

Faculty of physics

Quantum Field Theory in Curved Spacetime

Notes of lectures by Prof. Stefan Hofmann, Ph.D.

Academic year 2025/2026
Winter semester

Author:

Simon Bukovšek

Last change: February 5, 2026

Contents

1 Bundles and sections **3**

2 Distribution and differential operators **9**
2.1 Differential operators 11
2.2 Normally hyperbolic differential operators 14
2.3 Distributions on manifolds 15

3 Causal structure **18**
3.1 Quantum observables 24

4 Representations of quantum observables **27**
4.1 Canonical commutation relations 30

Index **31**

Disclaimer: these notes are not official and were not endorsed by professor. The content is almost entirely based on the lectures, but occasionally the notes were modified or extended slightly to improve clarity or add extra information. In particular, the mistakes are almost surely mine. If you find any, please let me know at simon.bukovsek@tum.de or Simon.Bukovsek@campus.lmu.de.

Overview

This course focuses on the mathematical point of view of quantum field theory in general spacetimes. The chapters will cover the following topics:

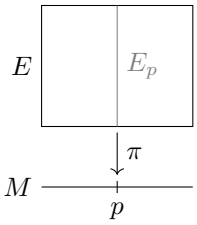
- preliminaries: model-thinker approach, data storage devices and read-out (bundles, sections);
- classical field theory: PDE, normally hyperbolic differential operators, focus on global aspects, initial value problem;
- quantum functional approach;
- applications: Unruh effect, Hawking radiation.

Chapter 1

Bundles and sections

The main goal of this chapter is to introduce the mathematical language of bundles and sections, which will be used throughout the course to describe fields in curved spacetime. Bundles act as data storage devices, while sections represent the read-out of the stored data.

Definition 1.1 (fibred manifold)**.** A *fibred manifold* is a triple (E, π, M) where E and M are smooth manifolds and $\pi\colon E \to M$ is a surjective submersion. The manifold E is called the *total space*, M the *base space*, and π the *projection map*.



For each point $p \in M$, the preimage $\pi^{-1}(p)$ is called the *fibre* over p, denoted by E_p. The maps and manifolds will always be smooth, unless said otherwise. There are many examples of fibred manifolds.

Example 1.

1. Let M and F be manifolds. The triple $(M \times F, \mathrm{pr}_1, M)$ is a fibred manifold, where $\mathrm{pr}_1\colon M \times F \to M$ is the projection onto the first factor. The fibre over $p \in M$ is $E_p = \{p\} \times F \cong F$. This is called the *trivially fibred manifold* with fibre F.
2. Consider $\mathrm{SL}(2, \mathbb{R})$ and $H \subset \mathbb{C}$ as the subset $\mathrm{Im}\,(z) > 0$. Define $\pi : \mathrm{SL}(2, \mathbb{R}) \to H$ with

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \frac{ia_{11} + a_{12}}{ia_{21} + a_{22}}$$

   which is clearly surjective. Hence, $(\mathrm{SL}(2, \mathbb{R}), \pi, H)$ is a fibred manifold.

Actually, since π is a submersion, locally around each point $e \in E$, the fibred manifold looks like a trivially fibred manifold.

Definition 1.2 (Adapted coordinate system)**.** Let (E, π, M) be a fibred manifold with $\dim M = n$ and $\dim E = n + m$. Let $y : \mathcal{U} \to \mathbb{R}^{n+m}$ be a coordinate system on $\mathcal{U}^{\text{open}} \subset E$. The chart $(\mathcal{U}, y)$ is called an *adapted coordinate system* if

$$\text{pr}_1(y(a)) = \text{pr}_1(y(b))$$

for all $a, b \in \mathcal{U}$ with $\pi(a) = \pi(b)$, where $\text{pr}_1 : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is the projection onto the first n coordinates.

Essentially, we want the points that live in the same fibre to have the same first m coordinates.

Definition 1.3 (Global trivialization)**.** Let (E, π, M) be a fibred manifold. A (global) *trivialization* of (E, π, M) is a pair (F, t) consisting of a manifold F called a *typical fibre* of (E, π, M) and a diffeomorphism $t : E \to M \times F$ such that the following diagram commutes (i.e., $\text{pr}_1 \circ t = \pi$):

$$\begin{array}{ccc} E & \xrightarrow[\cong]{t} & M \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\text{pr}_1} & \\ & M. & \end{array}$$

A fibred manifold that admits a global trivialization is called a *trivial bundle*.

Remark. Consider $(M \times F, \text{pr}_1, M)$. This is a trivial fibred manifold, However, let $g : M \times F \to F$ such that $g(p, \bullet) : F \to F$ is a diffeomorphism for each $p \in M$. Then the map

$$\begin{aligned} t : M \times F &\to M \times F \\ (p, f) &\mapsto (p, g(p, f)) \end{aligned}$$

gives another trivialization of the same fibred manifold with the same typical fibre F. Seemingly, there are many trivializations of the same trivial fibred manifold. But trivial bundles of different trivializations must be diffeomorphic.

Example 2. Consider again $(\text{SL}(2, \mathbb{R}), \pi, H)$ and set $N_1 = 1/\sqrt{a_{11}^2 + a_{21}^2}$. Define $S_1 : \text{SL}(2, \mathbb{R}) \to S^1$ with $S_1(A) = N_1 \cdot (a_{11}, a_{21})$ and (S^1, t_1) with the diffeomorphism

$$\begin{aligned} t_1 : \text{SL}(2, \mathbb{R}) &\to H \times S^1 \\ t_1(A) &\mapsto (\pi(A), S_1(A)). \end{aligned}$$

This is a trivialization of $(\text{SL}(2, \mathbb{R}), \pi, H)$ with typical fibre S^1. We could also use $N_2 = 1/\sqrt{a_{12}^2 + a_{22}^2}$ and $S_2 : \text{SL}(2, \mathbb{R}) \to S^1$ with $S_2(A) = N_2 \cdot (a_{12}, a_{22})$ to define another trivialization (S^1, t_2) with $t_2(A) = (\pi(A), S_2(A))$.

We will usually name the fibred manifolds only with the projection map, i.e., we will write $\pi : E \to M$ instead of (E, π, M). Most of the time, we won't be able to find a global trivialization of a fibred manifold. However, we can always find local trivializations.

Definition 1.4 (Local trivialization)**.** Let $\pi : E \to M$ be a fibred manifold and $p \in M$. A *local trivialization* of π around p is a triple (W_p, F_p, t_p) consisting of a neighbourhood $W_p \subset M$ of p, a manifold F_p and a diffeomorphism $t_p : \pi^{-1}(W_p) \to W_p \times F_p$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(W_p) & \xrightarrow[\cong]{t_p} & W_p \times F_p \\ & {\scriptstyle \pi|_{\pi^{-1}(W_p)}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & W_p. & \end{array}$$

A fibred manifold which has at least one local trivialization around each point of the base space is called a *fibre bundle*. Now we have a typical fibre only locally, i.e., F_p can depend on the point $p \in M$. However, we have the following result for connected base spaces.

Lemma 1.1. *Let $\pi : E \to M$ be a bundle. Then there is a manifold F so that for each local trivialization (W_p, F_p, t_p) of π around $p \in M$, the manifolds F and F_p are diffeomorphic.*

Example 3.

1. Let TM denote the tangent manifold of a smooth manifold M and $\tau_M : TM \to M$. Then (TM, τ_M, M) is a bundle with typical fibre $\mathbb{R}^{\dim M}$. Let $V \in TM$ be a vector field with local representation $V = V^a \partial_a|_p$ (using Einstein summation convention and abbreviation $\partial_a = \partial/\partial x^a$) and $x^a : M \to \mathbb{R}$ are coordinate functions around $p \in M$. Let $\gamma : \mathbb{R} \to M$ with initial data $\gamma(0) = p$ and $\dot\gamma(0) = V$:

$$x^a(\gamma(0)) = x^a(p), \quad (\dot{x^a} \circ \gamma)(0) = V^a.$$

   The local coordinates on TM around V are given by $(x^a, \dot x^a)$ and $(W_p, \mathbb{R}^{\dim M}, t_p)$ is a local trivialization of $\tau_M : TM \to M$ around p with

$$\begin{aligned} t_p : \tau^{-1}(W_p) &\to W_p \times \mathbb{R}^{\dim M} \\ V &\mapsto (\tau_M(V), \dot x(V)). \end{aligned}$$

   It is a homework to show that this is indeed a diffeomorphism and that TM satisfies all topological requirements of a bundle. Additionally, show that its dual, the cotangent bundle T^*M, is also a bundle with typical fibre $\mathbb{R}^{\dim M}$.

2. Consider $M = \mathbb{R}^n$. The tangent bundle TM is diffeomorphic to $TM \cong \mathbb{R}^m \times \mathbb{R}^m$. If $x : M \to \mathbb{R}^m$ is a global chart (M, x) then $(\mathbb{R}^m, t)$ with $t = (\tau, \dot x) : TM \to M \times \mathbb{R}$ is a global trivialization. In other words, $(TR^m, \tau_{\mathbb{R}^m}, \mathbb{R}^m)$ is trivial.

3. The tangent bundle (TS^1, τ, S^1) is trivial. This does not hold for even n in (TS^n, τ, S^n). Hence, (TS^2, τ, S^2) is not trivial.

Proposition 1.2. *Let M and F be manifolds, E a set and $\pi : E \to M$ a function so that for each $p \in M$, the preimage $\pi^{-1}(p)$ is equipped with a manifold structure. Furthermore, assume that for each $p \in M$, there is a neighbourhood $W_p \subset M$ of p and a bijection $t_p : \pi^{-1}(W_p) \to W_p \times F$ satisfying*

1. $\mathrm{Pr}_1 \circ t_p = \pi|_{\pi^{-1}(W_p)}$,
2. *for each $q \in W_p$, the map $\mathrm{Pr}_2 \circ t_p|_{\pi^{-1}(q)} : \pi^{-1}(q) \to F$ is a diffeomorphism.*

Then E can be equipped with a unique manifold structure such that (E, π, M) is a bundle with typical fibre F and local trivializations (W_p, F, t_p).

Definition 1.5 (Section)**.** Let $\pi : E \to M$ be a bundle. A map $\Phi : M \to E$ is called a *section* of π if $\pi \circ \Phi = \mathrm{id}_M$.

The set of all sections of π is denoted by $\Gamma(\pi)$.

Example 4.

1. The trivial bundle $(M \times F, \mathrm{pr}_1, M)$ has a coordinate system (x, u) around $\pi^{-1}(p) \subset E$ and a section $\Phi \in \Gamma(\pi)$. Then $(x^a \circ \pi)(\Phi(p)) = x^a(p)$. Additionally, we have $(u^\alpha \circ \Phi)(p) = \Phi^\alpha(p)$, where $\Phi^\alpha : M \to \mathbb{R}$ are smooth functions.
2. For a tangent bundle (TM, τ_M, M), a section $V \in \Gamma(\tau_M)$ is a vector field on M. Thus, we can also denote $\Gamma(\tau_M) = (M)$. For an abstract chart with coordinates $(x^a, \dot{x}^a)$ around $\tau_M^{-1}(p)$, we have $V^\alpha = \dot{x}^\alpha \circ V$ (where the projection onto the fibre is understood). Then $V = V^\alpha \, \partial/\partial x^\alpha$.
3. Let T^0S^2 be the open subset of TS^2 containing all non-zero tangent vectors. Then $(T^0S^2, \tau_{S^2}, S^2)$ is a bundle. Does τ_{S^2} permit sections? No! There are no non-vanishing vector fields on S^2 (Hairy ball theorem).

