if_j=-|M_{ab\to ij}|^2(f_af_b-f_if_j)=|M_{ab\to ij}|^2(f_af_b-f_a^{\mathrm{eq}}f_b^{\mathrm{eq}}).$$

The unpolarized cross-section for the reaction $a+b\to i+j$ is

$$\sigma=\frac{1}{Fg_ag_b}\sum_{\mathrm{spins}}\int\frac{\mathrm{d}^3p_i}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}_i}}\frac{\mathrm{d}^3p_j}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}_j}}|M_{ab\to ij}|^2(2\pi)^4\delta^{(4)}(p_a+p_b-p_i-p_j),$$

where $F=\sqrt{(p_ap_b)^2-m_a^2m_b^2}$ is called the baryon flux factor. We can also introduce the *Møller velocity*

$$v_{\mathrm{M\o l}}=\frac{F}{E_aE_b}=\sqrt{(\mathbf{v}_a-\mathbf{v}_b)^2-(\mathbf{v}_a\times\mathbf{v}_b)^2/c^2}.$$

Now, we can use this cross-section to rewrite the collision term:

$$\hat{C}[f_a]=-\int\frac{\mathrm{d}^3p_a}{(2\pi)^3}\frac{\mathrm{d}^3p_b}{(2\pi)^3}g_ag_b\sigma v_{\mathrm{M\o l}}(f_af_b-f_a^{\mathrm{eq}}f_b^{\mathrm{eq}}).$$

To further simplify this expression, we define the thermally averaged cross-section:

$$\langle\sigma v_{\mathrm{M\o l}}\rangle=\frac{\displaystyle\int\frac{\mathrm{d}^3p_a}{(2\pi)^3}g_af_a^{\mathrm{eq}}\frac{\mathrm{d}^3p_b}{(2\pi)^3}g_bf_b^{\mathrm{eq}}\sigma v_{\mathrm{M\o l}}}{\displaystyle\int\frac{\mathrm{d}^3p_a}{(2\pi)^3}g_af_a^{\mathrm{eq}}\frac{\mathrm{d}^3p_b}{(2\pi)^3}g_bf_b^{\mathrm{eq}}}=\frac{\displaystyle\int\frac{\mathrm{d}^3p_a}{(2\pi)^3}g_af_a^{\mathrm{eq}}\frac{\mathrm{d}^3p_b}{(2\pi)^3}g_bf_b^{\mathrm{eq}}\sigma v_{\mathrm{M\o l}}}{n_a^{\mathrm{eq}}n_b^{\mathrm{eq}}}.$$

The terms in the denominator are just the equilibrium number densities $n_a^{\rm eq}$ and $n_b^{\rm eq}$. Furthermore, at the time when the WIMPs depart from the equilibrium density, they are still in the kinetic equilibrium with the SM particles (but not in the chemical equilibrium). This means that their distribution function is modified only slightly

$$f_a(E,t) = f_a^{\rm eq} e^{-\alpha_a(t)}, \quad n_a = n_a^{\rm eq} e^{-\alpha_a(t)}.$$

With this, we can write

$$f_a f_b - f_a^{\rm eq} f_b^{\rm eq} = f_a^{\rm eq} f_b^{\rm eq} (e^{-\alpha_a(t)-\alpha_b(t)} - 1).$$

The collision term becomes

$$\hat{C}[f_a] = (e^{-\alpha_a+\alpha_b}) \langle \sigma v_{\rm M\o l} \rangle n_a^{\rm eq} n_b^{\rm eq} = - \langle \sigma v_{\rm M\o l} \rangle (n_a n_b - n_a^{\rm eq} n_b^{\rm eq}).$$

The Boltzmann equation for the number density is thus

$$\boxed{\frac{{\rm d}n_a}{{\rm d}t} + 3Hn_a = - \langle \sigma v_{\rm M\o l} \rangle (n_a n_b - n_a^{\rm eq} n_b^{\rm eq}).}$$

2.6 The Boltzmann's equation for the "yield"

We will assume that the collision term is zero and that the total number of particles in a comoving volume is conserved:

$$\frac{{\rm d}N}{{\rm d}t} = \frac{\rm d}{{\rm d}t}(na^3) = 0.$$

The question is: is the production or destruction of particles efficient? We can normalize the number density to the entropy density using ${\rm d}S/{\rm d}t = 0$

$$0 = \frac{{\rm d}(na^3)}{{\rm d}t} = \frac{\rm d}{{\rm d}t}\left(\frac{nS}{s}\right) = S\frac{\rm d}{{\rm d}t}\left(\frac{n}{s}\right).$$

This means that $n \propto s$. The proportionality constant is called the *yield* $y = n/s = N/S$. If there is particle creation or annihilation the yield changes. We can rewrite the Boltzmann equation for the yield:

$$\frac{{\rm d}y_a}{{\rm d}t} = \frac{\dot{n}_a s - n_a \dot{s}}{s^2} = \frac{1}{s^2}\left(-3Hn_a s - s\langle \sigma v \rangle (n_a n_b - n_a^{\rm eq} n_b^{\rm eq}) - n_a(-3HS)\right).$$

We introduce the equilibrium yield $y_a^{\rm eq} = n_a^{\rm eq}/s$, and the Boltzmann equation for the yield becomes

$$\boxed{\frac{{\rm d}y_a}{{\rm d}t} = -s\langle \sigma v \rangle (y_a y_b - y_a^{\rm eq} y_b^{\rm eq}).}$$

The equilibrium number density is

$$n^{\rm eq} = \begin{cases} \frac{\zeta(3)}{\pi^2} g_{\rm eff} T^3 & \text{relativistic particles,} \\ g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} & \text{non-relativistic particles,} \end{cases} \qquad g_{\rm eff} = \begin{cases} g & \text{bosons,} \\ \frac{3}{4}g & \text{fermions,} \end{cases}$$

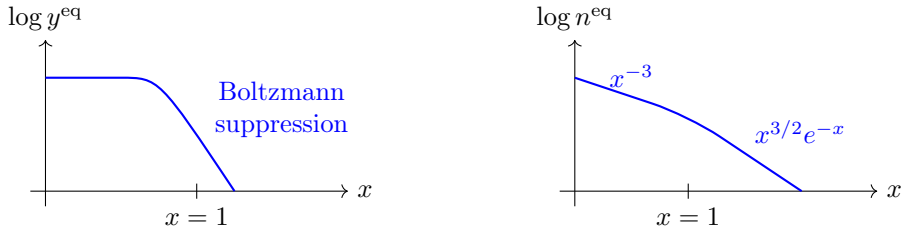
and the entropy density is

$$s = \frac{2\pi^2}{45} g_{*,s} T^3.$$

Combining these two expressions, we get the equilibrium yield:

$$y^{\mathrm{eq}} = \begin{cases} 0.278\, \frac{g_{\mathrm{eff}}}{g_{*,s}}, & \text{relativistic particles,} \\ 0.145\, \frac{g}{g_{*,s}} x^{3/2} e^{-x}, & \text{non-relativistic particles,} \end{cases}$$

where $x = m/T$.



In the radiation dominated era, the parametric dependences are $a \sim t^{1/2}$, $t \sim T^{-2}$ and $a \sim T^{-1}$. This means that aT and a/x are constant. We can thus rewrite the time derivative as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a}{x}\right) = \frac{\dot{a}}{x} - \frac{a}{x^2}\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = Hx.$$

With this, we rewrite the Boltzmann equation for the yield as

$$\frac{\mathrm{d}y_a}{\mathrm{d}x} = -\frac{s(x)}{xH(x)} \langle \sigma v \rangle (x) \left(\frac{y_a(x) y_b(x)}{y_a^{\mathrm{eq}} y_b^{\mathrm{eq}}} - q \right) y_a^{\mathrm{eq}} y_b^{\mathrm{eq}}.$$

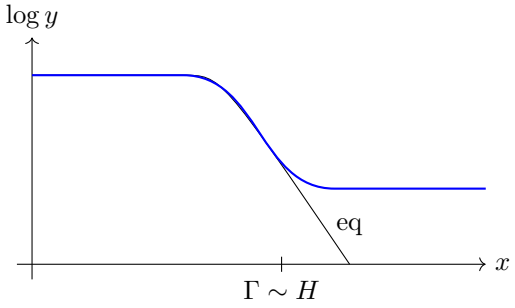
We can further simplify the prefactor using $s(x) y_b^{\mathrm{eq}} = n_b^{\mathrm{eq}}$ to get

$$\frac{x}{y_a^{\mathrm{eq}}} \frac{\mathrm{d}y_a}{\mathrm{d}x} = -\frac{\langle \sigma v \rangle\, n_b^{\mathrm{eq}}}{H(x)} \left(\frac{y_a(x) y_b(x)}{y_a^{\mathrm{eq}} y_b^{\mathrm{eq}}} - q \right) = -\frac{\Gamma_{\mathrm{ann}}}{H} \left(\frac{y_a(x) y_b(x)}{y_a^{\mathrm{eq}} y_b^{\mathrm{eq}}} - q \right).$$

This equation holds for a general equation $\mathrm{DM}_a + \mathrm{DM}_b \longrightarrow \mathrm{SM}_i + \mathrm{SM}_j$. However, in many models, the dark matter particle is its own antiparticle ($\mathrm{DM} + \mathrm{DM} \rightleftharpoons \mathrm{SM} + \mathrm{SM}$). In this case, we have $a = b$ and

$$\frac{x}{y_{\mathrm{eq}}} \frac{\mathrm{d}y}{x} = -\frac{\Gamma_{\mathrm{ann}}}{H} \left(\frac{y^2}{y_{\mathrm{eq}}^2} \right).$$

This is the Ricatti differential equation, which has no closed form solutions and even numerically it can be tedious to solve. Since $\Gamma \propto n_{\mathrm{eq}}$ and $H \sim T^2$, the prefactor Γ/H can simplify if $T \gg m$. In this case, the prefactor is very large, and the yield follows closely the equilibrium yield (which is constant). For very low temperatures ($T \ll m$), the prefactor becomes very small, and the yield becomes constant again (the freeze-out).



In this model, we have completely ignored the temperature dependence of $\langle\sigma v\rangle$. This can play an important role in determining the relic abundance of dark matter, however the general features of the solution remain the same. There are two options for the temperature dependence of the cross-section:

$$\sigma v \propto v^p, \quad p = \begin{cases} 0 & s\text{-wave}, \\ 2 & p\text{-wave}. \end{cases}$$

Also keep in mind that $\langle v\rangle \propto T^{1/2} \propto x^{-1/2}$. We can expand the behaviour of different terms in the prefactor around $x = 1$:

$$H(x) = H_{(x=1)}x^{-2}, \quad s(x) = s_{(x=1)}x^{-3}, \quad \langle\sigma v\rangle(x) = \langle\sigma v\rangle_{(x=1)} \begin{cases} 1 & s\text{-wave}, \\ x^{-1} & p\text{-wave}. \end{cases}.$$

The Riccati equation reads

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\lambda x^{-n-2}(y^2 - y_{\mathrm{eq}}^2),$$

where $n = p/2$ and

$$\lambda = \frac{\langle\sigma v\rangle s}{H}_{x=1} \simeq 0.264 \left(\frac{g_{*,s}}{g_*^{1/2}}\right) M_{\mathrm{Pl}} m \langle\sigma v\rangle_{(x=1)}.$$

The equation can be rewritten terms of the difference to the equilibrium yield $\Delta = y - y_{\mathrm{eq}}$:

$$\Delta' \approx -\lambda x^{-n-2}\Delta(\Delta + 2y_{\mathrm{eq}}) - y'_{\mathrm{eq}}.$$

At early times when $x \ll 1$, Δ is small and Δ' is also small. Thus, we write

$$0 \approx \lambda x^{-n-2}\Delta 2y_{\mathrm{eq}} - y'_{\mathrm{eq}}$$

and

$$\Delta \approx -\frac{y'_{\mathrm{eq}}}{2y_{\mathrm{eq}}}\frac{x^{n+2}}{\lambda} \approx \frac{x^{n+2}}{2\lambda}.$$

In late times $x \gg x_{\mathrm{fo}}$, we have $y_{\mathrm{eq}} \ll y \approx \Delta$:

$$\Delta' \approx \lambda x^{-n-2}\Delta(\Delta + 2y_{\mathrm{eq}}) - y'_{\mathrm{eq}}.$$

We can ignore the y'_{eq} and y_{eq} terms and write

$$\int_{x_{\mathrm{fo}}}^{\infty} \Delta^{-2}\,\mathrm{d}\Delta = -\int_{x_{\mathrm{fo}}}^{\infty} \lambda x^{-n-2}\,\mathrm{d}x.$$

In the boundary terms, we can neglect $\Delta(x_{\rm fo})$ and get

$$y(\infty) = \Delta(\infty) \approx \frac{n+1}{\lambda} x_{\rm fo}^{n+1} = \frac{3.79\,(n+1) x_{\rm fo}^{n+1}}{(g_{*,s}/g_*^{1/2}) M_{\rm Pl} m \langle\sigma v\rangle_{(x=1)}} \sim \frac{10^{-18}\,{\rm GeV}^{-1}}{m \langle\sigma v\rangle_{(x=1)}}.$$

The number density of the dark matter today is $n_0 = s_0 y_\infty$ where $s_0 = 2170\,{\rm cm}^{-3}$. The relic density parameter is thus

$$\Omega_{{\rm dm},0} h^2 = \frac{m n_0}{\rho_{\rm c}} h^2 = 1.07 \cdot 10^9\,{\rm GeV}^{-1} \frac{(n+1) x_{\rm fo}^{n+1}}{(g_{*,s}/g_*^{1/2}) M_{\rm Pl} \langle\sigma v\rangle_{(x=1)}}.$$

This expression has no explicit dependence on m, and it is implicit in $x_{\rm fo}$. For s-wave annihilation ($n = 0$) and $x_{\rm fo} \approx 20$, we get

$$\Omega_{{\rm dm},0} h^2 \approx \frac{2 \cdot 10^{-10}\,{\rm GeV}^{-2}}{\langle\sigma v\rangle_{(x=1)}} = 0.12, \quad \Longrightarrow \quad \langle\sigma v\rangle_{(x=1)} = 10^{-9}\,{\rm GeV}^{-2}.$$

There were many assumptions we made deriving this result. Most notably, we assumed the dark matter interaction with the Standard Model is contact-like. Another thing that could happen, is that the dark matter particles are still being produced after $T < m$ simply because there still are SM particles in the Boltzmann tail with enough energy to produce them.

2.7 Three exceptions (Griest, Seckel 1991)

(i) **Coannihilations**
Suppose there are more particles in the dark sector χ_i with masses $m_{\chi_1} < m_{\chi_2} < \ldots$ If the lightest mass is much smaller than the others, only the lightest particle contributes to the relic abundance. All the others will follow the equilibrium distribution for longer and thus have a negligible number density today. The ratio is

$$\frac{n_{\chi_2}}{n_{\chi_1}} \sim \left(\frac{m_{\chi_2}}{m_{\chi_1}}\right)^{3/2} e^{-(m_{\chi_2} - m_{\chi_1})/T}.$$

However, if the masses are similar $m_{\chi_2} \approx m_{\chi_1}$, then both particles decouple at the same time and both contribute to the relic abundance. The possible reactions between these two particles (χ_a, χ_b) and the Standard Model particles (x, y) are

- $\chi_a \chi_b \to xy$, σ_{ab} (annihilation),
- $\chi_a x \to \chi_b y$, $\sigma_{a\to b}$ (conversion),
- $\chi_a \to \chi_b xy$, $\Gamma_{a\to b}$ (decay).

A single species in the dark sector follows the Boltzmann equation

$$\frac{{\rm d}n_a}{{\rm d}t} + 3Hn_a = -\sum_{x,y,b} \{\langle\sigma_{ab} v_{ab}\rangle (n_a n_b - n_a^{\rm eq} n_b^{\rm eq}) + \\ - [\langle\sigma_{a\to b} v_{ax}\rangle n_a n_x - \langle\sigma_{b\to a} v_{by}\rangle n_b n_y] - \Gamma_{a\to b} n_a + \Gamma_{b\to a} n_b\}.$$

However, if we consider the total number density of all dark sector particles $n = \sum_a n_a$, then the conversion and decay terms cancel out (they do not change the total number of dark sector particles)

$$\frac{\mathrm{d}n}{\mathrm{d}t} + 3Hn = -\sum_{x,y,a,b} \langle \sigma_{ab} v_{ab} \rangle \, (n_a n_b - n_a^{\mathrm{eq}} n_b^{\mathrm{eq}}).$$

If a and b have similar masses, then $n_a/n \simeq n_a^{\mathrm{eq}}/n^{\mathrm{eq}}$ and thus

$$\frac{\mathrm{d}n}{\mathrm{d}t} + 3Hn = -\sum_{x,y,a,b} \langle \sigma_{ab} v_{ab} \rangle \, \frac{n_a^{\mathrm{eq}} n_b^{\mathrm{eq}}}{n_{\mathrm{eq}}^2} (n^2 - n_{\mathrm{eq}}^2).$$

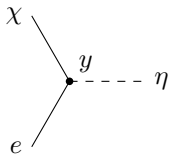
The equilibrium number densities can be absorbed into an effective cross-section

$$\langle \sigma v \rangle_{\mathrm{eff}} = \sum_{x,y,a,b} \langle \sigma_{ab} v_{ab} \rangle \, \frac{n_a^{\mathrm{eq}} n_b^{\mathrm{eq}}}{n_{\mathrm{eq}}^2}$$

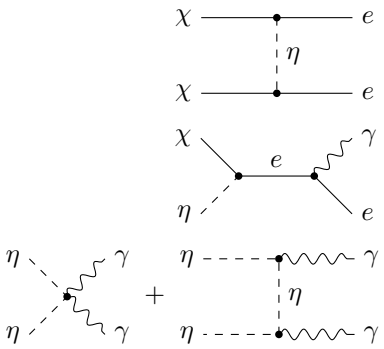
and the relic abundance can be computed as before using this effective cross-section

$$\Omega_{\mathrm{dm},0} \simeq \frac{2 \cdot 10^{-10}\,\mathrm{GeV}^2}{\langle \sigma v \rangle_{\mathrm{eff}}}.$$

Example 1. Let's consider a dark sector with two particles χ (bino) and η (selectron), with the interaction to the Standard model electron via the Yukawa coupling y:



The possible annihilation channels are



The effective cross-section is then

$$\langle \sigma v \rangle_{\mathrm{eff}} \sim y^2 C_{\chi\chi} \left(\frac{n_\chi^{\mathrm{eq}}}{n^{\mathrm{eq}}} \right)^2 + y^2 e^2 C_{\chi\eta} \frac{n_\chi^{\mathrm{eq}} n_\eta^{\mathrm{eq}}}{n_{\mathrm{eq}}^2} + e^4 C_{\eta\eta} \frac{(n_\eta^{\mathrm{eq}})^2}{n_{\mathrm{eq}}^2}.$$

The equilibrium number densities are related as

$$n^{\mathrm{eq}} = n_\chi^{\mathrm{eq}} + n_\eta^{\mathrm{eq}} = n_\chi^{\mathrm{eq}} \left(1 + \frac{n_\eta^{\mathrm{eq}}}{n_\chi^{\mathrm{eq}}}\right) = n_\chi^{\mathrm{eq}}(1+R)$$

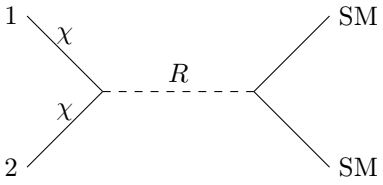
where $R = n_\chi^{\mathrm{eq}}/n_\eta^{\mathrm{eq}} = e^{(-m_\eta + m_\chi)/T}$. Using this, we can rewrite the effective cross-section as

$$\langle \sigma v \rangle_{\mathrm{eff}} \sim \frac{y^2 C_{\chi\chi} + y^2 e^2 C_{\chi\eta} R + e^4 C_{\eta\eta} R^2}{(1+R)^2}.$$

If $m_\eta \gg m_\chi$, then only the first term contributes ($\chi\chi \to e^+e^-$). If $m_\eta \approx m_\chi$, all terms contribute however if also $y \ll e$, then the last term dominates ($\eta\eta \to e^+e^-$). Even if the process $\chi\chi \to e^+e^-$ is very weak, the relic abundance can be influenced by χ converting to an intermediate particle η which then readily decays to the standard model particles (the so-called *coannihilation*).