Hence, we need a local notion of sections.

Definition 1.6 (Local section)**.** Let (E, π, M) be a fibred manifold. A *local section* of π is a map $\Phi : W \to E$ where W is an open submanifold of M such that $\pi \circ \Phi = \mathrm{id}_W$.

We will denote the set of all local sections of π by $\Gamma_{\mathrm{loc}}(\pi)$.

Definition 1.7 (Bundle morphism)**.** Let (E, π, M) and (H, ρ, N) be two bundles. A *bundle morphism* from π to ρ is a pair of smooth maps $(f, \overline{f})$ with $f : E \to H$ and $\overline{f} : M \to N$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{f} & H \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\rho} \\ M & \xrightarrow{\overline{f}} & N. \end{array}$$

The map $\overline{f}$ is called the projection of f.

Equip H with an abstract chart (y^a, v^α) and introduce $f^a = y^a \circ f$ and $f^\alpha = v^\alpha \circ f$. Then there exist $\overline{f}^a$ satisfying $\overline{f}^a = y^a \circ \overline{f}$ and $f^a = \overline{f}^a \circ \pi$.

Example 5. Consider two tangent bundles (TM, τ_M, M) and (TN, τ_N, N). If $f : M \to N$ is a smooth map, then the pair (f_*, f) is a bundle morphism from τ_M to τ_N, where $f_* : TM \to TN$ is the push-forward (the differential) of f. The push-forward works as usual $f_*(v)(h) = v(h \circ f)$ for $v \in T_pM$ and $h \in C^\infty(N)$. Choose an abstract chart around $V \in TM$ with coordinate functions $(x^a, \dot{x}^a)$ and an abstract chart around $f_*(V) \in TN$ with coordinate functions $(y^a, \bar{y}^a)$. Let $p = \tau_M(V)$, where $V = V^a \, \partial/\partial x^a|_p$. Then

$$f_*(V)(h) = V^a \frac{\partial f^b}{\partial x^a}(p) \frac{\partial h}{\partial y^b}(f(a)), \quad f_*(V) = V^a \frac{\partial f^b}{\partial x^a} \frac{\partial}{\partial y^b}.$$

For homework, you can dualize this example.

Definition 1.8 (Vector bundle)**.** A *vector bundle* is a quintuple (E, π, M, σ, μ) where
- (E, π, M) is a bundle,
- $\sigma : E \times_M E \to E$ satisfies $\sigma(E_p \times E_p) \subset E_p$ for all $p \in M$,
- $\mu : \mathbb{R} \times E \to E$ satisfies $\mu(\mathbb{R} \times E_p) \subset E_p$ for all $p \in M$,
- $(E_p, \sigma|_{E_p \times E_p}, \mu|_{\mathbb{R} \times E_p})$ is a real vector space for all $p \in M$,
- for each $p \in M$, there is a local trivialization $(W_p, \mathbb{R}^n, t_p)$ such that $t_p|_{E_q} : (E_q, \sigma|_{E_q \times E_q}, \mu|_{\mathbb{R} \times E_q}) \to \{q\} \times (\mathbb{R}^n, +, \cdot)$ combined with the projection to the second component is a vector space isomorphism for all $q \in W_p$.

The map σ represents the addition of vectors and μ represents the multiplication by scalars. In the last condition, we identify σ and μ projected on a single fibre with a real vector space.

Example 6.

1. For any base manifold M, the trivial line bundle $(M \times \mathbb{R}, \mathrm{pr}_1, M)$ is a vector bundle. A generalization of this is the n-plane bundle $(M \times \mathbb{R}^n, \mathrm{pr}_1, M)$.
2. The tangent bundle (TM, τ_M, M) is a vector bundle. (It is a homework to show this.) Each fibre T_pM has the structure of a vector space. If (W, X) is a chart around $p \in M$, then
$$(\tau_M|_{\tau_M^{-1}(W)}, \dot{x}) : \tau_M^{-1}(W) \to W \times \mathbb{R}^{\dim M}$$
is a local trivialization around $p \in M$. Since $\dot{x} \circ \mathrm{Pr}_2 : \tau_M^{-1}(W) \to \mathbb{R}^{\dim M}$ is linear, this is a linear trivialization.

Definition 1.9 (Dual bundle)**.** Let (E, π, M) be a vector bundle with fibres E_p. The *dual bundle* of π is the bundle with fibres E_p^* determined by (E^*, π^*, M).

Next, we want to make a multilinear generalization of these constructions.

Definition 1.10 (Tensor bundle)**.** Let (E, π, M) and (F, ρ, M) be vector bundles with fibres E_p and F_p. The *tensor bundle* of π and ρ is the vector bundle, determined by $(E \otimes F, \pi \otimes \rho, M)$ with fibres $E_p \otimes F_p$.

Example 7. Consider the tensor bundle $(TM^{\otimes 2}, \tau_M^{\otimes 2}, M)$ with fibres $T_pM \otimes T_pM$ for all $p \in M$. By definition, locally the fibres contain bilinear map $T *_p M \otimes T_p^*M \to \mathbb{R}$. These maps are called

tensors of type $(2,0)$, which, on a chart (W,x) around $p \in M$, enjoy the following representation:

$$T = T^{ab} \left(\frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b} \right) \Big|_p .$$

Chapter 2

Distribution and differential operators

In this chapter, we will consider the semi-Riemannian space-time manifold (M, g) with the tangent bundle (TM, τ_M, M) and a vector bundle (E, π, M) over M. Moreover, there is a connection ∇ on E compatible with the metric g on M (*Levi–Civita connection*) and a connection ∇^E on E (covariant derivative).

Definition 2.1 (Symmetrized covariant differentiation)**.** Using the ingredients listed above, we define the *symmetrized covariant differentiation* D^E as the map

$$\mathrm{D}^E : \Gamma(S^k T^*M \otimes E) \to \Gamma(S^{k+1} T^*M \otimes E)$$
$$\mathrm{D}^E(\alpha \otimes s)(v_1, \dots, v_{k+1}) \coloneqq \sum_{a=1}^{k+1} \left(\nabla_{v_a} \alpha \otimes s + \alpha \otimes \nabla^E_{v_a}\right)(v_1, \dots, v_{a-1}, v_{a+1}, \dots, v_{k+1})$$

for all $\alpha \in \Gamma(S^k T^*M)$, $s \in \Gamma(E)$ and $v_1, \dots, v_{k+1} \in TM$.

Proposition 2.1. *Symmetrized covariant differentiation is well-defined, linear, and satisfies the following*

(1) for the trivial vector bundle $(M \times \mathbb{R}, \pi, M)$ and $\phi \in \Gamma(\pi)$ it holds that $\mathrm{D}^\pi \phi = \mathrm{d}\phi$,

(2) for $\alpha \in \Gamma(S^k T^*M)$ and $\beta \in \Gamma(S^l T^*M)$ the generalized Leibniz rule holds:

$$\mathrm{D}^E((\alpha \vee \beta) \otimes s) = ((\nabla \alpha) \vee \beta) \otimes s + \alpha \vee \mathrm{D}^E(\beta \otimes s),$$

where $\alpha \vee \beta = \alpha \otimes \beta + \beta \otimes \beta$,

(3) in a chart (U, x) it holds that

$$\mathrm{D}^E(\alpha \otimes s)|_U = (\mathrm{d}x^a \vee \nabla_{\partial_a} \alpha) \otimes s + \mathrm{d}x^a \vee \alpha \otimes \nabla^E_{\partial_a} s.$$

Proof. (1) On the line bundle, we have $\mathrm{D}^E\phi(V) = \nabla_V \phi = \mathrm{d}\phi(V) = V(\phi)$ and abstracting V away gives the result.

(2) Follows for instance from (3).

(3) By direct computation we have

$$\begin{aligned}\left(\mathrm{D}^E(\alpha\otimes s)\right)(v_1,\dots,v_{k+1}) &= \sum_{l=1}^{k+1}(\nabla_{v_l}\alpha\otimes s+\alpha\otimes\nabla^E_{v_l}s)(v_1,\dots,\widehat{v_l},\dots,v_{k+1})\\ &= \sum_{l=1}^{k+1}\left(\mathrm{d}x^a(V_l)\nabla_{\partial_a}\alpha\otimes s+\mathrm{d}x^a(V_l)\alpha\otimes\nabla^E_{\partial_a}s\right)(v_1,,,\widehat{v_l},,,v_{k+1})\\ &= \mathrm{d}x^a\vee\left(\nabla_{\partial_a}\alpha\otimes s+\alpha\otimes\nabla^E_{\partial_a}s\right)(v_1,\dots,\widehat{v_l},\dots,v_{k+1}).\end{aligned}$$

□

Consider $S^lT^*M\otimes E$ and equip the fibres with a Riemannian metric h and denote the induced norm by $\|\bullet\|$. For every compact subset $K\subset M$ and every $\ell\in\mathbb{N}_0$, consider

$$P_{K,\ell}(s) \coloneqq \sup_{p\in K}\left\|(\mathrm{D}^E)^{\circ\ell}s|_p\right\|_h.$$

Lemma 2.2. *The maps $P_{K,\ell}$ are well-defined for all compact $K\subseteq M$ and for all numbers $\ell\in\mathbb{N}_0$.*

These maps are gauges in the sense that $P_{K,\ell}(as)=aP_{K,\ell}(s)$ for all $a\geq 0$ and $P_{K,\ell}$ satisfies the triangle inequality.

Definition 2.2 (Test space)**.** Let $\pi:E\to M$ be a vector bundle. By $\Gamma^k_c(\pi)$ we denote the space of all k-times continuously differentiable sections of π with compact support. The space $\Gamma^k_c(\pi)$ is referred to as the *test space* of π and its elements are called *test sections*.

Definition 2.3 (Convergence of test sections)**.** Let $\pi:E\to M$ be a vector bundle and $\psi,\phi_n\in\Gamma_c(\pi)$. The sequence $(\phi_n)_{n\in\mathbb{N}}$ converges to ψ if the following conditions are satisfied:

(1) there is a compactum $K\subset M$ that includes the supports of all ϕ_n,

(2) for all $\ell\in\mathbb{N}_0$, it holds that

$$P_{K,\ell}(\psi-\phi_n)\xrightarrow{n\to\infty}0.$$

2.1 Differential operators

Definition 2.4 (Differential operator)**.** Let $\pi : E \to M$ and $\rho : F \to E$ be vector bundles over the same base manifold (M, g) and $P_g : \Gamma(\pi) \to \Gamma(\rho)$ be a linear map. The map P_g is called a *differential operator of order ℓ* if

(1) for any open subset $U \subset M$ there is a linear map $P_{g,U} : \Gamma(\pi|_{\pi^{-1}(U)}) \to \Gamma(\rho|_{\rho^{-1}(U)})$ such that for all $s \in \Gamma(\pi)$ it holds that $P_{g,U}(s|_U) = (P_g s)|_U$,

(2) relative to any abstract chart (U, x) and for any local base-sections $e_\alpha \in \Gamma(\pi|_{\pi^{-1}(U)})$ and $f_\beta \in \Gamma(\rho|_{\rho^{-1}(U)})$ the section $P_g s$ has the local representation

$$(P_g s)|_U = \sum_{r=0}^{\ell} \frac{1}{r!} (P_{g,U}^{\alpha_1,\dots,\alpha_r})_\alpha^\beta \frac{\partial^r s^\alpha}{\partial x^{\alpha_1} \dots \partial x^{\alpha_r}} f_\beta$$

with locally defined functions $(P_{g,U}^{\alpha_1,\dots,\alpha_r})_\alpha^\beta \in C^\infty(U)$ totally symmetric in $\alpha_1, \dots, \alpha_r$.

Definition 2.5 (Leading symbol)**.** Let $\pi : E \to M$ and $\rho : F \to M$ be vector bundles over the same spacetime (M, g) and $P_g : \Gamma(\pi) \to \Gamma(\rho)$ be differential operator of order $k \in \mathbb{N}_0$ represented in an abstract chart (U, x) of M and the corresponding local base sections. Then the (global) section $\sigma^k(P_g)$ in $\Gamma(S^k T^*M \otimes \rho \otimes \pi^*)$, locally represented by

$$\sigma^k(P_g)|_U = \frac{1}{k!} (P_{g,U}^{a_1,\dots,a_n})_\alpha^\beta \partial_{a_1} \vee \dots \vee \partial_{a_k} \otimes f_\beta \otimes e^\alpha$$

is called the *leading symbol* of the differential operator P_g.

Consider

$$\begin{aligned}(\mathrm{D}^E)^k s|_U &= \frac{\partial^k s^\alpha}{\partial x^{a_1} \dots \partial x^{a_k}} \mathrm{d}x^{a_1} \vee \dots \vee \mathrm{d}x^{a_k}\, e_\alpha + \text{lower order derivatives} \\ &= k! \frac{\partial^k s^\alpha}{\partial x^{a_1} \dots \partial x^{a_k}} \mathrm{d}x^{a_1} \otimes \dots \otimes \mathrm{d}x^{a_k} \otimes e_\alpha.\end{aligned}$$

Introduce a global section $X \in \Gamma(S^k TM \otimes \mathrm{Hom}(E, F))$ locally expressed as

$$X|_U = \frac{1}{k!} (X^{a_1 \dots a_k})_\alpha^\beta \partial_{a_1} \wedge \dots \wedge \partial_{a_k} \otimes f_\beta \otimes e^\alpha.$$

This suggests that we can define differential operators of order $k \in \mathbb{N}_0$ out of sections $X \in \Gamma(S^k TM \otimes \mathrm{Hom}(E, F))$ and the operator D^E:

$$\mathrm{Pair}(X, (\mathrm{D}^E)^{\circ k} s)|_U = k! (X^{a_1 \dots a_k})_\alpha^\beta \frac{\partial^k s^\alpha}{\partial x^{a_1} \dots \partial x^{a_k}} f_\beta + \text{lower derivative terms.}$$

The operation Pair means contracting free indices or, equivalently, X eating $(\mathrm{D}^E)^{\circ k} s$.

Definition 2.6 (Standard ordered quantization)**.** Let

$$X \in \bigoplus_{k\in\mathbb{N}} \Gamma(S^k TM \otimes \mathrm{Hom}(E,F))$$

be global sections where elements are in

$$\bigcup_{k\in\mathbb{N}} S^k TM \otimes \mathrm{Hom}(E,F)$$

and let $\hbar$ be the reduced Planck constant $h/2\pi$. The *standard ordered quantization* of X is the map $Q_{\mathrm{std}} : \Gamma(\pi) \to \Gamma(\rho)$ defined by the direct sum

$$Q_{\mathrm{std}}(X)_s = \sum_{r\in\mathbb{N}_0} \frac{1}{r!}\left(\frac{\hbar}{i}\right)^r \mathrm{Pair}\left(X^{(r)}, \frac{1}{r!}(D^E)^{\circ r} s\right)$$

for $s \in \Gamma(\pi)$. Here, $X = \sum_r X^{(r)}$ with only finitely many non-zero

$$X^{(r)} \in \Gamma(S^r TM \otimes \mathrm{Hom}(E,F)).$$

Example 8. For $V = \mathrm{Vec}(\mathbb{R}^n)$ and $\mathrm{id} : \mathbb{R}^n \times \mathbb{C} \to \mathbb{R}^n \times \mathbb{C}$, the standard ordered quantization if $V \otimes \mathrm{id}$ is

$$Q_{\mathrm{std}}(V \otimes \mathrm{id}) = (-i\hbar)\nabla_V \psi,$$

where ψ is an L^2-normalizable element in $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$.

Theorem 2.3. *(Global symbol calculus) On two vector bundles $\pi : E \to M$ and $\rho : F \to M$, the standard ordered quantization prescription provides a (filtration preserving) $\Gamma(\mathrm{pr}_1)$-linear isomorphism*

$$Q_{\mathrm{std}} : \bigoplus_{k\in\mathbb{N}_0} \Gamma(S^k TM \otimes \rho \otimes \pi^*) \to \mathrm{DOpe}(\pi,\rho)$$

so that $X \in \Gamma(S^k TM \otimes \rho \otimes \pi^*)$ has the following leading symbol:

$$\sigma^k(Q_{\mathrm{std}}(X)) = (-i\hbar)^k X.$$

The filtration preserving property means that Q_{std} maps sections in $\Gamma(S^k TM \otimes \rho \otimes \pi^*)$ to differential operators of order k. In the infinite direct sum, only finitely many components are non-zero. Note also that $\Gamma(\mathrm{pr}_1) \cong \mathcal{C}^\infty(M,\mathbb{R})$.

Sketch of a proof. Local expressions for $(\mathrm{D}^E)^{\circ k} s$ and X show that $Q_{\mathrm{std}}(X)$ is indeed a differential operator. Consider a non-zero part $X^{(x)}$ of X. Again from local representation, we see that the differential operator $Q_{\mathrm{std}}(X^{(k)})$ has order k. The standard ordered quantization is $\mathcal{C}^\infty(M,\mathbb{R})$-linear, because for $f \in \mathcal{C}^\infty$ it holds $Q_{\mathrm{std}}(fX) = fQ_{\mathrm{std}}(X)$. The filtration preserving property also follows from the local representation. Finally, the leading symbol is calculated explicitly:

$$Q_{\mathrm{std}}(X^{(k)}) = \frac{1}{k!}(-i\hbar)^k \mathrm{Pair}\left(X^{(k)}, \frac{1}{k!}(D^E)^{\circ k} s\right) = \frac{(-i\hbar)^k}{k!}(X^{(k);a_1,\ldots,a_k})^\beta_\alpha \frac{\partial^k s}{\partial x^{a_1}\ldots\partial x^{a_k}} f_\beta.$$

The lower order terms were dropped. It is left as an exercise to show that Q_{std} is an isomorphism. □

Consider two vector bundles (E, π, M) and $(E^* \otimes \bigwedge^n T^*M, \pi^* \otimes A^n\tau_M^*, M)$ and $s \in \Gamma_c(\pi)$, $\mu \in \Gamma(\pi^* \otimes A^n\tau)$. We define the pairing

$$\text{Pair}(s, \mu) = \int_M \mu(s).$$

Lemma 2.4. *This pairing is bilinear and non-degenerate. Moreover, for $f \in \mathcal{C}^\infty(M, \mathbb{R})$, $s \in \Gamma(\pi)$ and $\mu \in \Gamma(\pi^* \otimes A^k\tau_M^*)$ the following holds:*

$$\text{Pair}(fs, \mu) = \text{Pair}(s, f\mu).$$

Proof. Exercise. □

Proposition 2.5. *Let (E, π, M) and (F, ρ, M) be vector bundles over the same n-dimensional base space M, and let $P \in \text{DOpe}^k(\pi, \rho)$ be a differential operator of order $k \in \mathbb{N}_0$. Then $P : \Gamma_c(\pi) \to \Gamma_c(s)$ has a unique adjoint operator $P^* : \Gamma_c(\rho^* \otimes A^n\tau_M^*) \to \Gamma_c(\pi^* \otimes A^n\tau_M^*)$ satisfying*

$$\text{Pair}(Ps, \mu) = \text{Pair}(s, P^*\mu)$$

with respect to the pairing Pair *and vector bundles $\rho^* \otimes A^n\tau_M^*$, $\pi^* \otimes A^n\tau_M^*$. Moreover, this unique adjoint operator is a differential operator of order k.*

Proof. Let $\{(U_i, x_i)\}_{i\in\mathcal{I}}$ be an abstract atlas, and let $e_{i,\alpha} \in \Gamma_{U_i}(\pi|_{\pi^{-1}(U_i)})$, $f_{i,\beta} \in \Gamma_{U_i}(\rho|_{\rho^{-1}(U_i)})$. Let $\{\chi_i\}i \in \mathcal{I}$ be a partition of unity subordinate to the cover $\{U_i\}_{i\in\mathcal{I}}$. Locally, we have

$$Ps|_{U_i} = \sum_{r=0}^{k} \frac{1}{r!} \left(P_{U_i}^{a_1\dots a_r}\right)_\alpha^\beta \frac{\partial^r s_i^\alpha}{\partial x_i^{a_1} \dots \partial x_i^{a_r}} f_\beta$$

where $s_i^\alpha = e_i^\alpha(s|_{U_i})$. The coordinatization of the local representation of μ is given by

$$\mu|_{U_i} = \mu_{i,\beta} f^\beta |\mathrm{d}x_i^1 \wedge \dots \wedge \mathrm{d}x_i^n|.$$

By direct computation, we find

$$\begin{aligned}
\text{Pair}(Ps, \mu) = \int_M \mu(Ps) &= \sum_{i\in\mathcal{I}} \int_{x_i(U_i)} \mathrm{d}^n x_i \, (\chi_i \mu Ps) \circ x_i^{-1} \\
&= \sum_{i\in\mathcal{I}} \int_{x_i(U_i)} \mathrm{d}x_i \left(\chi_i \mu_{i,\beta} \sum_{r=0}^{k} \frac{1}{r!} \left(P_{U_i}^{a_1\dots a_r}\right)_\alpha^\beta \frac{\partial^r s_i^\alpha}{\partial x_i^{a_1} \dots \partial x_i^{a_r}} \right) \circ x_i^{-1} \\
&= \sum_{i\in\mathcal{I}} \int_{x_i(U_i)} \mathrm{d}^n x_i \left[\sum_{r=0}^{k} \frac{(-1)^r}{r!} \frac{\partial^r}{\partial x_i^{a_1} \dots \partial x_i^{a_r}} \left(\chi_i \mu_{i,\beta} (P_{U_i}^{a_1\dots a_r})_\alpha^\beta \right) s^\alpha \right] \circ x_i^{-1}
\end{aligned}$$