(ii) **Resonances**

Let's look at the resonances that arise from the mediator particle.



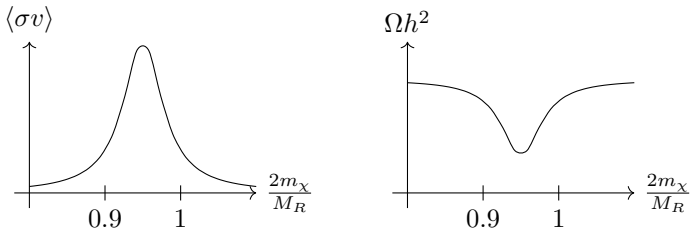
The cross-section can be parametrized as a sum of an s-wave and a p-wave term $\sigma v = a + bv^2$. According to the *Breit-Wigner formula*, the resonance is described as

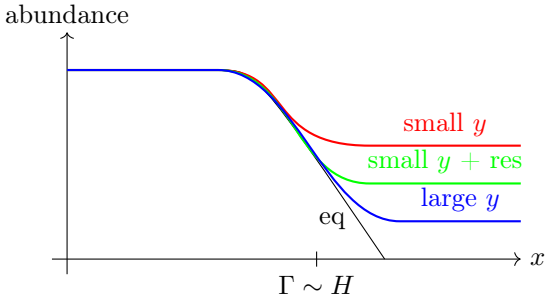
$$\sigma v = \frac{\alpha^2 s}{(M_R^2 - s)^2 + M_R^2 \Gamma_R^2}$$

where α is the coupling constant $\alpha = g^2/4\pi$, M_R and Γ_R are the mass and the decay width of the resonance and s is the Mandelstam variable $s = 2m_\chi^2 + 2(E_1E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2) \approx 4m_\chi^2/(1 - v^2/4)$. For a narrow resonance ($\Gamma_R \ll M_R$), it holds

$$\lim_{\gamma \to 0} \frac{\gamma}{x^2 + \gamma^2} = \pi\delta(x) \quad \Longrightarrow \quad \sigma v = \frac{\alpha^2 s}{M_R \Gamma_R} \pi \delta(M_R^2 - s).$$

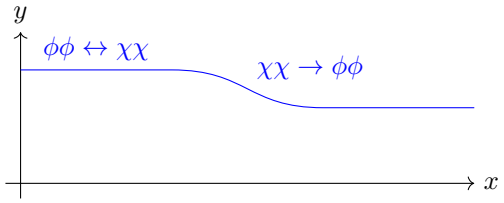
The thermally averaged cross-section will have the same shape as the Breit-Wigner distribution, and the relic abundance will have an inverted shape $\Omega h^2 \propto 1/\langle \sigma v \rangle$.





(iii) **Threshold effects**

Certain threshold effects can occur when the dark matter mass is close to half the mass the mediator mass. If $m_\chi > m_\phi$ the evolution of χ follows the usual route: first it is in thermal equilibrium with the SM particles, then it departs from the chemical equilibrium, and finally it freezes out.



However, if $m_\chi \sim m_\phi$, some dark matter particles can still be produced from the large momentum tail of the ϕ distribution even after the χ particles have departed from the chemical equilibrium. In this case, the Boltzmann equation for the number density becomes

$$\frac{\mathrm{d}n_\chi}{\mathrm{d}t} + 3Hn_\chi = -\langle\sigma v\rangle_{\chi\chi}\, n_\chi^2 + \langle\sigma v\rangle_{\phi\phi}\, n_\phi^2.$$

The equilibrium number densities of ϕ and χ are related as

$$\frac{n_\phi^{\mathrm{eq}}}{n_\chi^{\mathrm{eq}}} = e^{-(m_\phi - m_\chi)}$$

so in equilibrium, it holds

$$\langle\sigma v\rangle_{\phi\phi} = \langle\sigma v\rangle_{\chi\chi}\, e^{2(m_\chi - m_\phi)/T}.$$

The Boltzmann equation is then

$$\begin{aligned}\frac{\mathrm{d}n_\chi}{\mathrm{d}t} + 3Hn_\chi &= -\langle\sigma v\rangle_{\chi\chi}\, n_\chi^2 + \langle\sigma v\rangle_{\phi\phi}\, (n_\phi^{\mathrm{eq}})^2 \\ &= -\langle\sigma v\rangle_{\phi\phi}\, e^{-2(m_\phi - m_\chi)/T}(n_\chi^2 - (n_\chi^{\mathrm{eq}})^2) \\ &= -\langle\sigma v\rangle_{\mathrm{eff}}\, (n_\chi^2 - (n_\chi^{\mathrm{eq}})^2).\end{aligned}$$

Consequently, the dark matter relic abundance can be significantly altered

$$\Omega_{\mathrm{dm},0}h^2 \approx \frac{2\cdot 10^{-10}\,\mathrm{GeV}^2}{\langle\sigma v\rangle_{\mathrm{eff}}} = \frac{2\cdot 10^{-10}\,\mathrm{GeV}^2}{\langle\sigma v\rangle_{\phi\phi}} e^{2(m_\phi - m_\chi)/T}.$$

Chapter 3

Indirect dark matter searches

One of the ways of detecting the dark matter particles is to look at the motion of the Standard Model particles that are produced in the dark matter annihilation. In order to understand the signals, we need to talk about three topics: the production of the SM particles in the annihilation, their propagation through the galaxy, and finally their detection on the Earth.

3.1 Production

The first component is the source term $Q_i(\mathbf{r}, E)$. It is the number of particles of type i produced per unit volume, time and energy at the point $\mathbf{r}$ in the galaxy. The expression is given by

$$Q_i(\mathbf{r}, E) = \frac{1}{2}\left(\frac{\rho(\mathbf{r})}{M_{\mathrm{DM}}}\right)(\sigma v)_i \frac{\mathrm{d}N}{\mathrm{d}E}.$$

If the dark matter particle is self-conjugate (real scalar, Majorana fermion), there is a symmetry factor 1/2, and if the dark matter particle χ and antiparticle $\overline{\chi}$ are different, then the total dark matter density is $\rho_{\mathrm{tot}}(\mathbf{r}) = \rho_\chi(\mathbf{r}) + \rho_{\overline{\chi}}(\mathbf{r})$. If we assume that there is no asymmetry between particles and antiparticles, then $\rho_\chi = \rho_{\overline{\chi}} = \rho_{\mathrm{tot}}/2$ and the blue prefactor becomes 1/4. A common parametric choice for an isotropic dark matter density profile is:

$$\rho(r) = \frac{\rho}{\left(\frac{r}{r_{\mathrm{c}}}\right)^{\gamma}\left[1 + \left(\frac{r}{r_{\mathrm{c}}}\right)^{\alpha}\right]^{(\beta-\gamma)/\alpha}}.$$

Some possible options are given in the following table.

Profile	α	β	γ	r_{c} [kpc]
isothermal	2	2	0	20
Navaro–Frenk–White	1	3	1	3.5
Moore et al.	1.5	3	1.5	28

There is also the Einsato profile, which has an exponential dependence instead of a power-law one in the inner region:

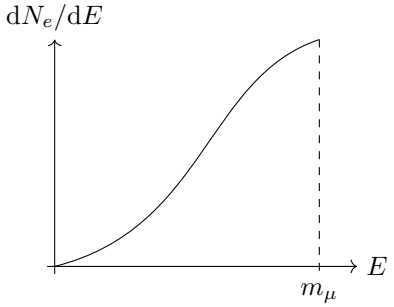
$$\rho(r) = \rho_0 \exp\left(-\frac{2}{\alpha}\left[\left(\frac{r}{r_{\mathrm{s}}}\right)^{\alpha} - 1\right]\right),$$

where $\alpha \approx 0.17$ and $r_\mathrm{s} = 20\,\mathrm{kpc}$.

We can perform two types of measurements: either a local measurement, which gives the dark matter density in the solar neighbourhood $\rho_\odot \approx 0.3\,\mathrm{GeV\,cm^{-3}}$, or a global measurement, which looks at the rotation curves of the galaxy. The measurements only give information about the dark matter distribution at large radii ($r \gtrsim 1\,\mathrm{kpc}$). The previously mentioned models mostly agree at this distance, but they differ significantly in the inner region of the galaxy.

The production rates per unit energy are simple for decays in photons and leptons

$$\begin{aligned}
\chi\chi \to \gamma\gamma: &\quad \frac{\mathrm{d}N}{\mathrm{d}E} = 2\delta(E - M_\mathrm{DM}),\\
\chi\chi \to e^+e^-: &\quad \frac{\mathrm{d}N}{\mathrm{d}E} = \delta(E - M_\mathrm{DM}) \quad \text{if } m_\mathrm{DM} \gg m_e,\\
\chi\chi \to \mu^+\mu^-: &\quad \frac{\mathrm{d}N}{\mathrm{d}E} \approx 2\delta(E - M_\mathrm{DM}).
\end{aligned}$$



However, the muons will further decay to electrons and neutrinos, which will give a continuous spectrum of electrons (see the graph above). When the dark matter annihilates to quarks or gauge bosons, these will hadronize and produce many secondary particles. The spectra can be computed using Monte Carlo event generators.

The freeze-out condition is $\langle\sigma v\rangle_\mathrm{fo} = 3 \cdot 10^{-26}\,\mathrm{cm^3\,s^{-1}}$. It is parametrized as

$$\sigma v_\mathrm{M\o l} = a + bv^2 + cv^4 + \dots$$

and the thermally averaged cross-section is

$$\langle\sigma v\rangle_\mathrm{M\o l} = \frac{\displaystyle\int \mathrm{d}^3\mathbf{v}_1 \int \mathrm{d}^3\mathbf{v} 2 e^{-m\mathbf{v}_1^2/(2T)} e^{-m\mathbf{v}_2^2/(2T)} \sigma v_\mathrm{M\o l}}{\displaystyle\int \mathrm{d}^3\mathbf{v}_1 \int \mathrm{d}^3\mathbf{v} 2 e^{-m\mathbf{v}_1^2/(2T)} e^{-m\mathbf{v}_2^2/(2T)}} = a + \frac{6b}{x}.$$

For the s-wave annihilation, we get $a = 3 \cdot 10^{-26}\,\mathrm{cm^3\,s^{-1}}$ and for the p-wave annihilation $b = x/6 \cdot 3 \cdot 10^{-26}\,\mathrm{cm^3\,s^{-1}} \approx 4 \cdot 10^{-25}\,\mathrm{cm^3\,s^{-1}}$. In the latter case, we can compare the velocity in the freeze-out epoch

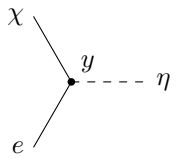
$$\frac{3}{2}k_\mathrm{B}T = \frac{1}{2}mv^2 \quad \Longrightarrow \quad \sqrt{\langle v^2\rangle} \approx \sqrt{3/x} \approx 0.4c$$

and the velocity in the galaxy today

$$\frac{v^2}{r} = \frac{GM}{r^2} \quad \Longrightarrow \quad \sqrt{\langle v^2\rangle} \approx 10^{-3}\,c.$$

We can see that the p-wave annihilation is suppressed by a factor of $\sim 10^{-5}$ today compared to the freeze-out epoch while in the s-wave case there is no suppression.

Consider a Majorana dark matter particle χ that annihilates to e_{R} and η.

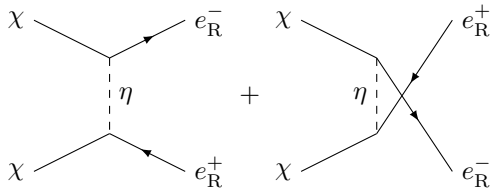


Assume the two χ particles are at rest and with zero angular momentum. The possible combinations of the spins are

$$S=0: \quad |00\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow),$$

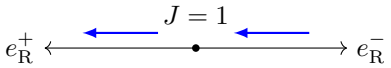
$$S=1: \quad \begin{cases} |11\rangle = \uparrow\uparrow, \\ |10\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), \\ |1-1\rangle = \downarrow\downarrow . \end{cases}$$

The total wave function of $\chi\chi$ is $\Psi = \Psi_{\mathrm{orbital}}\Psi_{\mathrm{spin}}$. Since the wave function must be antisymmetric and the orbital part is symmetric ($L=0$), the spin part must be antisymmetric ($S=0$ and hence $J=0$). The final state consists of e_{R} and $\overline{e}_{\mathrm{R}}$. Since these are both right-handed, their spins are parallel and $J=1$.



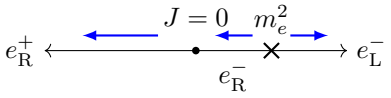
This cannot happen due to the spin conservation, so the annihilation is forbidden for massless electron:

$$\sigma v(\chi\chi \to e_{\mathrm{R}}^+ e_{\mathrm{R}}^-) = 0 \quad \text{if } v=0 \text{ and } m_e = 0.$$



Possible solutions are two. If the electron has a mass, there can be a helicity flip and the spins can be antiparallel. This gives a suppression factor of m_e^2/m_χ^2 (the same mechanism as for pion decay):

$$\sigma v(\chi\chi \to e_{\mathrm{R}}^+ e_{\mathrm{L}}^-) = a\left(\frac{m_e}{m_\chi}\right)^2 + bv^2 \ldots$$



Another option is to emit an additional photon in the final state. This can carry away the angular momentum and thus allow the annihilation even for massless electrons. The suppression factor is then α_{em}/π due to the additional vertex

$$\sigma v(\chi\chi \to e_{\mathrm{R}} e_{\mathrm{R}} \gamma) \sim \alpha_{\mathrm{em}}$$

and, similarly, the decay to two photons is suppressed by $(\alpha_{\mathrm{em}}/\pi)^2$:

$$\sigma v(\chi\chi \to \gamma\gamma) \sim \alpha_{\mathrm{em}}^2.$$

3.2 Propagation

We will be focusing on the propagation of charged cosmic rays (antiprotons, positrons). These particles are created in the dark matter annihilation and then propagate through the galaxy before reaching the Earth. Their motion is affected by several processes, which we will now describe. The number density of antiparticles per unit volume and time at the position $\mathbf{r}$ is given by the quantity $f(p^\mu, x^\mu) = f(E, \mathbf{r}, t)$, which we will assume does not depend on the 3-momentum. It is governed by the Boltzmann equation $\mathrm{d}f/\mathrm{d}t = C[f]$. The total time derivative can be expanded as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + (\nabla f)\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial f}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \nabla\cdot\mathbf{J} + \nabla_E J_E = \frac{\partial f}{\partial t} + \nabla\cdot(\dot{\mathbf{x}}f) + \nabla_E(\dot{E}f).$$

The first term is the net change in the number of particles, the second term is the change due to the motion in space and the last term is the change in the energy of the particles.

- The creation of antiparticles is given by the source term $Q_i(E, \mathbf{r}, t)$, while the destruction is due to the annihilations with the particles in the galactic disc. This rate can be modelled with a simple ansatz
$$\Gamma_{\mathrm{ann}} f(E, \mathbf{r}, t)\delta(z)2h,$$
where Γ_{ann} is the annihilation ratio and h is the half-thickness of the galactic disc (50 pc).
- The change in the energy of the particles is due to several processes.
  (i) Compton scattering with photons from CMB (10^{-3} eV), thermal radiation of dust (IR, 10^{-2} eV) or starlight (UV, vis, 1 eV).
  (ii) Synchrotron radiation due to magnetic fields in the galaxy.
  (iii) Ionizations with atoms.

  The energy term can be compacted into $\dot{E} = -b(E, \mathbf{r})$, which includes all three processes and depends on the position. The term for energy loss in the Boltzmann's equation becomes
$$\frac{\partial}{\partial E}(\dot{E}f) = -\frac{\partial}{\partial E}\left(b(E, \mathbf{r})f(E, \mathbf{r}, t)\right).$$
- The change in position also comes from multiple factors.
  (i) *Convection*: the particles are carried away by the moving fluid $\mathbf{J} = \mathbf{u}f$, where $\mathbf{u}$ is the velocity of the fluid. Antimatter particles can be carried away ba the stellar wind of the stars in the galactic disc (which is not spherically symmetric and is oriented away from the disc):
$$\mathbf{J}_{\mathrm{conv}} = \mathbf{v}_{\mathrm{c}} f(E, \mathbf{r}, t), \quad \mathbf{v}_{\mathrm{c}} = v_{\mathrm{c}}(z)\,\mathrm{sign}(z)\hat{\mathbf{e}}_z.$$

(ii) *Diffusion*: in the galaxy, there are complicated magnetic fields. Their effect is modelled by a random walk (*Fick's law*)

$$\mathbf{J}_{\text{diff}} = -K(E,\mathbf{r})\nabla f.$$

This law says that, on average, particles will move from regions with high density to regions of lower density. The proportionality factor can take the form

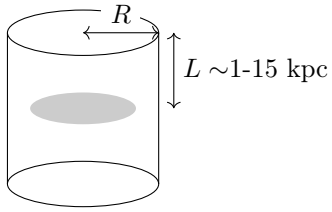
$$K(E,\mathbf{r}) = K_0\beta\mathcal{R}^\delta$$

where $\mathcal{R} = |\mathbf{p}|/Z$ is called the *rigidity* and Z is atomic number.

When we write everything together, we get the *master formula*

$$0 = \frac{\partial f}{\partial t} = Q(E,\mathbf{r},t) - 2h\delta(z)\Gamma_{\text{ann}}f + \frac{\partial}{\partial E}(b(E,\mathbf{r})f) - \nabla\cdot(\mathbf{v}_c f) + \nabla\cdot(K(E,\mathbf{r})\nabla f).$$

The parameters are determined by particle physics, atomic physics and some astrophysics. For the medium velocity $\mathbf{v}_c$ we can assume that its magnitude is independent of z. We set $f(E,\mathbf{r},t) = 0$ at the boundary of the cylinder (see figure below). Data fits best for cylinder height of $L \sim 4\,\text{kpc}$.



Normally, to determine the parameters δ, K_0, α, v_c, we use B/C (boron over carbon) ratio. In stars, fusion produces helium (H $\longrightarrow$ ^{4}He) and carbon (^{4}He $\longrightarrow$ ^{12}C). However, in cosmic rays, one sees Li, Be, B, etc., which are not produced in stars. They are produced by spallation of cosmic rays on other nuclei (C + p $\longrightarrow$ He + Li, C + p $\longrightarrow$ B + p). The flux of carbon can be measured: it propagates according to its own diffusion equation. This gives $f_C(\mathbf{r},E,t)$. We also know the process C + p $\longrightarrow$ B (the source term for boron) and can use the diffusion equation to compute $f_B(\mathbf{r},E,t)$. The parameters are then fitted to the measured B/C ratio.