Let's introduce an $(n+1)$-form

$$\omega_i := \sum_{r=0}^{k} \frac{(-1)^r}{r!} \frac{\partial^r}{\partial x_i^{a_1} \dots \partial x_i^{a_r}} \left(\chi_i \mu_{i,\beta} (P_{U_i}^{a_1\dots a_r})_\alpha^\beta \right) e_i^\alpha \otimes |\mathrm{d}x_i^1 \wedge \dots \wedge \mathrm{d}x_i^n|$$

which is a global section in $\Gamma_c(\pi^* \otimes A^n \tau_M^*)$. Since χ is a partition of unity, we can write

$$P^*\mu = \sum_{i \in \mathcal{I}} \omega_i$$

from which the assertion follows. The uniqueness of P^* will be discussed at the tutorial. □

Corollary 2.6. *The leading symbol $\sigma^k(p^*)$ of the adjoint operator of P is given by*

$$\sigma^k(P^*) = (-1)^k \sigma^k(P) \otimes \mathrm{id}_{\wedge^n T^*M}\,.$$

Proof. By direct computation. □

2.2 Normally hyperbolic differential operators

Let $V \in \Gamma(\tau_M)$ be a vector field on M. The metric connection $C(\nabla V) \in \mathcal{C}^\infty(M, \mathbb{R})$ is called the divergence of V denoted by $\mathrm{div}(V)$. In an abstract chart (U, x), it is expressed as

$$\mathrm{div}(V)|_U = \mathrm{d}x^a(\nabla_{\partial_a} V).$$

Definition 2.7 (Gradient and d'Alembert operator)**.** Let pr_1 be a vector bundle over a semi-Riemannian manifold (M, g).

(1) The *gradient* is the map $\mathrm{grad} : \Gamma(\mathrm{pr}_1) \to \Gamma(\tau_M)$ given by $\mathrm{grad}(f) = (\mathrm{d}f)^\sharp$.
(2) The *d'Alembert operator* of a section $f \in \Gamma(\mathrm{pr}_1)$ is a map $\Box_g : \Gamma(\mathrm{pr}_1) \to \Gamma(\mathrm{pr}_1)$ given by $\Box_g f = \mathrm{div}(\mathrm{grad}(f))$.

In the first definition, the musical isomorphism $\sharp : T^*M \to TM$ is used to raise indices. It is a homework to show that in an abstract chart (U, x), we have the following representations:

$$\begin{aligned}
\mathrm{grad}(f)|_U &= g^{ab} \frac{\partial f}{\partial x^a} \partial_b, \\
\mathrm{div}(V)|_U &= \frac{\partial V^a}{\partial x^a} + \Gamma^a_{ab} V^b, \\
\Box_g f|_U &= g^{ab} \left(\frac{\partial^2 f}{\partial x^a \partial x^b} - \Gamma^c_{ab} \frac{\partial f}{\partial x^c} \right).
\end{aligned}$$

Proposition 2.7. *Let (M, g) be a semi-Riemannian manifold. The d'Alembert operator $\Box_g$ is a second-order differential operator with leading symbol*

$$\sigma^2(\Box_g) = 2g^{-1} \in \Gamma(S^2 \tau_M).$$

Moreover, it is given by the standard ordered quantization

$$\Box_g = (-i\hbar)^2 Q_{\mathrm{std}}(2g^{-1}).$$

Hence, for $f \in \Gamma(\mathrm{pr}_1)$, it holds that

$$\Box_g f = \frac{1}{2} \mathrm{Pair}(g^{-1}, (\mathrm{D}^E)^{\circ 2} f).$$

Proof. Follows from direct computation on charts (U, x):

$$\begin{aligned}(\mathrm{D}^E)^{\circ 2} f|_U &= (\mathrm{D}^E)\mathrm{d}f|_U = \mathrm{D}^E(\operatorname{grad} f)^\flat|_U = \mathrm{d}x^r \vee \nabla_{\partial_r}(\partial_s f \mathrm{d}x^s) \\ &= \mathrm{d}x^r \vee (\partial_r \partial_s \mathrm{d}x^s + \partial_s f \nabla_{\partial_r} \mathrm{d}x^s) = (\partial_r \partial_x f - \Gamma^s_{rc} \partial_s f)\mathrm{d}x^r \vee \mathrm{d}x^c.\end{aligned}$$

We used $\nabla_{\partial_r} \mathrm{d}x^s ? - \Gamma^s_{rc} \mathrm{d}x^c$. Clearly $P_g = \Box_g$ is of second order and then by definition of the leading symbol, we have

$$\Box_g f|_U = \frac{1}{2} P^{rs}_U \partial_r \partial_s f + \text{lower order terms}$$

with $P^{rs}_U = 2g^{rs}|_U$. Thus, $\sigma^2(\Box_g) = 2g^{-1}|_U$. The second order quantization follows directly from the definition

$$Q_{\mathrm{std}}(2g^{-1})f = \frac{1}{2}(-i\hbar)^2 \mathrm{Pair}(2g^{-1}, (\mathrm{D}^M)^{\circ 2} f/2).$$

□

Definition 2.8 (Normally hyperbolic differential operator)**.** Let $\pi : E \to M$ be a vector bundle over a Lorentzian manifold (M, g). A differential operator $P_g : \Gamma(\pi) \to \Gamma(\pi)$ is called *normally hyperbolic* if it is second order and its leading symbol is given by $\sigma^2(P_g) = 2g^{-1} \otimes \mathrm{id}_E$.

Proposition 2.8 (*Generalized Weitzenböck formula*)**.** *Let $\pi : E \to M$ be a vector bundle over a Lorentzian manifold (M, g) and $P_g \in \mathrm{DOpe}^2(\pi)$ a normally hyperbolic differential operator. Then there is a unique connection ∇^E on E and a unique section $B \in \Gamma(\mathrm{End}(E))$ such that $P_g = \Box^E_g + B$.*

2.3 Distributions on manifolds

Definition 2.9 (Distribution)**.** A *distribution* u on M is a linear form on $\Gamma_{\mathrm{c}}(\mathrm{pr}_1)$ such that for every compact set $K \subset M$ there exist two constants $C > 0$ and $\ell \in \mathbb{N}_0$ with

$$|u(\varphi)|_{\mathbb{C}} \leq C P_{K,\ell}(\varphi)$$

for every $\varphi \in \Gamma_K(\mathrm{pr}_1)$. If the same $k \in \mathbb{N}_0$ can be chosen for all compact sets $K \subset M$, then u is said to be of *order* $k \leq \ell$.

Remark. Consider a sequence $(\varphi_n)_{n\in\mathbb{N}_0}$ in $\Gamma_{\mathrm{c}}(\mathrm{pr}_1)$. We say the sequence converges to $\varphi \in \Gamma_{\mathrm{c}}(\mathrm{pr}_1)$ if there is a compact set K such that the supports of all φ_n are contained in K and for all $k \in \mathbb{N}_0$ it holds that $P_{K,k}(\phi_n - \phi) \to 0$. If the sequence $(\varphi_n)_{n\in\mathbb{N}_0}$ converges to φ, then we have

$$|u(\phi_n) - u(\phi)|_{\mathbb{C}} = |u(\phi_n - \phi)|_{\mathbb{C}} \leq C P_{K,\ell}(\phi_n - \phi) \to 0.$$

Example 9.

(1) *Dirac's measure*:

$$\begin{aligned}\delta_p : \Gamma_c(\mathrm{pr}_1) &\to \mathbb{C},\\ \varphi &\mapsto \varphi(p).\end{aligned}$$

This measure is continuous and of order 0.

(2) Locally integrable densities: let $\mu : M \to |\bigwedge^{\dim M} |T^*M$. Any such μ can be interpreted as a distribution on M by setting $\mu : \Gamma_c(\mathrm{pr}_1) \to \mathbb{C}$ defined by

$$\varphi \mapsto \mu(\varphi) = \int_M \varphi\mu.$$

Since K is compact, $\mathrm{vol}_\mu(K)$ is finite and therefore $|\mu(\varphi)|_{\mathbb{C}} \leq P_{K,0}(\varphi)\,\mathrm{vol}_\mu(K)$. From this it follows that μ is a distribution of order 0.

Definition 2.10 (Distributional section)**.** Let $\pi : E \to M$ be a vector bundle with n-dimentional base manifold M and let $(E^* \otimes |\bigwedge^n |T^*M, \pi^* \otimes A^n\tau_M^*, M)$ be another vector bundle over the same base space. A *distributional section* of π is a continuous linear functional $s : \Gamma_c(\pi^* \otimes A^n\tau_M^*) \to \mathbb{C}$. The space of all distributional sections of π is denoted by $\Gamma^{-\infty}(\pi)$.

Remark. Continuity relies on using semi-norms of $\Gamma(\pi^* \otimes A^n\tau_M^*)$. For every compact set $K \subset M$ there are constants $C > 0$ and $\ell \in \mathbb{N}_0$ so that

$$|s(\omega)|_{\mathbb{C}} \leq C \max_{\ell' \leq \ell} P_{K,\ell'}(\omega)$$

for all $\omega \in \Gamma_K(\pi^* \otimes A^n\tau_M^*)$.

Definition 2.11 (Weak* topology)**.** The *weak* topology* on $\Gamma^{-\infty}(\pi)$ is the locally convex topology obtained from all the semi-norms $P_\omega(s) := |s(\omega)|_{\mathbb{C}}$, where $\omega \in \Gamma(\pi^* \otimes A^n\tau_M^*)$.

Definition 2.12 (Singular support)**.** Let (E, π, M) be a vector bundle and $s \in \Gamma^{-\infty}(\pi)$. The distributional section s is called *regular* in $p \in M$ if there is an open neighbourhood $U \subset M$ of p so that the restriction $s|_U$ is in $\Gamma_U(\pi)$. The *singular support* $\operatorname{sing\,supp} s$ of s consists of all points $p \in M$ at which it is not regular.

Theorem 2.9. *Let (E, π, M) be a vector bundle over an n-dimensional base space M and let s be a distributional section of in $\Gamma_c^{-\infty}(\pi)$ with compact support in M. Then s has finite global order and extends uniquely to a linear functional $s : \Gamma(\pi^* \otimes A^n\tau_M^*) \to \mathbb{C}$ which is continuous (in C^∞ topology). Conversely, if $s : \Gamma(\pi^* \otimes A^n\tau_M^*) \to \mathbb{C}$, then its restriction to $\Gamma_c(\pi^* \otimes A^n\tau_M^*)$ is a distributional section of compact support.*

Definition 2.13 (Differential operators on distributional sections)**.** Let (E, π, M) and (F, ρ, M) be vector bundles over the same base manifold M and let $P \in \mathrm{DOpe}(\pi, \rho)$ be a differential operator. Then the extension of P to distributional sections is $P : \Gamma^{-\infty}(\pi) \to \Gamma^{-\infty}(\rho)$ with $(Ps)\mu \mapsto s(P^*\mu)$ for all distributional sections $s \in \Gamma^{-\infty}(\pi)$ and all test sections $\mu \in \Gamma_c(\rho^* \otimes A^n \tau_M^*)$

Theorem 2.10. *Let π and ρ be two vector bundles over the same base space M and let $P \in \mathrm{DOpe}^k(\pi, \rho)$.*

(1) The previous definition gives well-defined distributional sections s of π. The map $P : \Gamma^{-\infty}(\pi) \to \Gamma^{-\infty}(\rho)$ is linear and continuous with respect to the weak topology.*

(2) The map P is the unique extension of $P : \Gamma_c(\pi) \to \Gamma_c(\rho)$ which is linear and weak-continuous.*

(3) With respect to the $\mathcal{C}^\infty(M, \mathbb{R})$-module structure of the space of distributional sections of π and ρ, the map $P : \Gamma^{-\infty}(\pi) \to \Gamma^{-\infty}(\rho)$ is a differential operator of order k, i.e., $P \in \mathrm{DOpe}^k(\Gamma^{-\infty}(\pi), \Gamma^{-\infty}(\rho))$.