L [kpc]	δ	K_0 [kpc^2/Myr]	v_c [km/s]
1	0.85	0.0016	13.5
4	0.70	0.0112	12
15	0.46	0.0765	5

Solution to the diffusion equation at the location of the solar system is

$$f_i(r_\odot,T) = \frac{Zv}{m^2}\int_0^{T_{\max}} G(T,T')\frac{\mathrm{d}N}{\mathrm{d}T}\,\mathrm{d}T'.$$

The quantity T is the kinetic energy, $G(T,T')$ is the Green's function of the diffusion equation and $(Zv/m^2)\mathrm{d}N/\mathrm{d}T$ is the source spectrum of the particle i. The flux of antimatter particles that reach the Solar system is

$$\Phi_i(T) = \frac{v(T)}{4\pi}f_i(T).$$

This is only the flux at the border of the Solar system. To get the flux at the Earth, we need to take into account the solar wind inside the heliosphere (once cosmic rays enter, they are affected by it). The effect is extremely complicated, so it has to be modelled by a very simple *force-field approximation*:

$$T_{\mathrm{IS}} = T_{\mathrm{ToA}} + \phi_{\mathrm{F}},$$

where T_{IS} is the interstellar kinetic energy, T_{ToA} is the kinetic energy at the top of the atmosphere and ϕ_{F} is the solar modulation potential (typically 500 MeV to 1.3 GeV, depending on the solar activity) also called the *Fick potential*. Using this model, the flux at the top of the atmosphere is related to the interstellar flux via the particle number conservation:

$$\Phi_{\mathrm{ToA}}(T_{\mathrm{ToA}}) = \frac{p_{\mathrm{ToA}}^2}{p_{\mathrm{IS}}^2}\Phi_{\mathrm{IS}}(T_{\mathrm{IS}}) = \frac{(T_{\mathrm{ToA}}^2 + 2mT_{\mathrm{ToA}})\Phi_{\mathrm{IS}}(T_{\mathrm{ToA}} + \phi_{\mathrm{F}})}{(T_{\mathrm{ToA}} + \phi_{\mathrm{F}})^2 + 2m(T_{\mathrm{ToA}} + \Phi_{\mathrm{F}})}.$$

We have used $p^2 = T^2 + 2mT$.

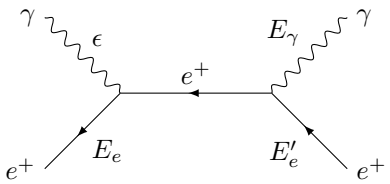
3.2.1 Positron propagation

Now, we will specifically focus on positrons. Their annihilations in the galactic disc are negligible, so we can drop the annihilation term in the master formula. The convection can also be neglected for positrons. However, energy losses are very important due to the synchrotron radiation and inverse Compton scattering. The diffusion equation for positrons becomes

$$\nabla \cdot (K(E, \mathbf{r})\nabla f) + \frac{\partial}{\partial E}(b(E, \mathbf{r})f) + Q(E, \mathbf{r}, t) = 0.$$

The energy loss for an inverse Compton scattering is

$$E_e - E_e' = E_\gamma - \epsilon$$



The rate of energy loss per unit energy of the scattered photon is

$$\frac{\mathrm{d}R_{\mathrm{loss}}(\epsilon, E_\gamma, \mathbf{r})}{\mathrm{d}E_\gamma} = (E_\gamma - \epsilon) f_{\mathrm{ISRF}}(E_\gamma, \mathbf{r}) \frac{\mathrm{d}\sigma(E_\gamma, \epsilon)}{\mathrm{d}E_\gamma}.$$

Here, $f_{\mathrm{ISRF}}(E_\gamma, \mathbf{r})$ is the interstellar radiation field (number density of photons per unit energy) and $\mathrm{d}\sigma/\mathrm{d}E_\gamma$ is the differential cross-section for the inverse Compton scattering. For the total energy loss, we have to integrate over both energies

$$b_{\mathrm{Compt}}(E_e, \mathbf{p}) = \int_0^\infty \mathrm{d}\epsilon \int_\epsilon^{E_{\max}} \mathrm{d}E_\gamma \, (E_\gamma - \varepsilon) \frac{\mathrm{d}\sigma}{\mathrm{d}E_\gamma} f_{\mathrm{ISRF}}(t, \mathbf{r}).$$

The differential cross-section for inverse Compton scattering is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_\gamma} = \frac{3}{4}\frac{\sigma_{\mathrm{T}}}{\gamma_e^2 \epsilon}\left(2q \log q + 1 + q - 2q^2 + \frac{1}{2}\frac{(q\Gamma)^2}{1 + q\Gamma}(1 - q)\right),$$

where

$$\gamma_e = \frac{E_e}{m_e}, \quad \Gamma = \frac{4\gamma\varepsilon}{m_e}, \quad q = \frac{E\gamma_e}{\Gamma(E_e - E_\gamma)}, \quad \sigma_{\mathrm{T}} = 0.67\,\mathrm{barn}.$$

The calculation for the synchrotron radiation is a bit different

$$b_{\mathrm{sync}}(E, \mathbf{r}) = \frac{4}{3}\sigma_{\mathrm{T}}\gamma_e^2\frac{B^2(\mathbf{r})}{2}.$$

It depends on the magnetic field $B(\mathbf{r})$ in the galaxy, which is not very well known. A common choice is

$$B \approx B_0 \exp\left[-\frac{(r - r_0)}{r_B} - \frac{|z|}{z_B}\right],$$

where r_0 =5-8 kpc, $r_B = 10\,\mathrm{kpc}$, $z_B \approx 2\,\mathrm{kpc}$ and $B_0 = 5\,\mu\mathrm{G}$. With these numbers the energy loss due to synchrotron radiation at the location of the Solar system is

$$b_{\mathrm{sync}} = 4 \cdot 10^{-17}\left(\frac{E_e}{\mathrm{GeV}}\right)^2 \mathrm{GeV\,s^{-1}}.$$

Combined with the Compton scattering, the total energy loss at the Solar system becomes

$$b = b_{\mathrm{Comp}} + b_{\mathrm{sync}} \approx 10^{-16}\,E_0^2\mathrm{GeV^{-1}\,s^{-1}}.$$

A very rough approximation for a lifetime of a positron would be to just assume that the energy loss is constant:

$$\Delta t \sim \frac{|\Delta E|}{b} \sim \frac{E}{b} \sim \frac{1}{10^{-16}\,E}\mathrm{GeV\,s}.$$

For a 1 TeV electron, the rough lifetime is about $\Delta t \sim 10^5$ years. For this time of travel, the energy loss is comparable to the whole energy.

Next, we will try to find some analytical solutions to the diffusion equation for positrons

$$\nabla(K(E)\nabla f) + \frac{\partial}{\partial E}(b(E)f) + Q(E, \mathbf{r}),$$

where $K(E) = K_0E^\delta$ and $b(E) = E^2/\tau$. Since we assumed that the diffusion coefficient doesn't depend on the position, the first term in the equation simplifies to $K(E)\nabla^2 f$. The next step is the introduction of the cylindrical coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

Remember that Bessel's functions satisfy the following differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - \alpha^2)y = 0.$$

In particular, the zeroth Bessel function J_0 satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}J_0 + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}J_0 + J_0 = 0.$$

Our boundary condition is $f(r = R) = 0$, so the solution can be expanded as

$$f(E, r, z) = \sum_n = 1^\infty A_n(E, z)J_0(\xi_n r/R),$$

where ξ_n is n-th zero of J_0. Using the ansatz for the diffusion equation gives

$$\sum_{n=1}^{\infty}\left(KA_n\left[\frac{\mathrm{d}^2 J_0}{\partial r^2}+\frac{1}{r}\frac{\partial J_0}{\partial r}\right]+K\frac{\partial^2 A_n}{\partial z^2}J_0+\frac{\partial A_n}{\partial E}J_0\right)=-Q(E,r,z).$$

The term in the square parenthesis simplifies to $-\xi_n^2/R^2\, J_0(\xi_n r/R)$ (this was the reason we used this ansatz). To extract the source term, we use the orthogonality of Bessel functions

$$\int_0^R \mathrm{d}r\, r J_0(\xi_n r/R)J_0(\xi_m r/R)=\frac{1}{2}J_1^2(\xi_n)R^2\delta_{mn}.$$

Applying this integral on our differential equation gives

$$-\frac{\xi_n^2}{R^2}K(E)A_n+K(E)\frac{\partial^2 A_n}{\partial^2 z}+b\frac{\partial A_n}{\partial E}=-\frac{2}{J_1^2(\xi_n)R^2}\int_0^R \mathrm{d}r\, rQ(E,r,z)J_0(\xi_n r/R).$$

Now we use another ansatz for A_n so that the boundary condition $f(z=\pm L)=0$ is satisfied

$$A_n(E,z)=\sum_{m=1}^{\infty}A_{n,m}(E)\sin(m\pi(z-L)/(2L)).$$

This set of functions is again orthogonal

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{2L}(z-L)\right)\sin\left(\frac{m\pi}{2L}(z-L)\right)=L\delta_{mn}.$$

The final relation of $A_{n,m}$ is

$$\frac{\mathrm{d}A_{n,m}}{\mathrm{d}E}-\frac{K(E)}{b(E)}\left(\frac{\xi_n^2}{R^2}+\frac{m^2\pi^2}{4L^2}\right)A_{n,m}=-\frac{1}{b}Q_{n,m},$$

where

$$Q_{n,m}=\frac{2}{J_1^2(\xi_n)R^2L}\int_0^R \mathrm{d}r\, r\int_{-L}^{L}\mathrm{d}z Q(r,z,E)J_0(\xi_n r/R)\sin(m\pi(z-L)/(2L)).$$

And finally, we arrive to the solution

$$A_{n,m}=\int_E^{E_{\max}}\mathrm{d}E'\,\frac{Q_{n,m}(E')}{b(E')}\exp\left(\left[\frac{\xi_n^2}{R^2}+\frac{m^2\pi^2}{4L^2}\right]\frac{K_0\tau}{\delta-1}\left(E^{\delta-1}-E'^{(\delta-1)}\right)\right)$$

which combines with the previous ansätze to give

$$f(E,r,z)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}A_{n,m}(E)J_0\left(\frac{\xi_n r}{R}\right)\sin\left(\frac{m\pi}{2L}(z-L)\right).$$

3.2.2 Antiproton propagation

The main difference between positron and antiproton propagation is that antiprotons do not lose energy as significantly as positrons. Therefore, we can neglect the energy loss term in the master formula. However, we still have to keep the diffusion and convection terms. The annihilation in

the galactic disc is also important for antiprotons, as they annihilate via the strong interaction $\sigma(p^+p^- \to \text{hadrons}) \propto \alpha_s^2$. The master formula for antiprotons becomes

$$0 = \frac{\partial f}{\partial t} = Q(T, \mathbf{r}) + \nabla \cdot (K(T, \mathbf{r}) \nabla f) - \nabla \cdot (\mathbf{v}_c f) - 2h\delta(z)\Gamma_{\text{ann}} f.$$

We used the kinetic energy T instead of the total energy E. The diffusion coefficient will be assumed to be constant in space $K(T, \mathbf{r}) = K(T)$ and the convection velocity is $\mathbf{v}_c = v_c \operatorname{sign}(z)\hat{\mathbf{e}}_z = v_c(2\theta(z) - 1)\mathbf{e}_z$. Hence, the derivative in the z direction will be $\partial \mathbf{v}_c / \partial z = (0, 0, 2v_c\delta(z))$. The annihilation term takes into account annihilations of antiprotons with protons and helium. The cross-sections are well measured

$$\sigma(p\overline{p}) = \begin{cases} 661\left(1 + 0.0115\, T^{-0.774} - 0.948\, T^{0.0151}\right), & T < 15.5\,\text{GeV}, \\ 36\, T^{-0.5}\,\text{mbarn}, & T \geq 15.5\,\text{GeV}. \end{cases}$$

The annihilation with helium is measured less accurately. It is useful, to use the liquid drop model to first estimate the radius of helium $R_A = R_H A^{1/3}$. Roughly, we can say that

$$\sigma(XA) \sim \sigma(Xp) A^{2/3}.$$

The annihilation rate is then

$$\Gamma_{\text{ann}} = \left(n_H + 4^{2/3} n_{He}\right) v \cdot \sigma(p\overline{p}).$$

The source term will be discussed in more detail later, but for now we will use

$$Q = \frac{1}{2} \frac{\rho_{DM}}{M_{DM}} \langle \sigma v \rangle_i \frac{\mathrm{d}N_{\overline{p},i}}{\mathrm{d}T}.$$

Using the same boundary conditions as for positrons $f(T, R, z) = 0$, the solution is given by the Bessel ansatz

$$f(T, r, z) = \sum_{n=1}^{\infty} A_n(T, z) J_0(\xi_n r / R).$$

The Laplacian on f is

$$\nabla^2 f = \sum_{n=1}^{\infty} \left[-\frac{\xi_n^2 A_n}{R^2} + \frac{\partial^2 A_n}{\partial z^2} \right] J_0(\xi_n r / R).$$

The master formula becomes

$$K(T) \left[-\frac{\xi_n^2 A_n}{R^2} + \frac{\partial^2 A_n}{\partial z^2} \right] - v_c \frac{\partial A_n}{\partial z} - 2(v_c + h\Gamma_{\text{ann}})\delta(z) A_n = -Q_n(T, z),$$

where

$$Q_n = \frac{2}{J_1^2(\xi_n) R^2} \int_0^R \mathrm{d}r\, r Q(T, z) J_0(\xi_n r / R).$$

The next step deviates from the positron case. We will perform the Laplace transform in the z direction. Defining

$$Y_n(s) = \mathcal{L}(A_n(z))(s) = \int_0^{\infty} A_n(z) e^{-zs}\, \mathrm{d}z$$

we get the other terms as

$$\begin{aligned}
v_\mathrm{c}\int_0^\infty \frac{\partial A_n(z)}{\partial z}e^{-zs}\,\mathrm{d}z &= v_\mathrm{c}(-A_n(0)+sY_n(s)),\\
\int_0^\infty \frac{\partial^2 A_n(z)}{\partial z^2}e^{-zs}\,\mathrm{d}z &= -A_n'(0)-sA_n(0)+s^2Y_n(s),\\
\int_0^\infty \delta(z)A_n(z)e^{-zs}\,\mathrm{d}z &= \frac{1}{2}A_n(0).
\end{aligned}$$

The solution needs to have the symmetry $A_n(-z)=A_n(z)$, $A_n'(T,0)=A_n'(T,-0)$, so (assuming continuous derivative) $A'(T,0)=0$. The final equation for Y_n is

$$\int_0^\infty Q(T,z)e^{-zs}\,\mathrm{d}z = Y_n(s)K\left(-\frac{\xi_n^2}{R^2}+s^2-\frac{v_\mathrm{c}}{K}s\right)-A_n(0)(h\Gamma_\mathrm{ann}+Ks)$$

with the solution

$$Y_n(s)=\frac{(h\Gamma_\mathrm{ann}+Ks)A_n(0)-\int_0^\infty Q_n(T,z)e^{-zs}\,\mathrm{d}z}{K\left(s^2-\frac{v_\mathrm{c}}{K}s-\frac{\xi_n^2}{R^2}\right)}.$$

The inverse Laplace transform is defined as

$$A_n(z)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}Y_n(s)e^{sz}\,\mathrm{d}s.$$

The value of γ is chosen so that it is greater than real values of all singularities of $Y_n(s)$. The singularities are the poles of the denominator

$$s_\pm=\frac{v_\mathrm{c}}{2K}\pm\sqrt{\left(\frac{v_\mathrm{c}}{2K}\right)^2+\frac{\xi_n^2}{R^2}}.$$

The integral can be solved using the residue theorem. Closing the contour to the left-half plane, we encompass both singularities. Let's denote

$$Y_n(s)=\frac{g(s)}{(s-s_+)(s-s_-)}$$

and then (after introducing $s_n=s_+-s_-$) the inverse Laplace transform becomes

$$A_n(z)=\frac{1}{s_n}\left[g(s_+)e^{s_+z}-g(s_-)e^{s_-z}\right]=\frac{e^{v_\mathrm{c}z/(2K)}}{s_n}\left[g_{s_+}e^{s_nz/2}-g(s_-)e^{-s_nz/2}\right].$$

Plugging in the definition of $g(s)$ gives

$$\begin{aligned}
A_n(T,z)=&\frac{e^{V_\mathrm{c}z/(2k)}}{Ks_n}\left\{\sinh(s_nz/2)\left[2h\Gamma_\mathrm{ann}+v_\mathrm{c}\right]+Ks_n\coth(s_nz/2)\right\}A_n(0)\\
&-\frac{2}{Ks_n}\int_0^\infty Q_n(T,z')e^{v_\mathrm{c}(z-z')/(2K)}\sinh(s_n(z-z')/2).
\end{aligned}$$

Applying the boundary condition $A_n(T,L)=0$ allows us to solve for $A_n(0)$. The solution for f is

$$f(T,r,z)=\sum_{n=1}^\infty e^{-v_\mathrm{c}z/(2K)}\frac{y_n(z)}{A_n\sinh(s_nL/2)}J_0(\xi_nr/R),$$

where

$$
\begin{aligned}
y_n(z) &= 2\int_0^L \mathrm{d}z'\, Q_n(T,z') e^{v_\mathrm{c}(L-z')/(2K)} \sinh\left(\frac{s_n}{2}(L-z')\right), \\
A_m &= 2h\Gamma_\mathrm{ann} + v_\mathrm{c} + K s_n \coth(s_n L/2), \\
s_n &= \sqrt{\frac{v_\mathrm{c}^2}{K^2} + \frac{4\xi_n^2}{R^2}}, \\
Q_n(T,z) &= \frac{2}{J_1^2(\xi_n) R^2} \int_0^R \mathrm{d}r\, r Q(T,z) J_0(\xi_n r/R).
\end{aligned}
$$

The benefit of the analytic approach is that it is very easy to see the dependence on the ratio v_c/K (convection over diffusion).