(4) $\mathrm{supp}(Ps) \subseteq \mathrm{supp}(s)$ and similarly for the singular support.

(5) $Ps|_U = P|_U s|_U$ is consistent for every open subset $U \subset M$.

Proof. We will prove (3) using induction. The base case is a differential operator of zeroth order $P \in \mathrm{DOpe}^0(\pi, \rho)$. We identify $\mathrm{DOpe}^0(\pi, \rho)$ with $\mathrm{Hom}_{\Gamma(\mathrm{pr}_1)}(\Gamma(\pi), \Gamma(\rho))$. For $s \in \Gamma^{-\infty}(\pi)$ and $f \in \Gamma(\mathrm{pr}_1)$ we have

$$P(fs)\mu = (fs)(P^*\mu) = s(fP^*\mu) = s(P^*(f\mu)) = (fPs)(\mu).$$

When we abstract $\mu \in \Gamma(\rho^* \otimes A^n \tau_M^*)$ away, we get $P(fs) = fPs$, so indeed it holds $P \in \mathrm{DOpe}^0(\Gamma^{-\infty}(\pi), \Gamma^{-\infty}(\rho))$. □

Chapter 3

Causal structure

Definition 3.1 (Index)**.** The *index* of a symmetric bilinear form b on a finite dimensional real vector space V is the largest integer q such that there is a subspace $W \subset V$ of dimension q on which the restriction $b|_{W\times W}$ is negative definite.

Definition 3.2 (Scalar and inner product)**.** A *scalar product* on a given real vector space V is a non-degenerate symmetric bilinear form $b : V \times V \to \mathbb{R}$. An *inner product* is a positive definite scalar product in V.

Definition 3.3 (Metric tensor)**.** Let M be a smooth manifold and $(T * M^{\otimes 2}, \tau_M^{*\otimes 2}, M)$ a tensor bundle over M. A section over this tensor bundle is called a *metric tensor* field on M if the elements of $T^*M^{\otimes 2}$ are symmetric non-degenerate bilinear forms $T_p(M) \otimes T_p(M) \to \mathbb{R}$ of constant index for all $p \in M$.

Definition 3.4 (Semi-Riemannian manifold)**.** Let $(T * M^{\otimes 2}, \tau_M^{*\otimes 2}, M)$ be a tensor bundle over M and let $\Gamma(\tau_M^{*\otimes 2})$ denote the space of smooth sections over this bundle. M is called a *semi-Riemannian manifold* if $\Gamma(\tau_M^{*\otimes 2})$ contains a metric tensor field on M.

Strictly speaking, a tuple (M, g) is a semi-Riemannian manifold if g is a metric tensor field on M.

Remark. Let (M, g) be a semi-Riemannian manifold. The common value ν of index g_p for all $p \in M$ is called the index of (M, g). If $\nu = 0$, then (M, g) is called a *Riemannian manifold*. If $\nu = 1$ and $\dim M \geq 2$, then (M, g) is called a *Lorentzian manifold*.

Definition 3.5 (Trichotomy)**.** Let (TM, τ_M, M) be the tangent bundle over a semi-Riemannian manifold (M, g). A tangent vector $v \in T_pM$ is called

- *time-like*vector!time-like if $g_p(v,v) < 0$,
- *light-like*vector!light-like if $g_p(v,v) = 0$ and $v \neq 0$,
- *space-like*vector!space-like if $g_p(v,v) > 0$ or $v = 0$.

The class into which v falls is called the *causal character* of v. The set of all light-like vectors in the fibre T_pM is called the *light cone* at p.

Remark. Let $\mathcal{T}$ be the set of all time-like vectors in a Lorentz vector space with scalar product g. For $u \in \mathcal{T}$, the set

$$\mathscr{C}(u) := \{v \in \mathcal{T} : g(u,v) < 0\}$$

is called the time-cone of V containing u. The opposite time-cone is given by

$$\mathscr{C}(-u) := \{v \in \mathcal{T} : g(u,v) > 0\}.$$

Since $u^{\perp_g}$ is space-like, $\mathcal{T}$ is the disjoint union of the two time-cones.

Lemma 3.1. *Time-like vectors in a Lorentzian space are in the same time-cone if and only if their scalar product is negative.*

Proof. Exercise. □

Proposition 3.2. *Let v, w be time-like vectors in a Lorentzian vector space (V, g). Then the following statements hold:*

(1) $|g(v,w)| \geq \|v\|_g \|w\|_g$ *(reverse Cauchy-Schwarz inequality),*

(2) If v and w are in the same time-cone, then there is a unique positive real number $\varphi \geq 0$ called the hyperbolic angle *between v and w so that $g(v,w) = -\|v\|_g \|w\|_g \cosh\varphi$.*

Corollary 3.3. *If v and w are vectors in the same time-cone in a Lorentzian vector space (V, g), then the $\|v+w\|_g \geq \|v\|_g + \|w\|_g$ (reverse triangle inequality) with equality if and only if v and w are linearly dependent with positive factor.*

Definition 3.6 (Causality relations)**.** Let (M, g) be a time-oriented Lorentzian manifold and $p, q \in M$. Time-oriented means that there is a continuous choice of time-cones in each tangent space T_pM. The *causality relations* between p and q are defined as follows:

(1) $p \ll q$ (read: p precedes q) if there is a future-directed time-like curve from p to q,

(2) $p < q$ (read: p causally precedes q) if there is a future-directed causal curve from p to q.

Definition 3.7 (Chronological and local future/past)**.** Let (M, g) be a time-oriented Lorentzian manifold and $A \subset M$.

(1) The subset
$$I^+_M(A) \coloneqq \{q \in M : \exists p \in A \text{ with } p \ll q\}$$
is called the *chronological future* of A in M. The *chronological past* $I^-_M(A)$ is defined analogously.

(2) The subset
$$J^+_M(A) \coloneqq \{q \in M : \exists p \in A \text{ with } p \leq q\}$$
is called the *causal future* of A in M. The *causal past* $J^-_M(A)$ is defined analogously.

The following relation holds:

$$I^+(A) = I^+(I^+(A)) = I^+(J^+(A)) = J^+(I^+(A)) \subset J^+(J^+(A)) = J^+(A).$$

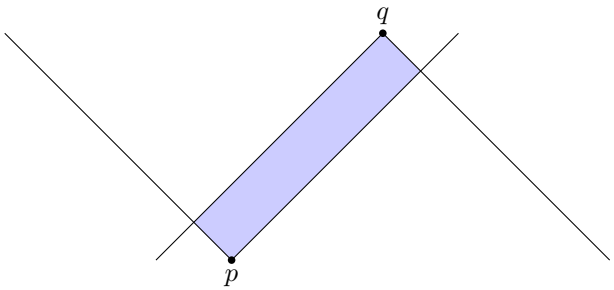
Most of the relations are trivial, the second one is a bit more tricky and is a homework.

Lemma 3.4. *Let (M, g) be a time-oriented Lorentzian manifold. The chronological future/past of any subset $A \subset M$ is open in M.*

Proof. Exercise. □

Remark.

(1) The chronological open *diamond* of two points $p, q \in M$ is defined as
$$I_M(p, q) \coloneqq I^+_M(p) \cap I^-_M(q).$$
These diamonds form a basis for the topology of M, called the *Alexandrov topology*.



Of course, there are also causal diamonds $J_M(p, q) = J^+_M(p) \cap J^-_M(q)$.

(2) We say that the *causal condition* holds on (M, g) if there are no closed causal curves in M.

Definition 3.8 (Strong causality condition)**.** The *strong causality condition* holds at $p \in M$ if for every open neighbourhood $U \subset M$ of p there is another open neighbourhood $V \subset U$ of p so that every causal curve segment with endpoints in V is completely contained in U.

We can interpret this definition as follows: if the strong causality condition holds at $p \in M$, then causal curves with starting points arbitrarily close to p that leave some fixed neighbourhood U of p can never return arbitrarily close to p.

Definition 3.9 (Arc length)**.** Let $I \subset \mathbb{R}$ be a closed interval and let $\alpha : I \to M$ be a piecewise $\mathcal{C}^1$ curve in a semi-Riemannian manifold (M, g). The *arc length* of α is defined by

$$L(\alpha) \coloneqq \int_I \mathrm{d}t \, \sqrt{|g(\dot{\alpha}(t), \dot{\alpha}(t))|}.$$

Remark. In a chart with coordinates $(x^1, \ldots, x^{\dim M})$, we have

$$\sqrt{|g(\dot{\alpha}(t), \dot{\alpha}(t))|} = \sqrt{\left| g_{ab}|_{\alpha(t)} (x^a \dot{\circ} \alpha)(t) (x^b \dot{\circ} \alpha)(t) \right|}.$$

The arc length of a light-like curve is always zero.

Definition 3.10 (Time separation)**.** Let (M, g) be a time-oriented Lorentzian manifold and let $p, q \in M$. The *time separation function* $\tau : M \times M \to [0, \infty]$ is defined by

$$\tau(p, q) \coloneqq \sup \{L(\alpha) : \alpha \text{ is future pointing causal curve segment from } p \text{ to } q\}$$

if q is in the causal future of p, $q \in J^+(p)$. If not then $\tau(p, q) = 0$.

For homework, consider the time separation of Minkowski spacetime $(\mathbb{R}^n, \eta)$.

Lemma 3.5.

1. *$\tau(p, q) > 0$ if and only if $p \ll q$,*
2. *the reversed triangle inequality holds: if $p \leq q \leq r$, then $\tau(p, r) \geq \tau(p, q) + \tau(q, r)$,*

In the following, we will first consider a local theory of wave equations, where the existence and uniqueness of solutions can be guaranteed. Later, we will extend these results to a global theory.

Definition 3.11 (Fundamental solution)**.** Let (E, π, M) be a vector bundle over a time-oriented Lorentzian manifold (M, g) and let $P_g : \Gamma^{-\infty}(\pi) \to \Gamma^{-\infty}(\pi)$ be a normally hyperbolic differential operator. A *fundamental solution* of P_g at $x \in M$ is a distributional section

$$F_x \int \Gamma^{-\infty}(\pi) \otimes E_x^* | \bigwedge^m | T_x^* M$$

so that $P_g F_x = \delta_x$ with δ_x being Dirac's measure at x. If $\mathrm{supp}(F_x) \subseteq J_M^{\pm}(x)$, then F is called an *advanced* (for $J_M^+(x)$) or *retarded* (for $J_M^-(x)$) fundamental solution.