3.2.3 Antinuclei

In this section, we will study the production and propagation of antinuclei. We will mostly focus on antideuterons, antihelium-3 and antihelium-4 and antitritium (which decays weakly into antihelium-3). Historically, the most important process for the production of antinuclei is the collision of two protons $pp \to pp\,p\overline{p}X$, where X stands for any other particles produced in the collision. This is the minimal process that can produce an antiproton (due to baryon number conservation). Antiprotons have been discovered in cosmic rays, but none of the other antinuclei have been observed yet. Other possible processes for antinuclei production are

$$
\begin{aligned}
pp &\to \overline{pn}\, pn\, pp\, X, \\
pp &\to \overline{d}\, pn\, pp\, X, \\
pp &\to {}^3\overline{\mathrm{He}}\, ppnn\, pp\, X.
\end{aligned}
$$

Let's look at the simplest case of antiproton production $pp \to ppp\overline{p}X$. The threshold energy for this process can be calculated using the Mandelstam variable s. The minimal value for s is the sum of the masses of the final state particles squared $s^2 \geq 16m_p^2$. In the rest frame of one of the protons (target), the four-momenta of the projectile and the target are

$$
k^\mu_\mathrm{proj} = (m+T, \mathbf{k}), \quad k^\mu_\mathrm{target} = (m,0,0,0),
$$

where T is the kinetic energy of the projectile proton. The Mandelstam variable is

$$
s = (k_\mathrm{proj} + k_\mathrm{target})^2 = 2m^2 + 2mT.
$$

The threshold condition $s \geq 16m_p^2$ gives $T \geq 6m_p$. In the centre-of-mass frame, the four-momenta of the projectile and the target are

$$
k^\mu_\mathrm{proj} = (E^*, p^*, 0, 0), \quad k^\mu_\mathrm{target} = (E^*, -p^*, 0, 0),
$$

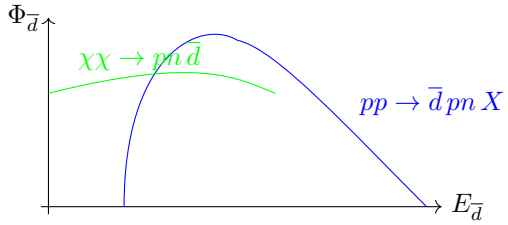
where E^* is such that $\sqrt{s} = 4m_p$ (the production threshold). This gives $E^* = 2m_p$. We can calculate the γ factor between the lab and the centre-of-mass frame from the relation

$$
\begin{pmatrix} E^* \\ -p^* \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix}.
$$

This gives $E^* = m\gamma$, hence $\gamma = 2$. To produce antideuterons, $pp \to p\overline{p}n\overline{n}ppX$, it has to hold $\sqrt{s} \geq 6m^2$ and, consequently,

$$
4m^2 + 2mT \geq 36m^2, \qquad T \geq 16m.
$$

In the centre of mass frame, it holds $E^* = 3m$ and hence the γ factor between the lab and the centre-of-mass frame is $\gamma = 3$. If the antideuteron is produced at rest in the centre-of-mass frame, the energy of the antiproton in the lab frame is $E = 3m$. From this we expect that there won't be many deuterons produced at low energy.



However, when dark matter particles decay into deuterons, they can be produced at low energies. The process $\chi\chi \to pn\overline{pn}$ can happen at $\sqrt{s} \geq 2m_d$ so the flux of antideuterons from dark matter annihilation is expected to be higher at low energies (green on the sketch above). However, the overall flux is expected to be very low, so a lot of effort is put into precisely calculating and measuring the production of antideuterons from colliding protons in order to be able to detect the smallest discrepancies that could point towards dark matter.

One of the leading models of antideuteron production is the *coalescence model*. In this model, an antiproton and an antineutron (produced from the dark matter) that are close enough in momentum space can fuse to form an antideuteron $\overline{pn} \to \bar{d}\gamma$. The binding energy for the deuteron is $B_{\bar{d}} = 2.2\,\mathrm{MeV}$, which is very small compared to the mass of the deuteron $m_{\bar{d}} = 1.88\,\mathrm{GeV}$. The probability of formation of an antideuteron from a dark matter annihilation is then given by

$$\langle X\bar{d}(k_{\bar{d}})|\chi\chi\rangle = \sum_{k_{\overline{p}},k_{\overline{n}}} \langle \bar{d}(k_{\bar{d}})|\overline{pn}\rangle \, \langle X\overline{p}(k_{\overline{p}})\overline{n}(k_{\overline{n}})\chi\chi\rangle \,.$$

In order for an antideuteron to form, the momenta of the antiproton and antineutron have to be close enough, as well as their coordinate positions. The momentum condition is usually modelled as $|\mathbf{k}_{\overline{p}} - \mathbf{k}_{\overline{n}}| < p_0$. Writing the same condition in terms of the relative four-momentum $\Delta^\mu = k^\mu_{\overline{p}} - k^\mu_{\overline{n}}$ gives $-\Delta^2 < p^2$. Total antideuteron production per event with available energy Q can be expressed as

$$N_{\bar{d}} = \int \mathrm{d}N_{\overline{p}}\,\mathrm{d}N_{\overline{n}}\,\theta(\Delta^2 + p_0^2) = \int \mathrm{d}^3\mathbf{k}_{\overline{p}}\,\mathrm{d}^3\mathbf{k}_{\overline{n}}\,\theta(\Delta^2 + p_0^2)\frac{\mathrm{d}^3N_{\overline{p}}\,\mathrm{d}^3N_{\overline{n}}}{\mathrm{d}^3\mathbf{k}_{\overline{p}}\,\mathrm{d}^3\mathbf{k}_{\overline{n}}}(k_{\overline{p}}, k_{\overline{n}}).$$

We can integrate this expression over all possible deuteron momenta and require the energy conservation $\delta(\mathbf{k}_{\bar{d}} - \mathbf{k}_{\overline{p}} - \mathbf{k}_{\overline{n}})$:

$$\begin{aligned}\frac{\mathrm{d}^3N_{\bar{d}}}{\mathrm{d}^3\mathbf{k}_{\bar{d}}} &= \int \mathrm{d}^3\mathbf{k}_{\overline{p}}\,\mathrm{d}^3\mathbf{k}_{\overline{n}}\,\theta(\Delta^2 + p_0^2)\frac{\mathrm{d}^3N_{\overline{p}}\,\mathrm{d}^3N_{\overline{n}}}{\mathrm{d}^3\mathbf{k}_{\overline{p}}\,\mathrm{d}^3\mathbf{k}_{\overline{n}}}(\mathbf{k}_{\overline{p}}, \mathbf{k}_{\overline{n}})\delta(\mathbf{k}_{\bar{d}} - \mathbf{k}_{\overline{p}} - \mathbf{k}_{\overline{n}}) \\ &= \int \mathrm{d}^3\mathbf{k}_{\overline{p}}\,\theta(\Delta^2 + p_0^2)|_{\mathbf{k}_{\overline{n}}=\mathbf{k}_{\bar{d}}-\mathbf{k}_{\overline{p}}}\frac{\mathrm{d}^3N_{\overline{p}}\mathrm{d}^3N_{\overline{n}}}{\mathrm{d}^3\mathbf{k}_{\overline{p}}\mathrm{d}^3\mathbf{k}_{\overline{n}}}(\mathbf{k}_{\overline{p}}, \mathbf{k}_{\bar{d}} - \mathbf{k}_{\overline{p}}).\end{aligned}$$

In the centre-of-mass frame, the Heaviside condition reads

$$(\Delta^2 + p_0^2)|_{\mathbf{k}_{\overline{n}}=\mathbf{k}_{\bar{d}}-\mathbf{k}_{\overline{p}}} = -|2\mathbf{k}_{\overline{p}} - \mathbf{k}_{\bar{d}}| + p_0^2.$$

This corresponds to a ball in the momentum space of radius $p_0/2$ located at $\mathbf{k}_{\bar{d}}/2$. In reality, this ball is very small, since p_0 is of order MeV and $k_{\bar{k}}$ is of order GeV or TeV. Therefore, we

can approximate the Heaviside condition with a delta function:

$$\frac{\mathrm{d}^3 N_{\overline{d}}}{\mathrm{d}^3 \mathbf{k}_{\overline{d}}} = \frac{\mathrm{d}^3 N_{\overline{p}} \mathrm{d}^3 N_{\overline{n}}}{\mathrm{d}^3 \mathbf{k}_{\overline{p}} \mathrm{d}^3 \mathbf{k}_{\overline{n}}} (\mathbf{k}_{\overline{d}}/2, \mathbf{k}_{\overline{d}}/2) \int \mathrm{d}^3 \mathbf{k}_{\overline{p}} \theta(\Delta^2 - p_0^2)|_{\mathbf{k}_{\overline{n}} = \mathbf{k}_{\overline{d}} - \mathbf{k}_{\overline{p}}}.$$

To make this Lorentz invariant, we multiply by the energy of the deuteron

$$E_{\overline{d}} \frac{\mathrm{d}^3 N_{\overline{d}}}{\mathrm{d}^3 \mathbf{k}_{\overline{d}}} = E_{\overline{p}} E_{\overline{n}} \frac{\mathrm{d}^3 N_{\overline{p}} \mathrm{d}^3 N_{\overline{n}}}{\mathrm{d}^3 \mathbf{k}_{\overline{p}} \mathrm{d}^3 \mathbf{k}_{\overline{n}}} (\mathbf{k}_{\overline{d}}/2, \mathbf{k}_{\overline{d}}/2) \underbrace{\frac{E_{\overline{d}}}{E_{\overline{p}} E_{\overline{n}}} \int \mathrm{d}^3 \mathbf{k}_{\overline{p}} \theta(\Delta^2 - p_0^2)|_{\mathbf{k}_{\overline{n}} = \mathbf{k}_{\overline{d}} - \mathbf{k}_{\overline{p}}}}_{V_{\mathrm{coal}}}.$$

The integral in the bracket is the coalescence volume V_{coal}. Since it is Lorentz invariant, we can evaluate it in the centre-of-mass frame

$$V_{\mathrm{coal}} = \frac{m_{\overline{d}}}{m_{\overline{p}} m_{\overline{n}}} \frac{\pi}{6} p_0^3.$$

Combining this result back to the previous expression gives

$$\gamma_{\overline{d}} \frac{\mathrm{d}^3 N_{\overline{d}}}{\mathrm{d}^3 \mathbf{k}_{\overline{d}}} = \gamma_{\overline{p}} \gamma_{\overline{n}} \frac{\mathrm{d}^3 N_{\overline{p}} \mathrm{d}^3 N_{\overline{n}}}{\mathrm{d}^3 \mathbf{k}_{\overline{p}} \mathrm{d}^3 \mathbf{k}_{\overline{n}}} (\mathbf{k}_{\overline{d}}/2, \mathbf{k}_{\overline{d}}/2) \cdot \frac{\pi p_0^3}{6}.$$

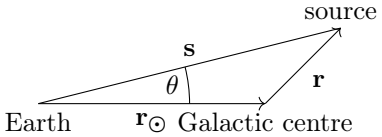
There have been several experiments trying to measure the coalescence momentum p_0 (BaBar ($B\overline{B}$), ALICE (pp)). The results vary quite a lot depending on the experiment and the process used to create the antideuterons. Seemingly, the coalescence momentum depends on the energy scale of the process. There are also two main experiments trying to measure the antideuteron and anti-helium flux in cosmic rays: GAPS and AMS-02. GAPS is a balloon-borne experiment that uses the exotic atom technique to identify antideuterons. AMS-02 is a magnetic spectrometer located on the International Space Station that identifies particles based on their charge-to-mass ratio.

3.3 Gamma rays

In this section, we will study the production and propagation of gamma rays in the galaxy, produced via the dark matter annihilation or decay in the centre of the Galaxy (called *prompt γ-rays*). The other sources are gamma rays produced via the inverse Compton scattering of electrons and from the dark matter halos of distant galaxies and galaxy clusters.

3.3.1 Prompt γ-rays

Assume the Sun is located at a distance $r_\odot$ from the Galactic centre. We want to calculate the flux of gamma rays coming from a direction that makes an angle θ with respect to the direction towards the Galactic centre. The distance from the Galactic centre to the source is $\mathbf{r}$ and the distance from the Sun to the source is $\mathbf{s}$.



The number of photons produced at $\mathbf{r}$ per unit energy and time is

$$\frac{\mathrm{d}N_{\mathrm{em}}}{\mathrm{d}E\,\mathrm{d}t} = Q(E,\mathbf{r}) = \frac{1}{2}\left(\frac{\rho(\mathbf{r})}{M}\right)^2 \sum_i \langle\sigma v_i\rangle \frac{\mathrm{d}N_i}{\mathrm{d}E}.$$

The number of photons received at Earth per area $\mathrm{d}S$ is

$$\frac{\mathrm{d}N_{\mathrm{rec}}}{\mathrm{d}t\,\mathrm{d}E} = \frac{\mathrm{d}S}{4\pi\mathbf{s}^2}\frac{\mathrm{d}N_{\mathrm{em}}}{\mathrm{d}E\,\mathrm{d}t}.$$

With the first equation, we can calculate the number of photons produced in the whole line of sight

$$\frac{\mathrm{d}N^{\mathrm{tot}}}{\mathrm{d}t\,\mathrm{d}E\,\mathrm{d}S} = \int \mathrm{d}V\,\frac{1}{4\pi s^2}\frac{\mathrm{d}N_{\mathrm{em}}}{\mathrm{d}E\,\mathrm{d}t} \quad\Longrightarrow\quad \frac{\mathrm{d}N^{\mathrm{tot}}}{\mathrm{d}t\,\mathrm{d}E\,\mathrm{d}S\,\mathrm{d}\Omega} = \int \frac{\mathrm{d}s}{4\pi}\frac{\mathrm{d}N_{\mathrm{em}}}{\mathrm{d}E\,\mathrm{d}t} = \frac{\mathrm{d}\Phi}{\mathrm{d}E\,\mathrm{d}\Omega}.$$

The differential flux of gamma rays from dark matter annihilation is then

$$\frac{\mathrm{d}\Phi}{\mathrm{d}E\,\mathrm{d}\Omega} = \frac{1}{8\pi}\frac{1}{M^2}\sum_i \langle\sigma v_i\rangle \frac{\mathrm{d}N_i}{\mathrm{d}E}\int \mathrm{d}s\,\rho^2(r(s,\theta)),$$

where

$$r(s,\theta) = \sqrt{s^2 + r_\odot^2 - 2sr_\odot\cos\theta}.$$

The term in front of the integral depends on the particle physics and the integrand depends on astrophysics (called *J-factor*). Astrophysicists usually use the galactic coordinates (ℓ, b). The galactic latitude b is the angle above the galactic plane and the galactic longitude ℓ is the angle in the galactic plane with respect to the direction towards the Galactic centre. The full galactic polar coordinates consist of three numbers (s,ℓ,b) and the translation to Cartesian coordinates is

$$\begin{aligned} x &= r_\odot - s\cos b\cos\ell,\\ y &= -s\cos b\sin\ell,\\ z &= s\sin b,\\ \mathrm{d}\Omega &= \mathrm{d}\cos b\,\mathrm{d}\ell. \end{aligned}$$

Usually, most of the contribution to the J-factor comes from the inner part of the Galaxy ($\theta < \theta_{\max}$), where the dark matter density is the highest. This corresponds to the solid angle

$$\Delta\Omega = 2\pi\int_0^{\theta_{\max}} \mathrm{d}\cos\theta.$$

The value of $\theta_{\max}$ can be taken to be around 0.1° and $J \sim (10^{22}\text{–}10^{25})\,\mathrm{GeV}^2\mathrm{cm}^{-5}$. The full flux is thus

$$\Phi = \int \mathrm{d}E\int \mathrm{d}\Omega\,\frac{\mathrm{d}\Phi}{\mathrm{d}E\,\mathrm{d}\Omega} \sim \frac{1}{8\pi M^2}\langle\sigma v\rangle N_\gamma^{\mathrm{prod}} J\Delta\Omega \sim N_\gamma^{\mathrm{prod}}(10^{-10}\text{–}10^{-7})\,\mathrm{cm}^{-3}\,\mathrm{s}^{-1}$$

and the total number of events is $N_{\mathrm{events}} = \Phi A_{\mathrm{eff}} T_{\mathrm{obs}}$. The overview of the current gamma-ray detectors is given in the table below.

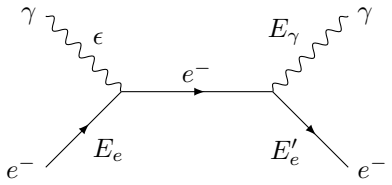
	Fermi-LAT	H.E.S.S.	CTA
Energy range	(0.1–300) GeV	(0.1–10) TeV	(0.01–10) TeV
A_{eff}	$\sim 1\,\text{m}^2$	$\sim 10^5\,\text{m}^2$	$\sim 10^6\,\text{m}^2$
T_{obs}	$10^8\,\text{s}$	$10^6\,\text{s}$	$10^6\,\text{s}$
N_{events}	10^2–10^5	10^5–10^8	10^6–10^9

The task of detecting with such big number of events seems very easy, but the main problem is the astrophysical background. The Galactic centre is a very active region with many gamma-ray sources (pulsars, supernova remnants, etc.). Therefore, it is very difficult to separate a possible dark matter signal from the astrophysical background, which can be much greater than the possible dark matter signal.

One of the biggest unknowns is the dark matter density profile in the inner part of the Galaxy. Different profiles can differ by several orders of magnitude in the Galactic centre, which translates directly to the uncertainty in the J-factor.

3.3.2 Inverse Compton scattering

The *inverse Compton scattering* (ICS) is the process where a high-energy electron scatters off a low-energy photon, transferring part of its energy to the photon. This process is very important for gamma-ray production in the Galaxy, as high-energy electrons produced from dark matter annihilation can up-scatter low-energy photons from the cosmic microwave background (CMB), starlight and infrared light to gamma-ray energies.



We will denote the energy of the incoming electron as E_e, the energy of the incoming photon as ϵ, the energy of the outgoing photon as E_γ and the energy of the outgoing electron as E'_e. We are interested in a few quantities, mainly the differential cross-section $\mathrm{d}\sigma_{\text{ICS}}/\mathrm{d}E_\gamma$, the flux of photons $cf_e(E_e, \mathbf{r})$ and the number density of target photons $f_{\text{ISRF}}(\varepsilon, \mathbf{r})$. The cross-section for ICS is given by the *Klein–Nishina formula*:

$$\frac{\mathrm{d}\sigma^{\text{IC}}(E_e, \varepsilon)}{\mathrm{d}E_\gamma} = \frac{3}{4}\frac{\sigma_{\text{T}}}{\gamma^2 \varepsilon}\left[2q\log q + 1 + q - 2q^2 + \frac{1}{2}\frac{(qT)}{1+q\Gamma}(1-q)\right],$$

where

$$\gamma = \frac{E_e}{m_e}, \quad q = \frac{E_\gamma}{\Gamma(E_e - E_\gamma)}, \quad \Gamma = \frac{4\gamma\epsilon}{m_e}, \quad \sigma_{\text{T}} = 0.67\,\text{barn}.$$

The rate at which photons are produced with energy E_γ at position $\mathbf{r}$ is given by

$$\frac{\mathrm{d}R_{\text{IC}}(\mathbf{r})}{\mathrm{d}E_\gamma} = \int_0^\infty \mathrm{d}\epsilon \int_{m_e}^\infty \mathrm{d}E_e\, f_e(E_e, \mathbf{r}) f_{\text{ISRF}}(\epsilon, \mathbf{r}) \frac{\mathrm{d}\sigma^{\text{IC}}(E_e, \epsilon)}{\mathrm{d}E_\gamma}.$$

The maximal energy that a photon can gain in the scattering is

$$E_\gamma^{\max} = \left(\frac{1}{E_e} + \frac{1}{4\gamma^2\epsilon}\right)^{-1}.$$

Different combinations of E_e and ϵ lead to different regimes of ICS.