Lemma 3.6. *Let (E, π, M) be a vector bundle over a time-oriented Lorentzian manifold (M, g) and let $U \subseteq M$ be sufficiently small open set for the usual construction of fundamental solutions of P_g.*

(1) The map $F_U^{\pm} : \Gamma_c(\pi^*|_U) \to \Gamma(\pi^*|_U)$ induces a dual map $(F_U^{+})^{\sim} : \Gamma_c^{-\infty}(\pi|_U) \to \Gamma^{-\infty}(\pi|_U)$ given by dualization that is defined as follows: for $J_2 \in \Gamma_c^{-\infty}(\pi|_U)$ and $J_1 \in \Gamma_c(\pi^*|_U)$ it holds that

$$((F_U^{\pm})^{\sim}(J_2))(J_1) = J_2(F_U^{\pm}(J_1)).$$

(2) For all source fields $J \in \Gamma_c^{-\infty}(\pi|_U)$, the dual fundamental solution satisfies the identity $P_g(F_U^{\pm})^{\sim}(J) = J$.

Proof.

(1) Intuitively clear.

(2) Let $J' \in \Gamma_c(\pi^*|_U)$. By direct computation,

$$(P_g(F^{\pm})^{\sim}(J))(J') = (F_U^{\pm})^{\sim}(J)(P_g^* J') = J((F_U^{\pm})(P_g J')) = J(P_g(F_U^{\pm})(J')) = J(J').$$

□

Lemma 3.7.

$$\operatorname{supp}((F_U^{\pm})^{\sim}(Q)) \subset F_U^{\pm}(\operatorname{supp}(Q)).$$

Consider a time-oriented Lorentzian manifold (M, g) with the following properties:

(1) causality condition holds on M;

(2) causal future of any point in M is a closed subset of M;

(3) time separation function is finite and continuous on $M \times M$.

If the first two conditions hold, then (M, g) is called a *causally simple spacetime*.

Theorem 3.8. *Let $\pi : E \to M$ be a vector bundle on a causally simple spacetime (M, g) on which the time separation function is finite and continuous. Let $P_g : \mathrm{DOpe}^2(\pi)$ be a normally hyperbolic differential operator and Φ a distributional section in $\Gamma^{-\infty}(\pi)$. Then any Φ with either past or future compact support and satisfying the homogeneous wave equation $P_g\Phi = 0$ must be identically zero.*

The intuition behind this theorem goes as follows. For all compactly supported $\varphi \in \Gamma_c(\pi^* \otimes |\wedge^m |T^*M)$ we want to show $\Phi(\varphi) = 0$. Locally, there exists $\psi \in \Gamma_U(\pi^* \otimes |\wedge^m |T^*M)$ such that $P_g^*\psi = \varphi$. What we would want to do is to write

$$\Phi(\varphi) = \Phi(P_g^*\psi) = P_g(\Phi(\psi)) = 0,$$

however we cannot guarantee that ψ has compact support. The actual proof is more technical.

Corollary 3.9. *For every normally hyperbolic differential operator $P_g : \mathrm{DOpe}^2(\pi)$ there is at most one solution at $p \in M$ with past compact support. Similarly, there is at most one solution with future compact support.*

Proof. Let $F_{1,p}$ and $F_{2,p}$ be two fundamental solutions at $p \in M$ with past compact support. Then $F_{1,p} - F_{2,p}$ solves the differential equation and still has past compact support. By the previous theorem, it must be identically zero and therefore $F_{1,p} = F_{2,p}$. □

We want to solve the Cauchy problem for normally hyperbolic differential operators on globally hyperbolic spacetimes:

$$P_g\varphi = J$$

subjected to initial conditions φ_{in} and $\dot{\varphi}_{\mathrm{in}}$ on a Cauchy hypersurface $\Sigma_{\mathrm{in}} \xrightarrow{\iota} M$, such that $\varphi_{\mathrm{in}}, \dot{\varphi}_{\mathrm{in}} \in \Gamma_{\mathrm{c}}(\iota^\sharp\pi)$, $\iota^\sharp\varphi = \varphi_{\mathrm{in}}$ and $\iota^\sharp\nabla^E_n = \dot{\varphi}_{\mathrm{in}}$ where $n \in \Gamma_{\Sigma_{\mathrm{in}}}(TM)$ (or equivalently $n \perp_g T\Sigma_{\mathrm{in}}$).

We have operators $G^\pm_U : \Gamma_{\mathrm{c}}(\pi|_U) \to \Gamma(\pi|_U)$ enjoy properties analogously to fundamental solutions $F^\pm_U$ of P_g and the dual maps $(G^\pm_U)^\sim : \Gamma_{\mathrm{c}}^{-\infty}(\pi^*|_U) \to \Gamma_{\mathrm{c}}^{-\infty}(\pi^*|_U)$. Introduce $G_U \coloneqq G^+_U - G^-_U$ and $G^\sim_U$ given by

$$((G_U)^\sim(J_2))(J_1) \coloneqq J_2(G_U(J_1)) = \int_{p\in U} \mathrm{d}\,\mathrm{vol}_g(p)\, J_2(p)(G_U(p))(J_1).$$

Lemma 3.10. *Let $\pi : E \to M$ be a vector bundle over Lorentzian manifold (M, g) and let $U \subset M$ be globally hyperbolic causal open subset of sufficiently small volume to accommodate the usual construction of fundamental solutions of P_g. Assume $\varphi \in \Gamma(\pi|_U)$ to be a solution to the inhomogeneous wave equation $P_g\varphi = J$ and let $Q \in \Gamma_{\mathrm{c}}(\pi^*|_U)$. Then*

$$\int_{p\in U} \mathrm{d}\,\mathrm{vol}_g(p)\, Q(p)\varphi(p) = \int_{\sigma\in\Sigma_{\mathrm{in}}} \mathrm{d}\,\mathrm{vol}_q(\sigma)\left[(\nabla^{*E}_n(G_U)^\sim(Q))(\sigma)\varphi_{\mathrm{in}}(\sigma) - (G_U)^\sim(Q)\dot{\varphi}_{\mathrm{in}}(\sigma)\right].$$

Theorem 3.11. *Let (M, g) be a causal Lorentzian manifold and $\pi : E \to M$ a vector bundle over (M, g) and $\iota : \sigma \to M$ a smooth space-like Cauchy hypersurface with future-directed unit normal vector field $n \in \Gamma_{(}\iota^\sharp\tau_M)$. Let J be a source modelled as a continuous section in $\Gamma^0(\pi)$ and $\varphi \in \Gamma^2(\pi)$ satisfying the inhomogeneous wave equation $P_g\varphi = J$ given by normally hyperbolic differential operator $P_g \in \mathrm{DOpe}^2(\pi)$. Then φ is uniquely determined by the initial conditions.*

3.1 Quantum observables

Definition 3.12 (C^*-algebra)**.** Let $\mathscr{A}$ be an associative $\mathbb{C}$-algebra, let $\|\bullet\|$ be a norm on the $\mathbb{C}$-vector space $\mathscr{A}$, and le $* : \mathscr{A} \to \mathscr{A}$ be a $\mathbb{C}$-antilinear map. Then the triple $(\mathscr{A}, \|\bullet\|, *)$ is called the *C^*-algebra* if the $\mathbb{C}$-vector space $(\mathscr{A}, \|\bullet\|)$ is complete and:
(1) $(a^*)^* = a$ (involution),
(2) $(a \circ b)^* = b^* \circ a^*$,
(3) $\|a \circ b\| \leq \|a\| \, \|b\|$ (sub-multiplicativity),
(4) $\|a^*\| = \|a\|$ (isometry),
(5) $\|a^* \circ a\| = \|a\|^2$ (C^*-property).

Remark. From properties (1), (2) and anti-linearity, $*$ is an involution. The C^*-algebra carries uniform topology.

Let $(H, \langle \bullet, \bullet \rangle)$ be a Hilbert space and let $\mathscr{A} = \mathcal{L}(H)$ be the algebra of all bounded operators on H. Equip $\mathcal{L}(H)$ with a norm

$$\|a\|_{\mathscr{A}} := \sup\{\|a\varphi\|_H : \varphi \in H, \|\varphi\|_H = 1\}.$$

Is this a C^*-norm? Here we focus on the C^*-property (5):

$$\|a\|^2_{\mathscr{A}} = \sup_{\|\varphi\|_H=1} \|a\varphi\|^2_H = \sup_{\|\varphi\|_H=1} \langle a\varphi, a\varphi\rangle_H = \sup_{\|\varphi\|_H=1} \langle \varphi, a^* \circ a\varphi\rangle \leq \sup_{\|\varphi\|_H=1} \|\varphi\| \, \|a^* \circ a\varphi\|_H =$$
$$= \|a^* \circ a\|_{\mathscr{A}} \leq \|a^*\|_{\mathscr{A}} \|a\|_{\mathscr{A}} = \|a\|_{\mathscr{A}} 2.$$

All together we have established that $\|a\|^2_{\mathscr{A}} \leq \|a * \circ a\|_{\mathscr{A}} \leq \|a\|^2_{\mathscr{A}}$, hence the C^*-property holds.

Let X be a locally compact Hausdorff space and set $\mathscr{A} = \mathcal{C}_c(X, \mathbb{C})$. Introduce $\|f\|_{\mathscr{A}} := \sup_{x \in X} |f(x)|_{\mathbb{C}}$. Since X is locally compact Hausdorff, the bound is always satisfied, so we could exchange sup with max. This satisfies the C^*-property with $*$ being complex conjugation and $\circ$ being pointwise multiplication: $\|f^* \cdot f\|_{\mathcal{A}} = \sup_{x \in X} |f(x)|^2_{\mathbb{C}} = \|f\|^2_{\mathscr{A}}$. This example is basically the same as the previous one.

Lemma 3.12. *Let $\mathscr{A}$ be an algebra with unit $\mathrm{id}_{\mathscr{A}}$. If $a \in \mathscr{A}$ is invertible, then it has a unique inverse $a^{-1} \in \mathscr{A}$. This inverse is again invertible with $(a^{-1})^{-1} = a$.*

If $a, b \in \mathscr{A}$ are invertible, then so is their product $a \circ b$ with $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$. If a is invertible so is a^* with $(a^*)^{-1} = (a^{-1})^*$. Proofs of these statements are left as exercise.