Type of photon	ϵ	E_e	$E_\gamma^{\rm may}$
starlight	1 eV	1 TeV	941 GeV
IRRD	0.1 eV	1 TeV	616 GeV
CMB	10^{-3} eV	1 TeV	15 GeV

3.3.3 Annihilation at cosmological distances

Gamma rays can also be produced from dark matter annihilation in distant galaxies and galaxy clusters. The produced gamma rays then travel through the expanding universe and can be detected on Earth. They are produced in so-called halos, which we define as the regions where the density is more than 200 times the critical density, $\Delta(z) > 200$. The mass of the halo, assuming spherical symmetry and radius r, can be expressed as

$$M = \frac{4\pi}{3}\rho_{\rm c}(z)\Delta(z)r_\Delta^3.$$

The observed photon on the Earth will be observed with a different energy than emitted, $E_{\rm o} = E_\gamma/(1+z)$. The differential flux of photons produced by a halo of mass M at the comoving distance χ is

$$\frac{{\rm d}\Phi}{{\rm d}E_{\rm o}} = \frac{1}{4\pi\chi(z)^2}\int_0^{r_\Delta}{\rm d}r\,4\pi r^2\frac{1}{2}\left(\frac{\rho(r)}{M}\right)^2\sum_i\langle\sigma v\rangle_i\left.\frac{{\rm d}N}{{\rm d}E_\gamma}\right|_{E_\gamma=E_0(1+z)}.$$

We can express the mass of the halo in terms of the density profile in order to eliminate the need of the normalization constant in the density profile (such as NFW). Hence, we define a new dimensionless function

$$\xi_M^2(r_\Delta/r_{\rm s}) = \frac{M}{\Delta(z)\rho(z)}\int_0^{r_\Delta}\int_0^{r_\Delta}{\rm d}r\,4\pi r^2\left(\frac{\rho(r)}{M}\right)^2 = \frac{\int_0^{r_\Delta}4\pi r^2\rho(r)^2\,{\rm d}r}{\int_0^{r_\Delta}4\pi r^2\rho(r)\,{\rm d}r}\frac{1}{\Delta(z)\rho_{\rm c}(z)}.$$

Since the factor is dimensionless, it can depend only on the ratio between two-dimensional quantities, the halo radius r_Δ and the scale radius $r_{\rm s}$ of the matter distribution. The ratio $r_\Delta/r_{\rm s} = c$ is called the *concentration parameter*. If there is no over-density ($\Delta = 0$, $\rho = \rho_{\rm c}$), then $\xi_M^2 = 1$. Hence, ξ_M^2 measures the enhancement in the flux due to the DM over-density. The final expression for the flux from a single halo is

$$\frac{{\rm d}\Phi}{{\rm d}E_{\rm o}} = \frac{\Delta(z)\rho_{\rm c}(z)}{4\pi\chi^2(z)}\frac{\xi_M^2(c)}{m_{\rm DM}^2}\sum_i\frac{\langle\sigma v\rangle_i}{2}\left.\frac{{\rm d}N_i}{{\rm d}E_\gamma}\right|_{E_\gamma=E_{\rm o}(1+z)}.$$

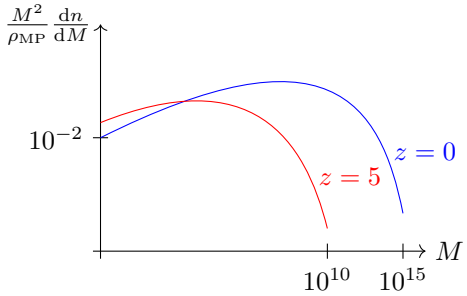
For a NFW profile,

$$\rho(r) = \frac{\rho_0}{\frac{r}{r_{\rm s}}\left(1+\frac{r}{r_{\rm s}}\right)^2},$$

the enhancement factor is

$$\xi_M^2(c) = \frac{c^3\left[1-\frac{1}{(1+c)^3}\right]}{9\left[\log(1+c)-\frac{c}{1+c}\right]^2}.$$

The distribution of masses of halos is sketched below.



Measurements show that the concentration follows a log-normal distribution

$$P(c) \propto \exp\left(-\frac{\log_{10}(c/c_\Delta)}{2\sigma^2_{\log c_\Delta}}\right).$$

Now, the total flux received on the Earth from the annihilation at cosmological distance χ is

$$\left.\frac{\mathrm{d}\Phi}{\mathrm{d}E_\mathrm{o}}\right|_\chi = \frac{1}{4\pi\chi(z)^2}\frac{1}{m^2_\mathrm{DM}}\Delta(z)\rho_\mathrm{c}(z)\int_{M_\mathrm{min}}^\infty \mathrm{d}M\,\frac{\mathrm{d}n}{\mathrm{d}M}M\int_0^\infty \mathrm{d}c\,P(c)\xi^2_M(c)\sum_i \frac{\langle\sigma v\rangle_i}{2}\left.\frac{\mathrm{d}N_i}{\mathrm{d}E_\gamma}\right|_{E_\gamma=E_\mathrm{o}(1+z)}.$$

We extract the astrophysical part into a single factor

$$\xi^2 = \frac{\Delta(z)\rho_\mathrm{c}(z)}{\rho_{\mathrm{DM},0}}\int_{M_\mathrm{min}}^\infty \frac{\mathrm{d}M}{\rho_{\mathrm{DM},0}}\frac{\mathrm{d}n}{\mathrm{d}M}M\int_0^\infty \mathrm{d}c\,P(c)\xi^2_M(c)$$

and write down the final expression for the flux from annihilation at cosmological distances

$$\left.\frac{\mathrm{d}\Phi}{\mathrm{d}E_\mathrm{o}}\right|_\chi = \frac{1}{4\pi\chi^2}\frac{\rho^2_{\mathrm{DM},0}}{m^2_\mathrm{DM}}\xi^2(\chi)\sum_i \frac{\langle\sigma v\rangle_i}{2}\left.\frac{\mathrm{d}N_i}{\mathrm{d}E_\gamma}\right|_{E_\gamma=E_\mathrm{o}(1+z)}.$$

The factor ξ is called the *clumping factor*, and it measures the enhancement in the flux due to the presence of halos compared to a smooth dark matter distribution. At redshifts $z \in [0, 5]$ it takes a value $\xi^2 \sim 10^5$. For the total photon flux, we need to integrate over all redshifts

$$\frac{\mathrm{d}\Phi_\mathrm{tot}}{\mathrm{d}E_\mathrm{o}\,\mathrm{d}\Omega} = \int_0^\infty \mathrm{d}\chi\,\xi^2\left.\frac{\mathrm{d}\Phi}{\mathrm{d}E_\mathrm{o}}\right|_\chi.$$

All we now need to do is combine the astrophysical quantities, expressed with the redshift, with the particle physics quantities expressed with the comoving distance. For this, we take the FLRW metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(\mathrm{d}\chi^2 + \chi^2\mathrm{d}^2\Omega).$$

For the radially moving photons, it holds $\mathrm{d}s^2 = 0$, $\mathrm{d}\Omega = 0$ and $\mathrm{d}t = a(t)\mathrm{d}\chi$. Hence,

$$\mathrm{d}z = \mathrm{d}(1+z) = \mathrm{d}\frac{a_0}{a(t)} = -\frac{a_0}{a^2}\dot{a}\,\mathrm{d}t = -(1+z)H(z)\,\mathrm{d}t$$

and

$$\mathrm{d}\chi = -\frac{\mathrm{d}t}{a(t)} = \frac{dz}{a(t)(1+z)H(z)} = \frac{\mathrm{d}z}{H(z)}.$$

With this we can obtain the following formula

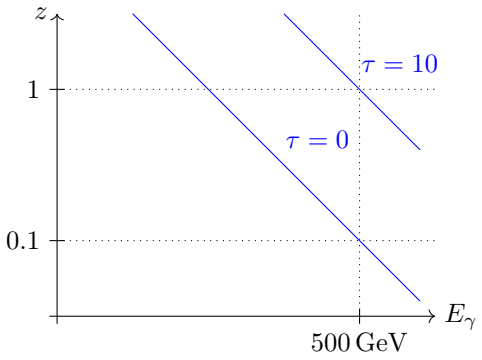
$$\left.\frac{\mathrm{d}\Phi}{\mathrm{d}E_\mathrm{o}}\right|_\chi = \frac{1}{4\pi}\frac{\rho^2_{\mathrm{DM},0}}{m^2_\mathrm{DM}}\sum_i \frac{\langle\sigma v\rangle_i}{2}\int_0^\infty \frac{\mathrm{d}z}{H(z)}\xi^2(z)\left.\frac{\mathrm{d}N_i}{\mathrm{d}E_\gamma}\right|_{E_\gamma=E_\mathrm{o}(1+z)}.$$

However, this is not the end of the story. These photons have to go a very long distance before the detection, and they can interact with the CMB radiation to produce an electron-positron pair. Assume we have a CMB photon with energy ϵ and a photon from DM annihilation coming at an angle θ. The production threshold is $s = (2m_e)^2 = 2E_\gamma\epsilon(1+\cos\theta)$. This can be expressed as

$$E_\gamma \sim \frac{2m_e^2}{\epsilon} \approx 1\,\mathrm{TeV}\left(\frac{\epsilon}{0.5\,\mathrm{eV}}\right)^{-1}.$$

This can be modelled by a simple attenuation factor, the optical thickness $\tau(E_\gamma, z)$, which gives the final correction to the flux formula

$$\left.\frac{\mathrm{d}\Phi}{\mathrm{d}E_\mathrm{o}}\right|_\chi = \frac{1}{4\pi}\frac{\rho_{\mathrm{DM},0}^2}{m_\mathrm{DM}^2}\sum_i \frac{\langle\sigma v\rangle_i}{2}\int_0^\infty \frac{\mathrm{d}z}{H(z)}\xi^2(z)\frac{\mathrm{d}N_i}{\mathrm{d}E_\gamma}e^{-\tau(E_\gamma,z)}\Big|_{E_\gamma=E_\mathrm{o}(1+z)}.$$



We can also look into other galaxies. The most promising are dwarf galaxies. One example is Segue 1 in the constellation of Leo. It is a dwarf spheroidal galaxy comprised of about 1000 stars. What is remarkable about it is that it has a very high mass-to-light ratio of about 3400 $M_\odot/L_\odot$, which makes it the most dark matter dominated galaxy known. When we look at the galaxy in the X-ray spectrum, we can see that there is very little emission, which means that there is very little astrophysical background. This set bounds on the dark matter annihilation cross-sections.

The observations from within our galaxy are a bit more complicated. There are many astrophysical sources of gamma rays, such as pulsars, blazars, AGNs, bremsstrahlung, pion decay, inverse Compton scattering, uniform extra-galactic background, Fermi bubbles and. Subtracting these backgrounds from the observed flux gives a spherically symmetric excess of gamma rays in the Galactic centre, that has energy spectrum strongly resembling dark matter annihilation into b quarks. Similar spectra are observed in other galaxies as well. However, a correlated signal is expected from dwarf galaxies, which is not observed. Moreover, the distribution can also be explained with a population of unresolved millisecond pulsars that come from tidally disrupted globular clusters at the close fly-by of the galactic centre. There are some experiments planned that could resolve the galactic centre better.

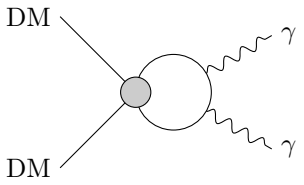
Particle physics tells us that the decay of dark matter into photons creates a rather narrow bump in the spectrum. The idea is to take the background spectrum of gamma radiation (which is approximately power-like) and find a bump in it. The main issue is that the power spectrum is not exactly power-like, so one first fits the background with a more complicated function and then focuses on the bump region. For maximizing the sensitivity, one uses only the region of the

sky that excludes the galactic disk (but includes the core). One can focus even more tightly to the galactic centre and two lobes above and below.

3.3.4 γ-ray production from dark matter

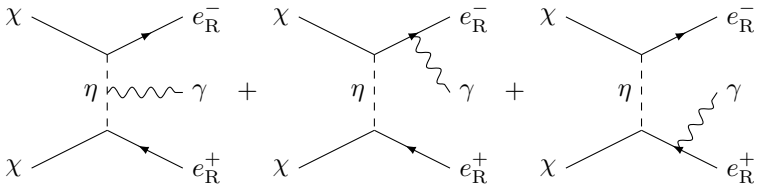
There are three ways to produce gamma rays from dark matter annihilation or decay: the decay into photons, the internal bremsstrahlung and the decay of the annihilation products.

- Since the dark matter is neutral, the annihilation DM + DM $\to \gamma + \gamma$ arises only at loop level.



  The emission lines are very distinct, and the cross-section is very small due to the loop suppression $\alpha^2/(16\pi^2)$: $\langle\sigma v\rangle \sim 10^{-30}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$.

- The internal bremsstrahlung arises when a charged particle in the final (or internal) state emits a photon.



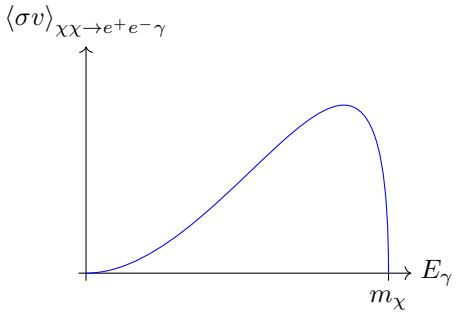
  The s-wave is suppressed by m_e^2/M_{DM}^2 in the final state, but the internal bremsstrahlung lifts this suppression. The internal propagator contributes a factor $1/(p_\eta^2 - m_\eta^2)$, where

$$p_\eta^2 = (p_\chi - p_e)^2 = m_\chi^2 + m_e^2 - 2p_\chi p_e \approx m\chi^2 - 2m_\chi E_e.$$

  The amplitude thus scales as

$$\mathcal{A}(\chi\chi \to e^+e^-\gamma) \sim \frac{1}{m_\chi^2 - m_\eta^2 - 2m_\chi E_e}.$$

  The regime of interest is when $m_\eta \sim m_\chi$. In this case the amplitude is enhanced for $E_e \to m_\chi$, which corresponds to high-energy photons ($E_\gamma \to m_\chi$).



The cross-section into three particles is suppressed by α/π compared to the two-particle final state.

- The dark matter can annihilate into other particles that can subsequently decay or hadronize into photons. For example, the annihilation into quarks leads to the production of neutral pions, which then decay into photons: $\pi^0 \to \gamma\gamma$. This process generally produces a box-shaped spectrum of gamma rays.

Chapter 4

Direct dark matter detection

The idea is that there is a halo of dark matter in the Galaxy and the Solar System is moving through it with velocity $v = 290\,\mathrm{km\,s^{-1}}$. Therefore, there is a flux of dark matter particles passing through the Earth. If a dark matter particle scatters off a nucleus in a detector, it can transfer some of its kinetic energy to the nucleus, causing it to recoil. The recoil can cause photon production or the increase of the temperature. The interaction can also produce other particles in the dark and visible sector. Recently, people also started considering the possibility of scattering off electrons: $\mathrm{DM} + e^- \to \mathrm{DM} + e^-$. A possible consequence of this scattering is the excitation or ionization of atoms in the detector. Hence, the options are:

(i) elastic scattering: $\mathrm{DM} + N \to \mathrm{DM} + N$,

(ii) inelastic scattering: $\mathrm{DM} + N \to \mathrm{DM}^* + N$,

(iii) production of other particles: $\mathrm{DM} + N \to \mathrm{DM} + N + X$,

(iv) ionization: $\mathrm{DM} + e^- \to \mathrm{DM} + e^-$,

(v) excitation: $\mathrm{DM} + A \to \mathrm{DM} + A^* \to A + \gamma$.

The main issue with the detection of these processes is the background.

(i) Cosmic ray interactions ($pN, \mu N, \nu N$): this can be avoided by going deep underground and/or shielding the detector with additional material. However, it is practically impossible to get rid of neutrino interactions.

(ii) Radioactive decays in the laboratory and the detector itself. This can be reduced by using very pure materials and shielding the detector and combining this with scintillation and ionization signals to distinguish between nuclear recoils and electron recoils.

Two main experiments searching for direct detection of dark matter are XENONnT and Lux-ZEPLIN.

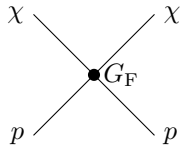
Let's estimate the event rate in a direct detection experiment. It is given by

$$R \sim \text{flux} \times \sigma \times \text{number of targets}.$$

Assume that the dark matter interacts with the proton with the Fermi interaction:

$$\sigma_{\chi p} \sim \frac{G_\mathrm{F}^2}{8\pi}\mu^2$$

where $\mu = m_p m_\chi/(m_p + m_\chi)$ is the reduced mass of the DM-proton system. For very heavy dark matter, $\mu \approx m_p$.



Using the values of $G_F \sim 10^{-5}\,\mathrm{GeV}^{-2}$, $m_p \sim 1\,\mathrm{GeV}$ and $\mathrm{GeV} \sim 2 \cdot 10^{-14}\,\mathrm{cm}^{-1}$, the cross-section is approximately

$$\sigma_{\chi p} \sim 4 \cdot 10^{-12}\,\mathrm{GeV}^{-2} \sim 10^{-39}\,\mathrm{cm}^2 \sim 10^{-3}\,\mathrm{pb}.$$

For the dark matter particle of mass $m_\chi \sim 100\,\mathrm{GeV}$ and the velocity $v \sim 200\,\mathrm{km\,s}^{-1} \sim 6 \cdot 10^{-4}\,c$, the kinetic energy of the particle is only

$$T \sim \frac{1}{2} m v^2 \sim 20\,\mathrm{keV} \sim (10^{-9}\,\mathrm{cm})^{-1}.$$

The size of the nucleus is much smaller than the de Broglie wavelength of the dark matter particle, so the scattering is coherent over the whole nucleus ($R \sim 10^{-13}\,\mathrm{m}\,A^{1/3}$). Therefore, the cross-section for scattering off a nucleus with mass number A is enhanced by a factor of A^2 (the amplitude is enhanced by A from the superposition of the contributions from all nucleons):

$$\sigma_{\chi N} \sim A^2 \sigma_{\chi p}.$$

For example, for $A \sim 100$, we have $\sigma_{\chi N} \sim 10\,\mathrm{pb}$.

This was the calculation of the cross-section. Now, we need to calculate the flux. It can be expressed as

$$\mathrm{flux} = \frac{\rho_{\mathrm{DM}}}{m_\chi} v,$$

where $\rho_{\mathrm{DM}} \sim 0.3\,\mathrm{GeV\,cm}^{-3}$ is the local dark matter density. For $m_\chi \sim 100\,\mathrm{GeV}$ and $v \sim 6 \cdot 10^{-4}\,c$, the flux is approximately $\mathrm{flux} \sim 6 \cdot 10^5\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$. This is quite a big number – in every second there are hundreds of thousands of dark matter particles passing through every square centimetre.

To calculate the rate, we will estimate the number of target nuclei a human. Mass of one person is 70 kg. For a very rough estimate, we will say that human is mostly made of carbon, which has atomic mass of $2 \cdot 10^{-26}\,\mathrm{kg}$. This gives 10^{27} nuclei in a human body. The rate is

$$\mathrm{rate} = 10^{-37}\,\mathrm{cm}^2 \times 6 \cdot 10^5\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1} \times 10^{-27} \sim 6 \cdot 10^{-5}\,\mathrm{s}^{-1} \sim 1 \text{ interaction per day.}$$

Hence, if DM interacts with protons with the weak interaction, then my body interacts with DM once per day! Of course, we interact with environmental cosmic rays and radioactive decay products much more frequently. The actual experiments run for a period of a year. These experiments haven't detected anything yet, so

$$\mathrm{rate} = 6 \cdot 10^5\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1} \times \sigma \times 1\,\mathrm{tonne}/(135\,M_{\mathrm{p}}) < 1$$

where $135\,M_{\mathrm{p}}$ is the mass of a xenon nucleus. This gives the bound on the cross-section

$$\sigma_{\chi N} < 10^{-41}\,\mathrm{cm}^2, \quad \sigma_{\chi p} < 10^{-45}\,\mathrm{cm}^2 = 1\,\mathrm{zb(zepto\ barn)}.$$

4.1 Nuclear recoils

Assume that all DM particles move with velocity $\mathbf{v}$. The differential number of nuclear recoils with energy E_R at time t is

$$\left.\frac{\mathrm{d}N}{\mathrm{d}E_\mathrm{R}}\right|_{\sigma \text{ fixed}} = N_\mathrm{T} t \frac{\rho_\mathrm{loc}}{M_\mathrm{DM}} v \frac{\mathrm{d}\sigma}{\mathrm{d}E_\mathrm{R}}(v, E_\mathrm{R}),$$

where N_T is the number of target nuclei in the detector and $\mathrm{d}\sigma/\mathrm{d}E_\mathrm{R}$ is the differential cross-section for DM-nucleus scattering. For a continuous distribution of velocities, we get

$$\left.\frac{\mathrm{d}N}{\mathrm{d}E_\mathrm{R}}\right|_{\sigma \text{ fixed}} = \int \mathrm{d}^3\mathbf{v}\, \frac{M_\mathrm{T}}{m_\mathrm{N}} t \frac{\rho_\mathrm{loc}}{M_\mathrm{DM}} v \frac{\mathrm{d}\sigma}{\mathrm{d}E_\mathrm{R}}(v, E_\mathrm{R}) f(\mathbf{v}),$$

where m_T is the total mass of the target and m_N is the mass of one target nucleus. We define the exposure $\varepsilon = M_\mathrm{T} t$ (usually given in tone-years). Moreover, we define the recoil rate normalized to unit target mass

$$\frac{\mathrm{d}R}{\mathrm{d}E_\mathrm{R}} = \frac{\rho_\mathrm{loc}}{m_\mathrm{DM} m_\mathrm{N}} \int_{v_\mathrm{min}}^{v_\mathrm{max}} \mathrm{d}^3\mathbf{v}\, f(\mathbf{v}) v \frac{\mathrm{d}\sigma}{\mathrm{d}E_\mathrm{R}}(v, E_\mathrm{R}).$$

The total number of events can be obtained from

$$N = \int_{E_\mathrm{R}^\mathrm{min}}^{E_\mathrm{R}^\mathrm{max}} \mathrm{d}E_\mathrm{R}\, \frac{\mathrm{d}R}{\mathrm{d}E_\mathrm{R}} \cdot \varepsilon.$$

The bounding values of velocity v_min and v_max are given by the kinematics of the scattering. Let $\mathbf{p}$ be the initial dark matter momentum in centre-of-mass system and $\mathbf{p'}$ the final momentum. The momentum of the nucleus will be denoted by $\mathbf{k}_\mathrm{N}$. In centre-of-mass, it holds $\mathbf{P} = \mathbf{p} + \mathbf{k}_\mathrm{N} = 0$ and the momentum transfer is defined via $\mathbf{q} = \mathbf{p}' - \mathbf{p} = \mathbf{k}_\mathrm{N} - \mathbf{k}'_\mathrm{N}$. The incoming momentum of the dark matter can also be expressed as $\mathbf{p} = -\mathbf{k}_\mathrm{N} = \mu\mathbf{v}$ where $\mathbf{v}$ is the DM velocity in the lab frame. For the outgoing momenta it holds

$$\mathbf{p}' = \mathbf{p} + \mathbf{q} = \mathbf{q} + \mu\mathbf{c}.$$

The momentum transfer can be estimated as

$$|\mathbf{q}|^2 = |\mathbf{p}|^2 + |\mathbf{p}'|^2 - 2\mathbf{p}\mathbf{p}' = 2|\mathbf{p}|^2(1 - \cos\theta).$$

The last equality holds for an elastic scattering $|\mathbf{p}| = |\mathbf{p}'|$. Thus,

$$\frac{|\mathbf{q}|^2}{2}\mu^2 v^2 (1 - \cos\theta).$$

In the lab frame, the momentum of the DM particle is $\mathbf{p}_\mathrm{lab}$ and the nucleus is stationary. The recoil momenta will be denoted by prime. Hence, $\mathbf{q} = \mathbf{k}_\mathrm{lab} - \mathbf{k}'_\mathrm{lab} = -\mathbf{k}'_\mathrm{lab}$. Hence, the modulus of the transferred momentum is the same as the recoil momentum of the nucleus in the lab frame. The recoil energy is

$$E_\mathrm{R} = \frac{1}{2}\frac{|\mathbf{k}'_\mathrm{lab}|^2}{m_\mathrm{N}} = \frac{\mu^2 v^2}{m_\mathrm{N}}(1 - \cos\theta) < \frac{2\mu^2 v^2}{m_\mathrm{N}}.$$

The velocity is hence bounded by below as

$$v > \sqrt{\frac{E_\mathrm{R} m_\mathrm{N}}{2\mu^2}}.$$

The upper bound is simply the escape velocity of the galaxy $v_{\rm esc} \sim 600\,{\rm km\,s^{-1}}$. The final expressions for the differential detection rate, the total rate and the number of events are

$$\frac{{\rm d}R}{{\rm d}E_{\rm R}} = \frac{\rho_{\rm loc}}{m_{\rm DM} m_{\rm N}} \int_{\sqrt{m_{\rm N} E_{\rm R}/(2\mu^2)}}^{v_{\rm esc}} {\rm d}^3\mathbf{v}\, f(\mathbf{v}) v \frac{{\rm d}\sigma}{{\rm d}E_{\rm R}}(v, E_{\rm R});$$
$$R = \int_{E_{\rm M}}^{\infty} {\rm d}E_{\rm R}\, \frac{{\rm d}R}{{\rm d}E_{\rm R}}; \qquad N = R\varepsilon.$$

The local density of the dark matter around the Earth is $\rho_{\rm loc} = 0.3\,{\rm GeV\,cm^{-3}}$. The velocity distribution is usually modelled as a Maxwell-Boltzmann distribution with a cut-off

$$f_{\rm halo}(\mathbf{v}) = \frac{1}{(2\pi\sigma_{\rm v})^{3/2} N_{\rm nor}} \exp\left(-\frac{|\mathbf{v}|^2}{2\sigma_v^2}\right)$$

for $v < v_{\rm esc} \sim 600\,{\rm km\,s^{-1}}$. The value for the velocity dispersion is $\sigma_v \sim 150\,{\rm km\,s^{-1}}$. The normalization constant $N_{\rm R}$ is defined by

$$\int {\rm d}^3\mathbf{v}\, f(\mathbf{v}) = 1.$$

However, this distribution holds in the galactic frame – for the observer on the Earth we have to correct for the Earth's motion in the Galaxy, $f_{\rm detector}(\mathbf{v}) = f_{\rm halo}(\mathbf{v} + \mathbf{v}_{\rm Earth}(t))$, where the Earth's velocity consists of three contributions

$$\mathbf{v}_{\rm Earth} = \mathbf{v}_{\rm LSR} + \mathbf{v}_{\rm pec} + \mathbf{v}_{\oplus,\odot}$$

where $\mathbf{v}_{\rm LSR}$ is the local standard of rest, $\mathbf{v}_{\rm pec}$ is the peculiar velocity and $\mathbf{v}_{\oplus,\odot}$ is the velocity of the Earth around the Sun. It is convenient to express these velocities in the galactic rectangular coordinates

- x: direction from the Sun to the galactic centre,
- y: direction of the galactic rotation,
- z: direction towards the north galactic pole.

In these coordinates, the velocity of the local standard of rest is $\mathbf{v}_{\rm LSR} = (0, v_0, 0)$ with $v_0 = 220\,{\rm km\,s^{-1}}$. The peculiar velocity of the Sun is $\mathbf{v}_{\rm pec} = (11.1, 12.2, 7.3)\,{\rm km\,s^{-1}}$. The Earth's orbital velocity around the Sun is

$$\mathbf{v}_{\oplus,\odot} = 29.8\,{\rm km\,s^{-1}}\,(\sin(\omega t), \cos(\omega t), 0)$$

where $\omega = 2\pi/(1\,{\rm year})$. The scattering amplitude is given by

$$\mathcal{M}(q^2) \sim \int {\rm d}^3\mathbf{r}\, e^{i\mathbf{q}\cdot\mathbf{r}} \langle f|V|i\rangle$$

where $\langle f|$ and $|i\rangle$ are the final and initial states, and V is the interaction potential between the dark matter and the nucleus

$$V(\mathbf{r}) = \sum_{\text{nucleus } N} \delta(\mathbf{r} - \mathbf{r}_N)\left(G_s^N + G_a^N \boldsymbol{\sigma}_N \cdot \boldsymbol{\sigma}_\chi\right).$$

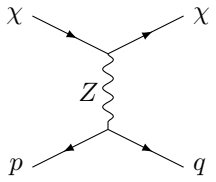
The first term corresponds to the spin-independent (independent under 3D rotations) and the second to the spin-dependent (axial). The amplitude can further be decomposed as $\mathcal{M}(q)^2 = \mathcal{M}(0)F(q^2)$. The form factor $F(q^2)$ takes into account the finite size of the nucleus. There are two options:

(i) DM interacts with the quarks directly: we need an effective theory for small q^2,

(ii) quarks are inside the nucleon (p or n): we need to calculate the matrix elements $\langle N|\bar{q}\Gamma q|N\rangle$,

(iii) nucleons inside the nucleus: we need nuclear matrix elements $\langle A|\sum_N \bar{N}\Gamma N|A\rangle$.

Example 2. Consider the interaction between a heavy Dirac fermion (like heavy neutrino) scattering with quarks via Z-boson exchange.



The interaction between the Z boson and the quarks is known, for the interaction with the dark matter we assume the splitting between axial and vector couplings:

$$\mathcal{M} = \sum_q \sqrt{2}G_{\mathrm{F}}[\overline{\chi}(V_\chi - A_\chi\gamma^5)\gamma_\mu\chi][\overline{q}(V_q - A_q\gamma^5)\gamma_\mu q]$$

where $V_q = T_{3q} - 2e_q \sin^2\theta_{\mathrm{W}}$, $A_q = T_{3q}$ and V_χ, A_χ are free parameters (unknown). For the interaction between DM and the nucleon, we separately consider vector and axial part. For the vector part, we have

$$\mathcal{M}_{\mathrm{V}} = \sum_q \sqrt{2}G_{\mathrm{F}}V_\chi V_q = [\overline{\chi}\gamma_\mu\chi][\overline{q}\gamma^\mu q].$$

When we put the amplitude between the initial and final states of the nucleon, we get

$$\langle p\chi|\mathcal{M}_{\mathrm{V}}|p\chi\rangle = \sqrt{2}G_{\mathrm{F}}V_\chi \langle\chi|\overline{\chi}\gamma_\mu\chi|\chi\rangle \sum_q V_q \langle p|\overline{q}\gamma^\mu q|p\rangle .$$

The term with dark matter particle is easy to interpret:

$$\langle\chi|\overline{\chi}\gamma_\mu\chi|\chi\rangle = \overline{u}_\chi(P')\gamma_\mu u_\chi(P).$$

The nucleon part is more complicated – one has to write a general ansatz with form factors

$$\langle p|\overline{q}\gamma^\mu q|p\rangle = \overline{u}_p(k')\left[F^{(p,q)}(Q^2)\gamma^\mu + \frac{i}{2}F_2^{(p,q)}(Q^2)\sigma^{\mu\nu}Q_\nu\right]u_p(k) \approx F_1^{(p,q)}(0)\overline{u}_p(k)\gamma^\mu u_p(k).$$

Here, k, k' denote the proton momentum, q denotes quark and $Q = k' - k$ is the momentum transfer. The second term is the Pauli form factor, which is suppressed for small momentum transfer. With this simplification, the expression becomes

$$\langle p\chi|\mathcal{M}_{\mathrm{V}}|p\chi\rangle = \sqrt{2}G_{\mathrm{F}}V_\chi \sum_q V_q[\overline{u}_\chi(p')\gamma^\mu u_\chi(p)][\overline{u}_p(k')\gamma_\mu u_p(k)].$$

The sum over quark vector couplings can be expressed in terms of proton (and neutron) couplings:

$$V_p = \sum_q V_q = \sum_q (T_{3q} - 2e_q \sin^2\theta_{\rm W}) = (1/2 - 2\sin^2\theta_{\rm W}).$$

From this we see $T_{3p} = 1/2$ and $e_p = +1$. Define

$$G_{\rm S}^p = \frac{G_{\rm F}}{\sqrt{2}}(1 - 4\sin^2\theta_{\rm W})V_\chi.$$

In order to connect this formula to the previous one, we expand the chiral representation of a spinor in non-relativistic limit

$$u^{(s)}(p) = \begin{pmatrix} \sqrt{p^\mu\sigma_\mu}\chi^{(s)} \\ \sqrt{p^\mu\overline{\sigma}_\mu}\chi^{(s)} \end{pmatrix} \approx \sqrt{m}\begin{pmatrix} (1 - \mathbf{p}\cdot\boldsymbol{\sigma}/(2m))\chi^{(s)} \\ (1 + \mathbf{p}\cdot\boldsymbol{\sigma}/(2m))\chi^{(s)} \end{pmatrix}$$

where $\sigma^\mu = (1, \boldsymbol{\sigma})$, $\overline{\sigma}^\mu = (1, -\boldsymbol{\sigma})$ and $\chi^{(1)} = (1,0)^T, \chi^{(2)} = (0,1)^T$. Using this, we can calculate

$$\overline{u}(p)\gamma^\mu u(p) = (2m, \mathbf{p} + 2i\mathbf{Q}\times\boldsymbol{\sigma}) \approx (2m, 0)$$

and the scattering amplitude simplifies dramatically,

$$\langle p\chi|M_V|p\chi\rangle = G_{\rm S}^p 4 m_\chi m_p.$$

Similarly, for the neutron we have

$$\langle n\chi|M_V|n\chi\rangle = G_{\rm S}^n 4 m_\chi m_n,$$

where

$$G_{\rm S}^n = \sqrt{2}G_{\rm F}V_\chi \sum_{udd} V_q F_i^{(n,q)}(Q^2).$$

The vector effective coupling of a neutron is $V_n = V_u + 2V_d = -1/2$. Now to the axial part. The amplitude is

$$\mathcal{M}_{\rm A} = -\sum_q \sqrt{2}G_{\rm F}A_\chi A_q[\overline{\chi}\gamma_\mu\gamma^5\chi][\overline{q}\gamma^\mu\gamma^5 q].$$

Putting this between the nucleon states, we get

$$\langle p\chi|\mathcal{M}_{\rm A}|p\chi\rangle = \sqrt{2}G_{\rm F}A_\chi \langle\chi|\overline{\chi}\gamma_\mu\gamma^5\chi|\chi\rangle \sum_q A_q \langle p|\overline{q}\gamma^\mu\gamma^5 q|p\rangle .$$

The dark matter part can be calculated as

$$\langle\chi|\overline{\chi}\gamma_\mu\gamma^5\chi|\chi\rangle = \overline{u}_\chi(P')\gamma_\mu\gamma^5 u_\chi(P).$$

For the nucleon part, we need to introduce axial form factors:

$$\begin{aligned}\langle p|\overline{q}\gamma^\mu\gamma^5 q|p\rangle &= \overline{u}_p(k')\left[F_A^{(p,q)}(Q^2)\gamma^\mu\gamma^5 + \frac{1}{2m_p}F_{A'}^{(p,q)}(Q^2)\sigma^{\mu\nu}\gamma^5 Q^\nu\right]u_p(k) \\ &\approx F_A^{(p,q)}(0)\overline{u}_p(k)\gamma^\mu\gamma^5 u_p(k).\end{aligned}$$

Returning to the amplitude, we have

$$\langle p\chi|\mathcal{M}_{\rm A}|p\chi\rangle \approx \sqrt{2}G_{\rm F}A_\chi \sum_q A_q F_{\rm A}^{(p,q)}(0)[\overline{u}_\chi(p')\gamma^\mu\gamma^4 u_\chi(p)][\overline{u}_p(k')\gamma_\mu\gamma^5 u_p(k)].$$

The sum can be written as

$$\sum_q A_q F_A^{(q,p)} = A_u \Delta_u^p + A_d \Delta_d^p + A_s \Delta_s^p \approx 0.68,$$

where Δ_q^p is the fraction of the proton spin carried by quark q. The values are $\Delta_u^p = 0.81$, $\Delta_d^p = -0.44$ and $\Delta_s^p = -0.11$. Interestingly, strange quark has a non-negligable contribution. These are obtained from observations of neutron decays, hyperon decays and $\nu p \to \nu X$. As before, we define

$$G_{\mathrm{A}}^p = \sqrt{2} G_{\mathrm{F}} A_\chi (A_u \Delta_u^p + A_d \Delta_d^p + A_s \Delta_s^p)$$

to get a more compact expression for the amplitude:

$$\langle p\chi | \mathcal{M}_{\mathrm{A}} | p\chi \rangle = G_{\mathrm{A}}^p [\overline{u}_\chi(p') \gamma^\mu \gamma^5 u_\chi(p)][\overline{u}_p(k') \gamma_\mu \gamma^5 u_p(k)].$$

In non-relativistic limit, we have $\overline{u}(p')\gamma^\mu\gamma^5 u(p) \approx (0, 2m\boldsymbol{\sigma})$. Thus, the amplitude becomes

$$\langle p\chi | \mathcal{M}_{\mathrm{A}} | p\chi \rangle = 4 m_\chi m_p G_{\mathrm{A}}^p (\boldsymbol{\sigma}_\chi \cdot \boldsymbol{\sigma}_p).$$

For the neutron, just replace $p \to n$.

Combining both vector and axial part together, we get exactly the spin-dependent and spin-independent terms we expected from before

$$\langle N\chi | \mathcal{M} | N\chi \rangle = 4 m_\chi m_N \left(G_{\mathrm{S}}^N + G_{\mathrm{A}}^N \boldsymbol{\sigma}_\chi \cdot \boldsymbol{\sigma}_N \right).$$

The mixed terms (axial from one and vector from the other particle) cancel perfectly because they are the product of terms with different parity.