Definition 3.13 (Spectral concepts)**.** Let $\mathscr{A}$ be an algebra with unit $\mathrm{id}_{\mathscr{A}}$. For each $a \in \mathscr{A}$ the set

$$\Gamma_{\mathscr{A}}(a) := \{\lambda \in \mathbb{C} : (\lambda \, \mathrm{id}_{\mathscr{A}} - a) \in \mathscr{A}^\times\}$$

(where $\mathscr{A}^\times$ denotes the set of invertible elements) is called the *resolvent set* of a. The set $\sigma_{\mathscr{A}}(a) := \mathbb{C} \setminus \Gamma_{\mathscr{A}}(a)$ is called the *spectrum* of a. For $\lambda \in \Gamma_{\mathscr{A}}(a)$ the inverse $(\lambda \, \mathrm{id}_{\mathscr{A}} - a)^{-1} \in \mathscr{A}$ is called the *resolvent* of a at λ. The real number

$$\rho_{\mathscr{A}}(a) := \sup\{|\lambda|_{\mathbb{C}} : \lambda \in \sigma_{\mathscr{A}}(a)\}$$

is called the *spectral radius* of a.

Proposition 3.13. *Let $(\mathscr{A}, \|\cdot\|, *)$ be a C^*-algebra with unit $\mathrm{id}_{\mathscr{A}}$ and consider some element $a \in \mathscr{A}$. Then $\sigma_{\mathscr{A}}(a)$ is a non-empty compact subset of $\mathbb{C}$ and the resolvent map $\Gamma_{\mathscr{A}}(a) \to \mathscr{A}$, $\lambda \mapsto (\lambda \mathrm{id}_{\mathscr{A}} - a)^{-1}$ is continuous. Moreover, it holds that*

$$\rho_{\mathscr{A}}(a) = \lim_{n\to\infty} \|a^{\circ n}\|_{\mathscr{A}}^{1/n} \leq \|a\|_{\mathscr{A}}.$$

The proof is long, and you can find it in any book on C^*-algebras.

Proposition 3.14. *Let $(\mathscr{A}, \|\cdot\|, *)$ be a C^*-algebra with unit $\mathrm{id}_{\mathscr{A}}$. For $\lambda \in \mathbb{C}$, $a \in \mathscr{A}$ it holds:*

(1) $\sigma_{\mathscr{A}}(\lambda \mathrm{id}_{\mathscr{A}} - a) = \lambda - \sigma_{\mathscr{A}}(a)$,
(2) $\sigma_{\mathscr{A}}(a^*)$ is he complex conjugate of $\sigma_{\mathscr{A}}(a)$,
(3) $\sigma_{\mathscr{A}}(a^{-1}) = (\sigma_{\mathscr{A}}(a))^{-1}$.

Proof.

(1) Exercise.

(2) Let $\lambda \in \sigma_{\mathscr{A}}(a^*)$, i.e., $(\lambda \mathrm{id}_{\mathscr{A}} - a^*)$ is not invertible. Hence, $A = (\overline{\lambda} \mathrm{id}_{\mathscr{A}} - a)$ is not invertible, since $(A^*)^{-1} = (A^{-1})^*$.

(3) In the lecture notes.

□

Definition 3.14 (Special classes)**.** Let $(\mathscr{A}, \|\cdot\|, *)$ be a C^*-algebra. An element $a \in \mathscr{A}$ is called *normal* if $a^* \circ a = a \circ a^*$. It is called *self-adjoint* if $a^* = a$. If $\mathscr{A}$ has an inverse $\mathrm{id}_{\mathscr{A}}$, then a is called *isometric* if $a^* \circ a = \mathrm{id}_{\mathscr{A}}$. It is called *unitary* if it is isometric and invertible.

Theorem 3.15. *Let $(\mathscr{A}, \|\cdot\|, *)$ be a C^*-algebra with unit $\mathrm{id}_{\mathscr{A}}$.*

(1) If a is a normal or self-adjoint element of $\mathscr{A}$, then $\rho_{\mathscr{A}}(a) = \|a\|_{\mathscr{A}}$.
(2) If a is an isometric or unitary element of $\mathscr{A}$, then $\|a\|_{\mathscr{A}} = 1$.
(3) For unitary a, it holds $\sigma_{\mathscr{A}}(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda|_{\mathbb{C}} = 1\}$.
(4) If a is self-adjoint, then $\sigma_{\mathscr{A}}(a) \subseteq [-\rho_{\mathscr{A}}(a), \rho_{\mathscr{A}}(a)]$ with $\rho_{\mathscr{A}}(a) = \|a\|_{\mathscr{A}}$. Consequently, $\sigma_{\mathscr{A}}(a \circ a) \subseteq [0, \|a\|_{\mathscr{A}}^2]$.
(5) For every $a \in \mathscr{A}$ and every polynomial $P(a)$, we have $\sigma_{\mathscr{A}}(P(a)) = P(\sigma_{\mathscr{A}}(a))$.

A small exercise is to show that $(\mathscr{A}, *)$ uniquely determines the norm $\|\cdot\|_{\mathscr{A}}$ (up to a constant factor).

Definition 3.15 (Positivity)**.** An element of a $*$-algebra $\mathscr{A}$ is called *positive* if it is self-adjoint and its spectrum is contained in $[0, \infty)$.

Theorem 3.16. *Let $(\mathscr{A}, \|\cdot\|, *)$ be a C^*-algebra and let $a \in \mathscr{A}$ be a self-adjoint element. Then a is positive if and only if there is a self-adjoint element $b \in \mathscr{A}$ so that $a = b \circ b$.*

In the proof, one constructs the square root of a using functional calculus.

Proposition 3.17. *Let $(\mathscr{A}, \|\cdot\|, *)$ be a C^*-algebra with unit $\mathrm{id}_{\mathscr{A}}$. Let $a \in \mathscr{A}$ be a self-adjoint element and $a_\pm = (|a| \pm a)/2$. Then*
(1) $a_\pm$ is positive,
(2) $a = a_+ - a_-$,
(3) $a_+ \circ a_- = 0$.

Theorem 3.18. *Let $(\mathscr{A}, \|\cdot\|, *)$ be a C^*-algebra and $A \in \mathscr{A}$. Then A is positive if and only if $A = b^* \circ b$ for some $b \in \mathscr{A}$.*

Chapter 4

Representations of quantum observables

Definition 4.1 (*-morphism)**.** Let $(\mathscr{A}, \|\cdot\|_{\mathscr{A}}, *_{\mathscr{A}})$ and $(\mathscr{B}, \|\cdot\|_{\mathscr{B}}, *_{\mathscr{B}})$ be two C^*-algebras. An algebra homomorphism $\pi : \mathscr{A} \to \mathscr{B}$ is called a **-morphism* if it satisfies $\pi(a^{*_{\mathscr{A}}}) = \pi(a)^{*_{\mathscr{B}}}$ for all $a \in \mathscr{A}$.

There are some good questions to ask about *-morphisms. Does a *-morphism preserve positivity? If a is positive, there is an element $b \in \mathscr{A}$ so that $a = b^* \circ b$. Hence,

$$\pi(a) = \pi(b^{*_{\mathscr{A}}} \circ_{\mathscr{A}} b) = \pi(b^{*_{\mathscr{A}}}) \circ_{\mathscr{B}} \pi(b) = \pi(b)^{*_{\mathscr{B}}} \circ_{\mathscr{B}} \pi(b)$$

from which the assertion follows.

Proposition 4.1. *Let $(\mathscr{A}, \|\cdot\|_{\mathscr{A}}, *_{\mathscr{A}})$ and $(\mathscr{B}, \|\cdot\|_{\mathscr{B}}, *_{\mathscr{B}})$ be two C^*-algebras with units $\mathrm{id}_{\mathscr{A}}$ and $\mathrm{id}_{\mathscr{B}}$, respectively. Let $\pi : \mathscr{A} \to \mathscr{B}$ be a unit-preserving *-morphism. Then $\|\pi(a)\|_{\mathscr{B}} \leq \|a\|_{\mathscr{A}}$.*

Lemma 4.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras and let π be a $*$-morphism from $\mathscr{A}$ to $\mathscr{B}$. Then π is continuous.*

Proposition 4.3. *Let $(\mathscr{A}, \|\cdot\|_{\mathscr{A}}, *_{\mathscr{A}})$ and $(\mathscr{B}, \|\cdot\|_{\mathscr{B}}, *_{\mathscr{B}})$ be two C^*-algebras with units $\mathrm{id}_{\mathscr{A}}$ and $\mathrm{id}_{\mathscr{B}}$, respectively. Let $\pi : \mathscr{A} \to \mathscr{B}$ be a unit-preserving *-morphism. Then the range $\pi(\mathscr{A})$ is a C^*-subalgebra of $\mathscr{B}$.*

Definition 4.2 (*-isomorphism)**.** A $*$-morphism between two C^*-algebras that is onto and one-to-one is called a **-isomorphism*.

Definition 4.3 (Representation of a C^*-algebra)**.** A representation of a C^*-algebra $(\mathscr{A}, \|\cdot\|, *)$ is a pair (H, π) where H is a complex Hilbert space and $\pi : \mathscr{A} \to \mathcal{L}(H)$ is a *-morphism. In this context, H is called the *representation space*, π the *representation map* and $\pi(\mathscr{A})$ are called the representatives of $\mathscr{A}$. A representation (H, π) is called *faithful* if π is *-isomorphism between $\mathscr{A}$ and $\pi(\mathscr{A})$.

Proposition 4.4. *Let (H, π) be a representation of a C^*-algebra $(\mathscr{A}, \|\cdot\|, *)$. The representation is faithful if and only if it satisfies the following equivalant conditions:*
(1) $\ker(\pi) = \{0\}$,
(2) $\|\pi(a)\|_{\mathcal{L}(H)} = \|a\|_{\mathscr{A}}$ *for all* $a \in \mathscr{A}$,
(3) for all strictly positive $a \in \mathscr{A}$, $\pi(a)$ *is strictly positive in* $\mathcal{L}(H)$.

Proof.

(1) $\implies$ (2) We can define the inverse morphism $\pi^{-1} : \pi(\mathscr{A}) \to \mathscr{A}$ by $\pi^{-1}(\pi(a)) = a$. Applying a previous estimate several times, we find

$$\|a\|_{\mathscr{A}} = \|\pi^{-1}(\pi(a))\|_{\mathscr{A}} \leq \|\pi(a)\|_{\mathcal{L}(H)} \leq \|a\|_{\mathscr{A}}.$$

Hence, $\|\pi(a)\|_{\mathcal{L}(H)} = \|a\|_A scr$.

(2) $\implies$ (3) If a is positive, ba a theorem (from last lecture), there is a self-adjoint $b \in \mathscr{A}$ so that $a = b^* \circ b$. But then

$$\|a\|_{\mathscr{A}} = \|b^* \circ b\|_{\mathscr{A}} = \|b\|^2_{\mathscr{A}} > 0.$$

So $\|a\|_{\mathscr{A}} > 0$. Be the first part of the proof, $\|\pi(a)\|_{\mathcal{L}(H)} = \|a\|_{\mathscr{A}} > 0$ since $\pi(a) \neq 0$. Thus π respects positivity.