Now, we can combine multiple nucleons inside the nucleus. The differential interaction rate is only slightly modified:

$$\frac{\mathrm{d}R}{\mathrm{d}E_{\mathrm{R}}} = \frac{\rho_{\mathrm{loc}}}{m_{\mathrm{DM}} m_{\mathrm{T}}} \int_{\sqrt{M_{\mathrm{N}} E_{\mathrm{R}}/(2\mu^2)}}^{v_{\mathrm{esc}}} \mathrm{d}\mathbf{v}\, f(\mathbf{v}) v \frac{\mathrm{d}\sigma_{\text{DM-T}}}{\mathrm{d}E_{\mathrm{R}}}(\mathbf{v}, E_{\mathrm{R}})$$

where m_{T} is the mass of the target. The scattering amplitude splits into spin-independent and spin-dependent parts in the same manner as in the nucleon case:

$$\mathcal{M}(q^2) \sim \int \mathrm{d}^3\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r}} \langle f | V | i \rangle \sum_{\text{nucleons } i} \delta(\mathbf{r} - \mathbf{r}_i) \left[G_{\mathrm{S}}^i + G_{\mathrm{A}}^i \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_\chi \right].$$

First, we need to embed quarks into nucleons, and then nucleons into the nucleus (target). For the spin-independent part, we introduce the nucleon mass form factor $F_{\mathrm{mass}}(q^2)$ so that

$$\overline{|\mathcal{M}(q^2)|} = |Z G_{\mathrm{S}}^p + (A - Z) G_{\mathrm{s}}^n|^2 |F_{\mathrm{mass}}(q^2)|^2.$$

Nucleon scatterings off of nuclei suggest that the mass form factor coincides well with the charge form factor $F_{\mathrm{mass}}(q^2) \equiv F_{\mathrm{em}}(q^2)$. The latter can be calculated from the electric charge distribution of a nucleus (*Woods–Saxon* distribution):

$$\rho(r) = \rho_0 \frac{1}{1 + \exp((r - R)/a)},$$

for which the form factor is

$$F(q^2) = \frac{1}{A} \int \mathrm{d}^3\mathbf{r}\, \rho(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} = \frac{3 j_1(qR)}{qR} e^{-(qa)^2/2}$$

where j_1 is the spherical Bessel function of the first kind, $R \approx 1.2\,\mathrm{fm}\,A^{1/3}$ is the effective nuclear radius and $a \approx 0.9\,\mathrm{fm}$ is the nuclear skin thickness.

For the spin-dependent part, we have to consider the evenness of nucleon spins inside the nucleus. For zero momentum transfer, the amplitude is

$$|\mathcal{M}(0)|^2 = \begin{cases} \frac{4(j+1)}{j}|G_\mathrm{A}^p \langle S_p\rangle + G_\mathrm{A}^n \langle S_n\rangle|^2, & \text{for even-odd nuclei} \\ 0, & \text{for } j \text{ even-even nuclei} \end{cases}.$$

The numerical values for the expected values of the spin are $\langle S_p\rangle \sim 0.0$ and $\langle S_n\rangle \sim 0.3$ (further digits depend on the type of nucleus). For ^{129}Xe, the spin-dependent scattering is dominated by scattering with the neutrons:

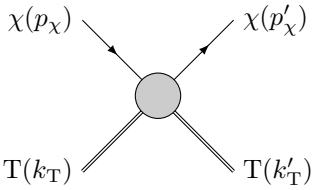
$$|\mathcal{M}(q^2)|^2 = \frac{j(j+1)}{f}\left[(G_\mathrm{A}^p + G_\mathrm{A}^n)\langle S_p + S_n\rangle F_\mathrm{spin}^0(q^2) + (G_\mathrm{A}^p - G_\mathrm{A}^n)\langle S_p - S_n\rangle F_\mathrm{spin}^\mathrm{i}(q^2)\right]$$

where $F_\mathrm{spin}^\mathrm{i}(q^2)$ is the spin-dependent form factor for the isospin. The form factor can be approximated as

$$F_\mathrm{spin}^0 \sim \exp(-r_0^2 q^2), \quad F_\mathrm{spin}^\mathrm{i} \sim \exp(-r_\mathrm{i}^2 q^2 + icq),$$

where $r_0 \sim 1.0\,\mathrm{fm}$, $r_\mathrm{i} \sim 1.2\,\mathrm{fm}$ and $c_\mathrm{i} \sim 0.3\,\mathrm{fm}$.

In summary, we are trying to calculate the differential cross-section for DM-nucleus scattering.



The differential cross-section is

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{4E_\chi E_\mathrm{T} v}(2\pi)^4\delta^{(4)}(p_\chi + k_\mathrm{T} - p'_\chi - k_\mathrm{T}) \underbrace{\frac{\mathrm{d}^3\mathbf{p}'_\chi}{(2\pi)^3 2E'_\chi}\frac{\mathrm{d}^3\mathbf{k}'_\mathrm{T}}{(2\pi)^3 2E'_\mathrm{T}}}_{\mathrm{d}_\mathrm{LIPS}} = \frac{|\mathcal{M}|^2}{16\pi s}\frac{\mathrm{d}\Omega}{4\pi}.$$

We denoted the Lorentz-invariant phase space as d_LIPS and the differential solid angle as $\mathrm{d}\Omega = 2\pi\mathrm{d}\cos\theta$. Since we want to calculate the differential with respect to the recoil energy E_R, we write

$$E_\mathrm{R} = \frac{\mu^2 v^2}{M_\mathrm{T}}(1-\cos\theta) \quad \Longrightarrow \quad \left|\frac{\mathrm{d}E_\mathrm{R}}{\mathrm{d}\cos\theta}\right| = \frac{\mu^2 v^2}{m_\mathrm{T}}.$$

Thus, the cross-section comes to the form

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{32\pi s}\frac{m_\mathrm{T}}{\mu^2 v^2}\mathrm{d}E_\mathrm{R}.$$

Since $s \approx (m_\mathrm{T} + m_\chi)^2$ for small momentum transfer and $\mu = m_\mathrm{T} m_\chi/(m_\mathrm{T} + m_\chi)$, we get

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{32\pi}\frac{1}{m_\chi^2 m_\mathrm{T}}\frac{1}{v^2}\mathrm{d}E_\mathrm{R}.$$

For further calculations, we will split the cross-section into spin-independent and spin-dependent parts. For the former, we can re-express the amplitude as the amplitude at zero momentum transfer times the form factor $|\mathcal{M}(q^2)|^2 = |\mathcal{M}(0)|^2 |F_{\text{mass}}(q^2)|^2$. At zero momentum transfer, we have

$$\sigma_0^{\text{SI}} = \int_{E_{\text{min}}}^{E_{\text{max}}} \mathrm{d}E_{\text{R}} \frac{|\mathcal{M}(0)|^2}{32\pi} \frac{1}{m_\chi^2 m_{\text{T}}} \frac{1}{v^2} = \frac{|M(0)|^2}{32\pi} \frac{1}{m_\chi^2 m_{\text{T}}} \frac{1}{v^2} (E_{\text{max}} - E_{\text{min}}) = \frac{|M(0)|^2}{32\pi} \frac{1}{m_\chi^2 m_{\text{T}}} \frac{1}{v^2} \frac{2\mu^2 v^2}{m_{\text{T}}}.$$

Thus, at any momentum transfer, the differential cross-section (for the spin independent part) is

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}E_{\text{R}}}\right)_{\text{SI}} = \sigma_0^{\text{SI}} \frac{m_{\text{T}}}{2\mu^2 v^2} F_{\text{SI}}^2(E_{\text{R}}).$$

The spin-independent (SI) part of the interaction between the dark matter and the nucleon is controlled by G_{s}^p and G_{s}^n:

$$\mathcal{M} \sim 4 m_\chi m_p |Z G_{\text{s}}^p + (A - Z) G_{\text{s}}^n|, \quad |\mathcal{M}|^2 = 16 m_\chi^2 m_p^2 |Z G_{\text{s}}^p + (A - Z) G_{\text{s}}^n|^2 F_{\text{SI}}^2(E_{\text{R}}).$$

Therefore, we obtained the formula

$$\boxed{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}E_{\text{R}}}\right)_{\text{SI}} = \frac{m_{\text{T}}}{2\pi v^2} |Z G_{\text{s}}^p + (A - Z) G_{\text{s}}^n|^2 F_{\text{SI}}^2(E_{\text{R}}).}$$

We can modify this equation slightly, and introduce the DM-to-proton scattering cross-section $\sigma_{\chi p,0}^{\text{SI}} = \mu^2 |G_{\text{s}}^p|^2 / \pi$ and obtain

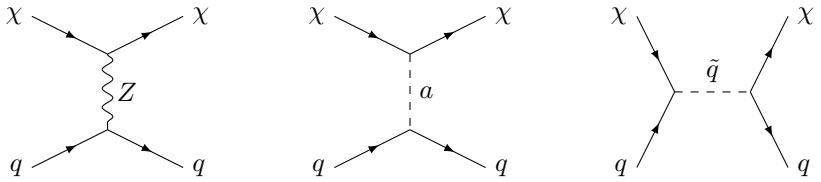
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}E_{\text{R}}}\right)_{\text{SI}} = \frac{m_{\text{T}}}{2\mu_p^2 v^2} |Z + (A - Z) G_{\text{s}}^n / G_{\text{s}}^p|^2 \sigma_{\chi p,0}^{\text{SI}} F_{\text{SI}}^2(E_{\text{R}}).$$

From the theory point of view, it is best to keep G_{s}^p and G_{s}^n as free parameters of the model (as in the boxed equation). However, for experimentalists, $\sigma_{\chi p,0}^{\text{SI}}$ and $G_{\text{s}}^n / G_{\text{s}}^p$ are more useful. There are many models that simplify the analysis. Experimentalists assume $G_{\text{s}}^n / G_{\text{s}}^p = 1$ (*isoscalar interaction*), so the only remaining parameter is $\sigma_{\chi p,0}^{\text{SI}}$. But this is a very strong assumption, because it also implies spin-independent interaction only. One other possible assumption (or model) is $G_{\text{s}}^n / G_{\text{s}}^p = Z/(Z - A)$ which implies constant spin-independent cross-section. Another model assumes the interaction of dark matter only with u and d quarks, so the scattering is dominated by neutron interactions and $G_{\text{s}}^p = 0$.

Let's now return to the spin-dependent part

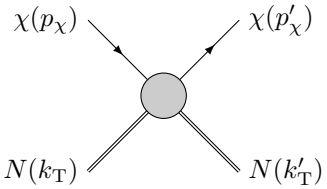
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}E_{\text{R}}}\right)_{\text{SD}} = |\langle S_p \rangle + \langle S_n \rangle G_{\text{a}}^n / G_{\text{a}}^p|^2 \frac{J_{\text{T}} + 1}{J_{\text{T}}} \frac{2 m_p}{3\mu_p^2 v^2} \sigma_{\chi p,0}^{\text{SD}} F_{\text{SD}}^2(E_{\text{R}}).$$

Comparing with the spin-dependent part, we replace $Z \to \langle S_p \rangle$ and $(A - Z) \to \langle S_n \rangle$ (we are scattering spins instead of masses). Possible spin-dependent interactions are depicted below.



The model on the right includes a supersymmetric particle $\tilde{q}$ (which now seems less and less likely as supersymmetry has not yet been detected). The model in the middle includes a pseudoscalar (which is an approach from this century) but the coupling scheme is a bit different and our previous discussion doesn't apply.

People consider a non-relativistic effective field theory (NREFT) of dark matter to nucleon interaction:



$$\mathcal{L}_\mathrm{int} = (\overline{\chi}\mathcal{O}_\chi\chi)(\overline{N}\mathcal{O}_N N),$$

similar to Fermi 4-interaction. There are many possibilities for operators $\mathcal{O}_\chi$ and $\mathcal{O}_N$, but they are restricted by the following.

(A) Galilean invariance: $\mathcal{O}$ can only depend on quantities that are Galilean invariant: $\mathbf{q} = \mathbf{p}' - \mathbf{p} = \mathbf{k}' - \mathbf{k}$ and $\mathbf{v} = \mathbf{v}_\chi - \mathbf{v}_N$.

(B) Energy-momentum conservation: $E_\mathrm{in} = E_\mathrm{out}$, so

$$\frac{1}{2}\mu\mathbf{v}^2 = \frac{1}{2}\mu(\mathbf{v} + \mathbf{q}/\mu)^2 \quad \Longrightarrow \quad \mathbf{v}\cdot\mathbf{q} = -\mathbf{q}^2/(2\mu).$$

Note also that $\mathbf{p} = m\chi\mathbf{V} + \mu\mathbf{v}$, $\mathbf{k} = m_N\mathbf{V} - \mu\mathbf{v}$ and $\mathbf{V} = 0$ in CMS.

(C) Hermiticity: the Hermitian conjugation changes initial for the final state. Hence, when we conjugate $\hat{\mathbf{q}} = \hat{\mathbf{p}}' - \hat{\mathbf{p}}$, we get $\hat{\mathbf{q}}^\dagger = \hat{\mathbf{v}} - \hat{\mathbf{v}}' = -\hat{\mathbf{q}}$. Thus, $\hat{\mathbf{q}}$ is anti-Hermitian and non-relativistic EFT has to be constructed using $i\hat{\mathbf{q}}$ as an operator, which is Hermitian. What about the operator constructed from the relative velocity $\mathbf{v}$. When we conjugate, we get $\mathbf{v}+\mathbf{q}/\mu$, which is not the same. Instead, we use the *perpendicular velocity* $\mathbf{v}^\perp = \mathbf{v}+\mathbf{q}/(2\mu)$. It holds $\mathbf{v}^\perp\cdot\mathbf{q} = 0$ and $\hat{\mathbf{v}}_\perp^\dagger = \hat{\mathbf{v}}_\perp$.

Thus, the list of quantities that are Hermitian and can be used to construct the NREFT are

$$i\hat{\mathbf{q}}, \hat{\mathbf{v}}^\perp, \hat{\mathbf{S}}_\chi, \hat{\mathbf{S}}_N.$$

A possible object that can enter our Hamiltonian is, for example, $\hat{\mathbf{S}}_N\cdot\hat{\mathbf{v}}^\perp$. When listing all the possible operators, we will classify them according to their discrete symmetries.

	T	P
$i\hat{\mathbf{q}}$	+	−
$\hat{\mathbf{v}}^\perp$	−	−
$\hat{\mathbf{S}}_\chi$	−	+
$\hat{\mathbf{S}}_N$	−	+

Let's list all possible operators:

- P-even:
  - T-even:
    - Spin-independent: $\mathbb{1}$, $\hat{\mathbf{v}}^\perp\cdot\hat{\mathbf{v}}^\perp$, $i\hat{\mathbf{S}}_N\cdot(\hat{\mathbf{q}}\times\hat{\mathbf{v}}^\perp)$.

    * Spin-dependent: $\hat{\mathbf{s}}_\chi \cdot \hat{\mathbf{S}}_N$, $\hat{\mathbf{S}}_\chi \cdot (\hat{\mathbf{q}} \times \hat{\mathbf{v}}^\perp)$, $(\hat{\mathbf{S}}_\chi \cdot \hat{\mathbf{q}})(\hat{\mathbf{S}}_N \cdot \hat{\mathbf{q}})$.
  - T-odd:
    * Spin-independent: $i\hat{\mathbf{S}}_N \cdot \mathbf{q}$.
    * Spin-dependent: $i\hat{\mathbf{S}}_\chi \cdot \mathbf{q}$.
- P-odd, T-even:
  - Spin-independent: $\hat{\mathbf{S}}_N \cdot \hat{\mathbf{v}}^\perp$.
  - Spin-dependent: $\hat{\mathbf{S}}_\chi \cdot \hat{\mathbf{v}}^\perp$, $i\hat{\mathbf{S}}_\chi \cdot (\hat{\mathbf{S}}_N \times \hat{\mathbf{q}})$.

The spin-dependent parts are only those that include $\hat{\mathbf{S}}_\chi$. In the Hamiltonian, we want to combine terms up to $\mathbf{q}^2$ and $\mathbf{q} \cdot \mathbf{v}^\perp$:

$$(\hat{\mathbf{S}}_\chi \cdot \hat{\mathbf{q}})(\hat{\mathbf{S}}_N \cdot \hat{\mathbf{v}}^\perp), \quad (\hat{\mathbf{S}}_\chi \cdot \hat{\mathbf{v}}^\perp)(\hat{\mathbf{S}}_N \cdot \hat{\mathbf{q}}), \quad (\hat{\mathbf{S}}_\chi \cdot \hat{\mathbf{q}})(\hat{\mathbf{S}}_N \times \hat{\mathbf{v}}^\perp) \cdot \hat{\mathbf{q}}, \quad \hat{\mathbf{S}}_\chi \cdot (\hat{\mathbf{S}}_N \times \hat{\mathbf{v}}^\perp).$$

There are in total 15 operators and the Hamiltonian is

$$\hat{H} = \sum_{i=1}^{15} c_i^n \hat{O}_i + c_i^p \hat{\mathcal{O}}_i.$$

Omitting the identity, there are 14 non-trivial operators and 28 different couplings (for each operator one for proton and one for neutron).

Consider an interaction with a pseudoscalar a (right diagram on the sketch above):

$$\mathcal{L} = -\frac{ig_\chi}{\sqrt{2}} a \overline{\chi} \gamma^5 \chi - i \sum_q g_q a \overline{q} \gamma^5 q.$$

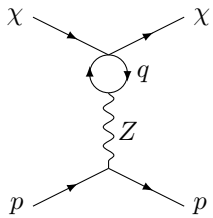
The operator has the form

$$\mathcal{O} = \frac{1}{\Lambda^2} (\overline{\chi} i \gamma^5 \chi)(\overline{q} i \gamma^5 q),$$

where Λ is the EFT scale. Doing the full computation gives that the interaction with pseudoscalar is suppressed by q^4:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_\mathrm{R}} = \frac{1}{128\pi} \frac{q^4}{\Lambda^4} \frac{m_\mathrm{T}}{m_\chi^2 m_N^2} \frac{1}{v^2} \sum_{N,N'} g_N g'_N F^{(N,N')}(q^2).$$

Current experiments are so sensitive that they could detect even 1-loop processes in the dark sector:



$\sim 10^{-45}\,\mathrm{cm}^2$.

Another possibility is the tree-level Higgs boson exchange, which is also of order $10^{-45}\,\mathrm{cm}^2$ if the coupling between Higgs and dark matter is of order $\mathcal{O}(1)$. A very recent development (from last week, January 2026) is a possibility that a recoil from an interaction between a light dark

matter particle and a nucleus kicks the nucleus out of the electron cloud and consequently ionizes the atom or produces scintillation in the material. It was first proposed by Migdal in 1939 and first reported on 15 January 2026. Assuming the velocity of the dark matter particle is smaller than the escape velocity of the Galaxy, we can infer the lower bound for the dark matter particle mass.