(3) $\implies$ (1) We will prove this by contradiction. Assume there is some $a \in \ker(\pi)$ with $a \neq 0_{\mathscr{A}}$. Then there is $b \in \ker \pi$ with $b \neq 0_{\mathcal{L}(H)}$,

$$\pi(b^{*_{\mathscr{A}}} \circ b) = \pi(b^{*_{\mathscr{A}}}) \circ \pi(b) = (\pi(b))^{*_{\mathcal{L}(H)}} \circ \pi(b) = 0_{\mathcal{L}(H)}.$$

Since π preserves positivity, $b^{*_{\mathscr{A}}} \circ b$ is positive. Consider now

$$\rho_{\mathscr{A}}(b^{*_{\mathscr{A}}} \circ b) = \|b^{*_{\mathscr{A}}} \circ b\|_{\mathscr{A}} = \|b\|^2_{\mathscr{A}}.$$

It is left as an exercise to show that $\pi(b^{*_{\mathscr{A}}} \circ b)$ is strictly positive, which is a contradiction to the assumption. Hence, $\ker(\pi) = \{0\}$.

□

Definition 4.4 (Invariant subspace)**.** Let (H, π) be a representation of a C^*-algebra and let H_s be a closed subspace of H. Then H_s is called an *invariant subspace* (or stable subspace) under π if $\pi(\mathscr{A})H_s \subset H_s$.

Lemma 4.5. *Let (H, π) be a representation of a C^*-algebra and let H_s be a closed subspace of H. Then π commutes with the orthogonal projection $P_s H = H_s$ if and only if H_s is invariant (stable) under π.*

Proof. For all $a \in \mathscr{A}$ it holds $\pi(a)H_\mathrm{s} = (\pi(a) \circ P_\mathrm{s})H \subset H_\mathrm{s}$. This implies $P_\mathrm{s} \circ \pi(a) \circ P_\mathrm{s} = \pi(a) \circ P_\mathrm{s}$. Since orhogonal projectors are self-adjoint (show this for homework), π is a *-morphism, so $\pi(a^{*_\mathscr{A}}) = (\pi(a))^{*_{\mathcal{L}(H)}}$ and

$$\pi(a) \circ P_\mathrm{s} = P_\mathrm{s} \circ \pi(a) \circ P_\mathrm{s} = P_\mathrm{s}^{*_{\mathcal{L}(H)}} \circ (\pi(a^{*_\mathscr{A}}))^{*_{\mathcal{L}(H)}} \circ P_\mathrm{s}^{*_{\mathcal{L}(H)}} = (P_\mathrm{s} \circ \pi(a^{*_\mathscr{A}}) \circ P_\mathrm{s})^{*_{\mathcal{L}(H)}}.$$

Since $a^{*_\mathscr{A}}$ is also an element of $\mathscr{A}$, it holds

$$(P_\mathrm{s} \circ \pi(a^{*_\mathscr{A}}) \circ P_\mathrm{s})^{*_{\mathcal{L}(H)}} = (\pi(a^{*_\mathscr{A}}) \circ P_\mathrm{s})^{*_{\mathcal{L}(H)}} = P_\mathrm{s}\pi(a).$$

□

Lemma 4.6. *Let (H, π) be a representation of a C^*-algebra and let H_s be a closed subspace of H. Let $\pi_\mathrm{s}(a) := P_\mathrm{s} \circ \pi(a) \circ P_\mathrm{s}$ for all $a \in \mathscr{A}$. Then $(H_\mathrm{s}, \pi_\mathrm{s})$ is a representation of $\mathscr{A}$. This is called the subrepresentation of (H, π) induced by H_s.*

Opportunity: if H_s is stable under π, then $H^\perp$ is also stable under π. We can construct a second subrepresentation $(H_\mathrm{s}^\perp, \pi_\mathrm{s}^\perp)$ where $\pi_\mathrm{s}^\perp(a) = P_\mathrm{s}^\perp \circ \pi(a) \circ P_\mathrm{s}^\perp$ for all $a \in \mathscr{A}$. Now H decomposes as $H = H_\mathrm{s} \oplus H_\mathrm{s}^\perp$ and every operator $\pi(a) \in \mathcal{L}(H)$ decomposes accordingly as

$$\pi(a) = \begin{pmatrix} \pi_\mathrm{s}(a) & 0 \\ 0 & \pi_\mathrm{s}^\perp(a) \end{pmatrix}.$$

This induces a decomposition of the representation:

$$(H, \pi) = (H_\mathrm{s}, \pi_\mathrm{s}) \oplus (H_\mathrm{s}^\perp, \pi_\mathrm{s}^\perp).$$

The trivial representation is the zero-*-morphism $\emptyset : \mathscr{A} \to \mathcal{L}(H)$, $\emptyset(a) = 0_{\mathcal{L}(H)}$ for all $a \in \mathscr{A}$.

Note. On a hilbert space H, it holds $\emptyset(a)H = 0_{\mathcal{L}(H)}H = \psi_0$ (the zero vector in H). If we denote the representation space by $R_H(\psi_0) = \mathbb{C}\psi_0$ (that is the ray in the Hilbert space spanned by ψ_0), then $(R_H(\psi_0), \emptyset)$ is a trivial representation of $\mathscr{A}$. Note also that the symbol $\emptyset$ represents the trivial representation, and $\varnothing$ represents the empty set.

Definition 4.5 (Non-degenerate representation)**.** Let (H, π) be a representation of a C^*-algebra and set $H_0 = \{\psi \in H : \pi(\mathscr{A})\psi = \psi_0\}$. Then (H, π) is called *non-degenerate* if $H_0 = R_H(\psi_0)$, i.e., if the only vector that is annihilated by all representatives is the zero vector.

Lemma 4.7. *Let (H, π) be a representation of a C^*-algebra and let $H_0 = \{\psi \in H : \pi(\mathscr{A})\psi = \psi_0\}$. Then H_0 is a closed subspace of H that is invariant under π and there is a subrepresentation (H_0, π_0) of (H, π) with $\pi_0 = P_{H_0} \circ \pi \circ P_{H_0}$. Alternatively, we say that a subset of bounded operators $\mathcal{B}(H) \subseteq \mathcal{L}(H)$ acts non-degenerately on H if $\{\psi \in H : \mathcal{B}(H)\psi = \psi_0\}$ is equal to the ray $R_H(\psi_0)$.*

Definition 4.6 (Cyclic vector)**.** A vector ψ_* in a Hilbert space H is called a *cyclic vector* for a set $\mathscr{B}(H)$ of bounded operators on H if the linear span generaed by $\mathscr{B}(H)\psi_*$ is dense in H.

A prime example of a cyclic vector is the vacuum state in quantum field theory.

Definition 4.7 (Cyclic representation)**.** Let (H, π) be a representation of a C^*-algebra and let ψ_* be a cyclic vector for $\pi(\mathscr{A}) \subset \mathcal{L}(H)$. Then the triple (H, π, ψ_*) is called a *cyclic representation*.

Lemma 4.8. *Every cyclic representation of a C^*-algebra is non-degenerate.*

4.1 Canonical commutation relations

Definition 4.8 (Weyl system)**.** A *Weyl system* is a triple $(V, W, \mathscr{A})$ consisting of a symplectic vector space (V, ω), a C^*-algebra $(\mathscr{A}, \|\cdot\|, *)$ with unit $\mathrm{id}_{\mathscr{A}}$ and a map $W : V \to \mathscr{A}$ so that for all $\varphi, \psi \in V$ the following properties hold:
(1) $W(0) = \mathrm{id}_{\mathscr{A}}$,
(2) $W(-\varphi) = W(\varphi)^*$,
(3) $W(\varphi) \circ W(\psi) = e^{-i\omega(\varphi,\psi)/2} W(\varphi + \psi)$.

Example 10. Let (V, ω) be an arbitrary symplectic vector space and $H = L^2(V, \mathbb{C})$ the Hilbert space of square-integrable complex-valued functions on V that vanish everywhere except for a countably many values in V. For an operator F, it holds $\operatorname{supp} F = S^{\text{finite}} \subset V$. Formally, $F = f \circ \chi_S$ where $f : V \to \mathbb{C}$ are functionals and $\chi_S : V \to S^{\text{finite}}$ is the characteristic function on S^{finite}. The scalar product is given by

$$(F, G)_H = \sum_{\varphi \in V} \overline{F}(\varphi) G(\varphi) = \sum_{\varphi \in S} \overline{f}(\varphi) g(\varphi), \quad \|F\|_H^2 = \sum_{\varphi \in S} |f(\varphi)|^2.$$

The C^*-algebra $\mathscr{A}$ is simply $\mathcal{L}(H)$, the algebra of all bounded operators on H. Define now the map $W : V \to \mathcal{L}(H)$ by

$$(W(\varphi)F)(\psi) = \exp(i\omega(\varphi, \psi)/2) F(\psi + \varphi).$$

This satisfies the Weyl relations:

(1) $W(0_V) = \mathrm{id}_{\mathscr{A}}$,

(2) $W(-\varphi)$ requires some computation:

$$\begin{aligned}(W(\varphi)F, G)_H &= \sum_{\psi \in V} \overline{(W(\psi)F)(\psi)} G(\psi) = \sum_{\psi \in V} \overline{\exp(i\omega(\varphi, \psi)/2) F(\psi + \varphi)} G(\psi) \\ &\overset{\chi = \psi + \varphi}{=} \sum_{\chi \in V} \overline{\exp(i\omega(\varphi, \chi - \varphi)) F(\chi)} G(\chi - \varphi) = \sum_{\chi \in V} \overline{F}(\chi) \exp(-i\omega(\varphi, \chi)/2) G(\chi - \varphi) \\ &= (F, W(-\varphi)G)_H.\end{aligned}$$

(3) By direct computation (for exercise).

Index

Symbols
*-isomorphism 27
*-morphism 27

A
adapted coordinate system 4
advanced fundamental solution 21
Alexandrov topology 20
arc length 21

B
base space 3
bundle morphism 6

C
C^*-algbera 24
causal character 19
causal condition 20
causal future 20
causal past 20
causality relations 19
causally simple spacetime 22
chronological future 20
chronological past 20
cyclic representation 30
cyclic vector 29

D
d'Alembert operator 14
differential operator of order ℓ 11
Dirac's measure 16
distribution 15
distributional section 16
dual bundle 7

F
faithful representation 28
fibre 3
fibre bundle 5
fibred manifold 3
fundamental solution 21

G
generalized Weitzenböck formula 15
gradient 14

I
index 18
inner product 18
invariant subspace 28
isometric element 25

L
leading symbol 11
Levi–Civita connection 9
light cone 19
light-like 19
local trivialization 5
Lorentzian manifold 18

M
metric tensor 18

N
non-degenerate action 29
non-degenerate representation 29
normal element 25
normally hyperbolic operator 15

O
order of distribution 15

P
positive element 25
projection map 3

R
representation map 28

representation space 28
resolvent 24
resolvent set 24
retarded fundamental solution 21
Riemannian manifold 18

S
scalar product 18
section 6
self-adjoint element 25
semi-Riemannian manifold 18
singular support 16
space-like 19
spectral radius 24
spectrum 24
standard ordered quantization 12
strong causality condition 20
subrepresentation 29
symmetrized covariant differentiation 9

T
tensor bundle 7
test sections 10
test space 10
time separation function 21
time-like 19
total space 3
trivial bundle 4
trivialization 4
trivially fibred manifold 3
typical fibre 4

U
unitary element 25

V
vector bundle 7

W
weak* topology 16
Weyl system 30