4.2 Dark matter in the Sun

In might happen, that χ loses enough energy in scatterings with nuclei in the stellar interior, that it becomes gravitationally bound to the star ($v < v_{\mathrm{esc}}$). Upon subsequent interactions, it keeps losing energy and alls to the centre. We could therefore have a concentration of dark matter particles in the core of the Sun. These particles can then annihilate into SM particles (e^-, γ, p, ν), which can be detected on Earth. As before, we have the following relation between the differential interaction rate and the differential cross-section:

$$\frac{\mathrm{dRate}}{\mathrm{d}E_{\mathrm{R}}} = \#\mathrm{targets} \times \mathrm{flux} \times \frac{\mathrm{d}\sigma}{\mathrm{d}E_{\mathrm{R}}} = \eta_i(r) \times \frac{\rho_\chi}{m_\chi} w(r) \times \frac{\mathrm{d}\sigma}{\mathrm{d}E_{\mathrm{R}}},$$

where $\eta_i(r)$ is the number density of target nuclei of type i at radius r inside the Sun and depends on the stellar structure. If we assume the velocity of the dark matter particles far away from the Sun is u, then the velocity at radius r inside the Sun is

$$w(r) = \sqrt{u^2 + v_{\mathrm{esc}}^2(r)}, \quad v_{\mathrm{esc}}^2(r) = \frac{2GM(r)}{r} = \frac{8\pi G}{r} \int_0^r \mathrm{d}r'\, r'^2 \rho(r').$$

We want to compute the rate of capture of dark matter particles with speed u at infinity, after colliding with a nucleus of type i at radius r inside the Sun:

$$\Omega = \int_{E_{\mathrm{R}}^{\min}}^{E_{\mathrm{R}}^{\max}} \eta_i(r) \frac{\rho_\chi}{m_\chi} w(r) \frac{\mathrm{d}\sigma}{\mathrm{d}E_{\mathrm{R}}}(E_{\mathrm{R}}, w(r)).$$

The limits of integration are determined by the kinematics of the scattering. The minimum recoil energy is simply the energy of a particle at infinity (as we require the energy after the collision to be less than zero), $E_{\mathrm{R}}^{\min} = mu^2/2$. The maximum energy that can be transferred in an elastic collision is

$$E_{\mathrm{R}} = \frac{\mu_i^2 w(r)^2}{m_i}(1 - \cos\theta) < \frac{2\mu_i^2 w(r)^2}{m_i} = E_{\mathrm{R}}^{\max}.$$

Now we integrate over the velocity distribution of dark matter particles far away from the Sun:

$$C_i(r) = \int \mathrm{d}^3\mathbf{w}\, \Omega(\mathbf{r}, \mathbf{w}) f(u(\mathbf{w})) = \int \mathrm{d}u\, uw f(u) \Omega(r, w(u)) = 4\pi \int_{u<u_{\max}} \mathrm{d}^3 u\, f(u) \frac{w}{u} \Omega(r, w).$$

To get the bound for $u_{\max}$, we return to the bound for the recoil energy $mu^2/2 < E_{\mathrm{R}} < 2\mu_i^2(u^2 + v_{\mathrm{esc}}^2)/m_i$, so it follows that

$$u_{\max} = v_{\mathrm{esc}}(r) \frac{2m_i m_\chi}{|m_i - m_\chi|}.$$

Finally, we integrate over the volume of the Sun and sum over all nuclear species to get the total capture rate:

$$C = \sum_i \int_0^{R_\odot} 4\pi r^2\, \mathrm{d}r \cdot \int \mathrm{d}^3\mathbf{u}\, \frac{f(\mathbf{u})}{|\mathbf{u}|}(u^2 + v_{\mathrm{esc}}^2) \int_{m_\chi u^2/2}^{2\mu_i^2(u^2+v_{\mathrm{esc}}^2)/m_i} \mathrm{d}E_{\mathrm{R}}\, \frac{\mathrm{d}\sigma}{\mathrm{d}E_{\mathrm{R}}}(E_{\mathrm{R}}, \sqrt{u^2 + v_{\mathrm{esc}}^2}) \eta_i(r) \frac{\rho_\chi}{m_\chi},$$

where, as before,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_\mathrm{R}} = \frac{m_i}{2\mu_i^2 w^2(r)}(\sigma_\mathrm{SI}^p F_{\mathrm{SI},i}^2(E_\mathrm{R}) + \sigma_\mathrm{SD}^p F_{\mathrm{SD},i}^2(E_\mathrm{R})).$$

The distribution of dark matter particles follows the Maxwell-Boltzmann distribution:

$$f_\chi(\mathbf{v}, \mathbf{r}) \propto e^{-m_\chi v^2/(2T(r))} e^{-m_\chi \phi(r)/T(r)}.$$

The nuclei in the Sun also have some velocity distribution, which can also kick already trapped dark matter particles out of the solar potential – evaporation with rate Γ_e (this is more relevant for light dark matter and is negligible for $m_\chi > 5\,\mathrm{GeV}$). The rate of change of the number of dark matter particles in the Sun is

$$\frac{\mathrm{d}N_\chi}{\mathrm{d}t} = C - \Gamma_\mathrm{e} - 2\Gamma_\mathrm{ann} \approx C - 2\Gamma_\mathrm{ann}.$$

The annihilation rate is given by

$$\Gamma_\mathrm{ann} = \int \frac{1}{2}\rho \cdot (\rho v)\sigma \,\mathrm{d}^3\mathbf{r} = \frac{1}{2}\sigma v \frac{\int \rho^2 \,\mathrm{d}V}{\left(\int \rho \,\mathrm{d}V\right)^2} N^2 = \frac{1}{2} C_\mathrm{A} N^2,$$

where

$$C_\mathrm{A} \sim 1.5 \cdot 10^{52}\,\mathrm{s}^{-1} \left(\frac{\sigma v}{3 \cdot 10^{-26}\,\mathrm{cm}^3\,\mathrm{s}^{-1}}\right) \left(\frac{m_\chi}{\mathrm{TeV}}\right)^{3/2}.$$

To make further calculations, we need to know the distribution of dark matter particles inside the Sun. The number density can be approximated as

$$n(r) = n_0 e^{-m_\chi \phi(r)/T(r)}, \quad \phi(r) = 4\pi \int_0^r \frac{GM(r')\,\mathrm{d}r'}{r'^2}.$$

This can be well modelled with

$$n(r) \propto e^{-r^2/R^2}, \quad R \approx 0.01\,R_\odot \left(\frac{100\,\mathrm{GeV}}{m_\chi}\right)^{1/2}.$$

The dynamics of the number of dark matter particles in the governed by the annihilation and capture processes:

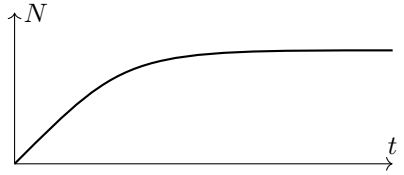
$$\frac{\mathrm{d}N_\chi}{\mathrm{d}t} = C - C_\mathrm{A} N_\chi^2.$$

The solution to this equation is

$$N_\chi(t) = \sqrt{\frac{C}{C_\mathrm{A}}} \tanh\left(\sqrt{C C_\mathrm{A}}\, t\right)$$

with the stationary solution $N_\chi = \sqrt{C/C_\mathrm{A}}$. The timescale to reach the stationary solution is

$$\tau = (C C_\mathrm{A})^{-1/2} \propto \rho_\mathrm{loc}^{-3/2} (\sigma v)^{-1/2} \sigma_{\chi N}^{-1/2}.$$



For spin independent cross-sections we find the equilibrium time to be much smaller than the age of the Sun $\tau_{\mathrm{SI}} \ll t_\odot$, while for spin-dependent interactions, the equilibrium time is comparable to the age of the Sun $\tau_{\mathrm{SD}} \sim t_\odot$.

Now, we will calculate the flux of annihilation products $(e^+, e^-, p, \gamma, \nu)$ at Earth:

$$\Phi = \frac{C_{\mathrm{A}} N^2/2}{4\pi d^2} = \frac{1}{8\pi d^2} C \tanh^2 \frac{t}{\tau} < \frac{C}{8\pi d^2}.$$

This process was once used to explain the solar neutrino problem, but now we know that neutrino oscillations are responsible for the deficit of solar neutrinos. The issue with detecting solar neutrinos is the large background from cosmic rays interacting with the Earth's atmosphere. However, the atmospheric neutrinos are isotropic (coming equally from all directions) while the solar neutrinos come from a specific direction (the Sun).

4.2.0Dark matter capture in neutron stars

Neutron stars are very compact objects with masses of order $M_{\mathrm{NS}} \sim 1.4 M_\odot$ and radii of order $R_{\mathrm{NS}} \sim 10\,\mathrm{km}$. The escape velocity from the surface is about $v_{\mathrm{esc}} \sim 0.6c$ (compare with the Sun, $618\,\mathrm{km\,s^{-1}}$) which requires a relativistic treatment of the capture process. The cross-section $\sigma_{\chi N}$ is comparatively large, and the capture rate is given by

$$C \approx \pi R^2 \frac{\rho_\chi}{m_\chi} \overline{v} \left(1 + \frac{v_{\mathrm{esc}}^2(r)}{\overline{v}^2}\right) \approx 1.9 \cdot 10^{25}\,\mathrm{s^{-1}} \left(\frac{\mathrm{GeV}}{m_\chi}\right)$$

where

$$\overline{v} \sim 220\,\mathrm{km\,s^{-1}}$$

is the velocity dispersion of dark matter particles in the Galaxy. The effective cross-section for scattering off neutrons is limited by the geometric cross-section of the neutron star, which is amplified by the gravitational focusing factor $1 + v_{\mathrm{esc}}^2/\overline{v}^2 \sim 10^6$. This approximation holds for $\sigma_{\chi n} \gtrsim 1 \cdot 10^{-45}\,\mathrm{cm^2}$. The number of dark matter particles in the neutron star evolves according to

$$\frac{\mathrm{d}N_\chi}{\mathrm{d}t} = C - C_{\mathrm{A}} N_\chi^2 = C - 2\Gamma_{\mathrm{ann}}.$$

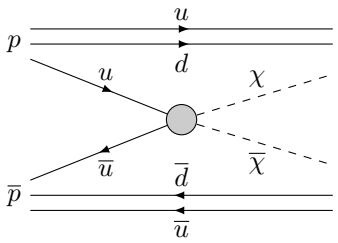
For a steady state, we have $\Gamma_{\mathrm{ann}} < 10^{25}\,\mathrm{s^{-1}}(1\,\mathrm{GeV}/m_\chi)$. The number of produced SM particles is huge, but the flux on Earth is small due to the large distance to neutron stars, $\Phi \propto d^{-2}$. Another way to detect dark matter in neutron stars is to measure their temperature. The capture of dark matter particles leads to heating of the neutron star. The standard astrophysics predicts that the neutron star cools down in about $10^5\,\mathrm{s}$ to about $1000\,\mathrm{K}$. The additional heating power from dark matter annihilation is

$$L \sim \Gamma_{\mathrm{ann}} 2 m_\chi \varepsilon,$$

where ε is the fraction of energy that is deposited in the neutron star (some energy is lost to neutrinos escaping the star). The energy lost in the outer space due to the black body radiation is governed by the Stefan–Boltzmann law, $L = 4\pi R^2 \sigma_{\mathrm{SB}} T^4$. If dark matter is captured and annihilates, then neutron star should not have a temperature smaller than $10^3\,\mathrm{K}$. However, this is very difficult to measure. Some hope is offered by the James Webb Space Telescope.

4.3 Dark matter searches at colliders

Consider a collision between a proton and anti-proton. One quark-antiquark pair could interact to produce a pair of dark matter particles, while the other (spectator) quarks would hadronize and produce jets. Measuring the momenta of the jets, we could infer the presence of dark matter particles from the missing transverse energy.



There is also the possibility of producing neutrinos, which would also escape the detector without interaction.

4.4 Axions

Axions were originally proposed to solve the strong-CP problem in QCD. The QCD Lagrangian contains a term that violates the CP symmetry (called the θ-term):

$$\mathcal{L}_{\text{QCD}} = \sum_q \overline{q}(i\not{D} - m_q)q - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + \frac{\theta}{32\pi^2}\tilde{G}^a_{\mu\nu}G^{\mu\nu}_a$$

where $\tilde{G}^a_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}G^{\mu\nu}_a$. The θ-term is the only CP-violating term in the QCD Lagrangian and contributes to the electric dipole moment (EDM) of the neutron:

$$d_n \simeq 5 \cdot 10^{-16}\,\theta r\,\text{cm}.$$

Experimental measurements of the neutron EDM give an upper limit $d_n < 1.8 \cdot 10^{-26}\,e\,\text{cm}$, which implies $\theta r < 10^{-10}$. A solution to this problem was proposed by Peccei and Quinn in 1977, who introduced a new global $U(1)_{\text{PQ}}$ *Peccei–Quinn symmetry* that is spontaneously broken at some high energy scale f_a. The breaking of this symmetry gives rise to a pseudo-Nambu–Goldstone boson, the axion a, which is massless if the symmetry is exact. However, this $\text{U}(1)_{\text{PC}}$ is anomalous under QCD, which gives the axion a small mass due to non-perturbative QCD effects:

$$m_a \sim \frac{\Lambda^3_{\text{QCD}}}{f_a} \sim 0.7\,\text{eV}\left(\frac{10^7\,\text{GeV}}{f_A}\right).$$

This gives new terms in the Lagrangian that dynamically cancel the θ-term, solving the strong-CP problem. The axion couples very weakly to standard model particles, with interaction strength suppressed by the large scale f_a. The axion potential is periodic, with formula

$$V(a) = m_a^2 f_a^2(1 - \cos(a/f_a)).$$

The minimum of the potential is at $a = 0$, which corresponds to $\theta = 0$ and thus solves the strong-CP problem. The relevant interaction Lagrangian terms for the interaction between the axion and standard model particles are:

$$\mathcal{L}_{af\overline{f}} = g_f\frac{m_f}{f_a}a\overline{f}\gamma^5 f, \quad \mathcal{L}_{a\gamma\gamma} = g_\gamma\frac{\alpha_{\text{EM}}}{\pi}\frac{a}{f_a}F_{\mu\nu}\tilde{F}^{\mu\nu}, \quad \mathcal{L}_{agg} = g_g\frac{\alpha_{\text{S}}}{8\pi}\frac{a}{f_a}\,\text{tr}(\mathbf{G}\tilde{\mathbf{G}}).$$

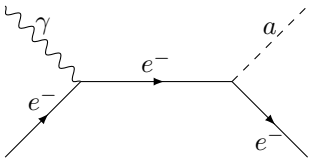
The axion decay time into two photons is

$$\tau_{a\to 2\gamma} = \frac{16\pi^2}{\alpha^2}\frac{\Lambda_{\mathrm{QCD}}^4}{m_a^5} \sim 10^{24}\,\mathrm{eV}\left(\frac{1\,\mathrm{eV}}{m_a}\right)^5.$$

For the axion to be a good dark matter candidate it has to be:

- electrically and colour neutral – true,
- slow at the time of structure formation – true for $m_a \gtrsim 1\cdot 10^{-22}\,\mathrm{eV}$,
- stable on cosmological timescales – true for $m_a \lesssim 25\,\mathrm{eV}$,

The mass can also be constrained by looking at stars and supernovae. Axions can be produced in stellar interiors via the Compton scattering or bremsstrahlung:



The luminance produced by axions in stars is

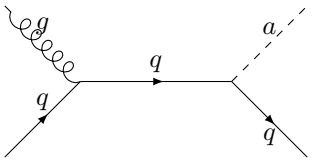
$$L_a \sim 6\cdot 10^{-4}\left(\frac{m_a}{1\,\mathrm{eV}}\right)^2 L_\odot.$$

This effect is even more pronounced in supernovae, where axions can be produced via nucleon bremsstrahlung $NN \to NNa$. The energy loss due to axion emission would shorten the duration of the neutrino pulse observed from supernovae. A rough estimate for neutrino and axion luminosities is

$$L_a \sim 10^{49}\,\mathrm{erg/s}\left(\frac{m_a}{1\,\mathrm{eV}}\right)^2, \quad L_\nu \sim 10^{53}\,\mathrm{erg/s}.$$

Hence, assuming $L_a < L_\nu$, the constraint on the axion mass is $m_a \lesssim 10^{-3}\,\mathrm{eV}$.

In the early universe, axions could be thermally produced via the following processes:



The cross-section for this process is approximately

$$\sigma_{qg} \sim |\mathcal{M}|^2 \sim |g_{\mathrm{s}}\alpha_{\mathrm{s}}/f_a|^2 \sim \frac{\alpha_{\mathrm{s}}^3}{\pi^2 f_a^2}.$$

The freeze-out will happen at $\Gamma \sim H$, so the freeze-out temperature T_{fo} is

$$\frac{T^3\alpha \mathrm{S}^3}{\pi^2 f_a^2} \sim \frac{T^2}{M_{\mathrm{Pl}}} \quad T_{\mathrm{fo}} \sim 10^{11}\,\mathrm{GeV}\left(\frac{f_a}{10^{12}\,\mathrm{GeV}}\right)^2.$$

At low energies, there is still some axion production via the pion process $\pi\pi \to \pi a$. The hot relic is about $\Omega_a = m_a/130\,\mathrm{eV}$. If axions are produced thermally, there are two problems:

- they would be hot dark matter, which is excluded by observations,
- if $\Omega_a \sim 20\,\%$, then $m_a \sim 10\,\mathrm{eV}$, which is too short-lived and excluded by stellar evolution.

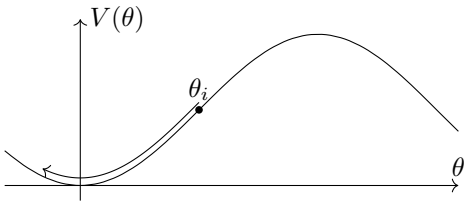
These problems are easy to circumvent if axions are produced non-thermally. There are quite a few models describing this, the most popular being the *misalignment mechanism*. This mechanism predicts that the axion field (here denoted by $\theta = a/f_a$, related to the strong-CP problem) is initially displaced from the minimum of its potential in the early Universe. The evolution of the axion field is governed by the equation of motion in an expanding Universe:

$$\dot{\theta} + 3H\theta = \text{collision term}.$$

The second term (the expansion of the universe) acts as friction, so the field settles in the minimum as the universe expands. The energy is converted into axion particles, whose density today is given by

$$\rho_\mathrm{c} = \frac{1}{2} m_a^2 f_a^2 \theta_i^2 (a_\mathrm{osc}/a_0),$$

where $a_0 = 1$ is the scale factor today and a_osc is the scale factor when the axion field starts oscillating.

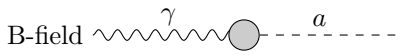


An estimate for the axion relic density produced via the misalignment mechanism is

$$\Omega_a h^2 \sim 0.4 \left(\frac{m_a}{10\,\mu\mathrm{eV}}\right)^{-1.18} \left(\frac{\theta_i}{\pi}\right)^2.$$

When the axion field is oscillating around the minimum, so is the neutron EDM. There is some hope that this oscillating EDM could be detected in experiments.

Axion can also be detected via its interaction with photons in a magnetic field in a plasma chamber (Primakoff effect), $\mathcal{L} \supset a\mathbf{E}\cdot\mathbf{B}$:



The probability of the conversion from an axion to a photon is

$$P(a \to \gamma; L) = \left(\frac{g_\gamma}{\pi f_a}\right)^2 B_0^2 \left(\frac{\sin(qL/2)}{q}\right)^2$$

where $q^2 = (m_a^2 - \omega_\mathrm{pl}^2)/(2E)$ and $\omega_\mathrm{pl} = \sqrt{n_e e^2/(\varepsilon_0 m_e)}$ is the plasma frequency.

4.5 Sterile neutrinos

The simplest extension of the Standard Model that can explain neutrino masses is to add right-handed neutrinos ν_s, which are singlets under the SM gauge group. There are no new symmetries

and only two new parameters, the Majorana mass M and the mixing θ_{as} between active and sterile neutrinos. The latter can be produced in the early Universe via mixing with active neutrinos. They should not be overproduced, which gives an upper bound on the mixing angle. The existence of lepton asymmetry can increase the production via the MSW mechanism. They are fermions, which means that they cannot have arbitrarily high number density in galaxies. The observed dark matter density in dwarf galaxies gives a lower bound on the mass of sterile neutrinos. They are not absolutely stable, as they can decay to active neutrino via W boson exchange. The two parameters (M and θ_{as}) are constrained by various observations:

- X-ray constraints from the decay $\nu_{\mathrm{s}} \to \nu_a \gamma$,
- lepton asymmetry,
- phase-space density constraints from dwarf galaxies.

There are very good prospects that in the following years the entire phase space of possible parameters will be observationally excluded.

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