

Faculty of physics

Quantum Field Theory

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Disclaimer: these notes are not official and were not endorsed by professor. The content is almost entirely based on the lectures, but occasionally the notes were modified or extended slightly to improve clarity or add extra information. In particular, the mistakes are almost surely mine. If you find any, please let me know at simon.bukovsek@tum.de or Simon.Bukovsek@campus.lmu.de.

Chapter 1

Review of basic paradigms

Action defines theory as:

$$S = \int \mathrm{d}^d x \, \mathcal{L}(\phi_i(\mathbf{x}, t), g_i),$$

where d is the number of dimensions (usually $d = 1 + 3$) and $\mathcal{L}$ is *Lagrangian density* which is a functional of fields ϕ_i and fundamental constants g_i. Index i denotes a type of field (scalar, vector, spinor, etc.). Some examples of g_i are α_{QED}, m_e, etc. In relativistic QFT we have:

1. ϕ_i are irreducible representations of the Poincaré group (translations + Lorentz group),
2. internal symmetries (e.g. U(1), SU(2), SU(3), discrete symmetries).

Lagrangian $\mathcal{L}$ is a singlet under 1) and 2) up to the total derivative.

1.1 Poincaré group

Now we will review the invariance under *Lorentz transformations* (LT) and space-time translations. Element of *Poincaré group* is $g = (\Lambda, a)$, where Λ is a Lorentz transformation and a^μ is a space-time translation. The element acts on a vector x^μ as:

$$x^\mu \xrightarrow{g} x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu.$$

Group element g is mapped to a representation of unitary operator $(\Lambda, a)\mathcal{U}(\Lambda, a)$. It can work on a state in *Hilbert space* $|\psi\rangle \in \mathcal{H}$ as

$$|\psi\rangle \xrightarrow{g} \mathcal{U}(\Lambda, a)\,|\psi\rangle$$

and on an operator $\mathcal{O}$ as

$$\mathcal{O} \xrightarrow{g} \mathcal{U}(\Lambda, a)\,\mathcal{O}\,\mathcal{U}^{-1}(\Lambda, a).$$

Ultimately, physics should be unchanged under g-transformation, i.e. $\langle\psi'|\psi\rangle = \langle\psi|\mathcal{U}^\dagger\mathcal{U}|\psi\rangle = \langle\psi|\psi\rangle$. This implies that $\mathcal{U}^\dagger\mathcal{U} = \mathbb{1}$, i.e. $\mathcal{U}$ is a unitary operator. Action is invariant under Lorentz transformations if $\mathcal{L}$ is a scalar operator. If we compare this to a 4-vector A^μ, we see that it transforms as

$$A^\mu(x) \xrightarrow{(\Lambda,0)} \mathcal{U}(\Lambda, 0)\,A^\mu(x)\mathcal{U}^{-1}(\Lambda, 0) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x).$$

1.2 Lorentz group

Lorentz group is a SO(1, 3) (special orthogonal) group. Morally, we can think of it as SO(4) in euclidean space. The letter O (orthogonal) indicates, it preserves distances $\mathbf{v}^\intercal\mathbf{v} = t^2+x^2+y^2+z^2$, the letter S (special) indicates that it has determinant equal to 1. The only difference with SO(1, 3) is that it preserves a different type of distance, namely $t^2 - \mathbf{x}^2$. We have

$$x_\mu x^\mu = x^\mu x^\nu \eta_{\mu\nu} = x_\mu x_\nu \eta^{\mu\nu},$$

where

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is *Minkowski signature metric*. With the condition that $x_\mu x^\mu$ is invariant under Lorentz transformations, we can derive the following condition for Λ:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \text{ and } x_\mu x^\mu = x'_\mu x'^\mu \implies \boxed{\Lambda^\intercal \eta \Lambda = \eta}.$$

Two examples of Lorentz transformations are:

$$\Lambda_1(\beta) = \begin{bmatrix} \cosh\beta & \sinh\beta & 0 & 0 \\ \sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which is a *Lorentz boost* in x direction, and

$$\Lambda_2(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which is a rotation in xy plane. Instead of *rapidity* β we could use the following transformation:

$$x \to \frac{x + vt}{\sqrt{1 - v^2/c^2}}, \quad t \to \frac{t + vx/c^2}{\sqrt{1 - v^2/c^2}}, \quad y \to y, \quad z \to z.$$

Inspection shows that $\sinh\beta = v/\sqrt{1 - v^2/c^2}$. In non-relativistic limit, the above transformation becomes $x \to x + vt$, $t \to t$, which is exactly *Galilean transformation*.

Two events x^μ and y^μ in space-time can have different space-time separations. We define the *invariant interval* as

$$\Delta s^2 = (x - y)^\mu (x - y)^\nu \eta_{\mu\nu} = (x^0 - y^0)^2 - (\mathbf{x} - \mathbf{y})^2.$$

We distinguish between three types of intervals:

- $\Delta s^2 > 0$ is a time-like interval,
- $\Delta s^2 = 0$ is a light-like interval,

- $\Delta s^2 < 0$ is a space-like interval.

The most important idea from this section is the following:

Two events with space-like separation ($\Delta s^2 < 0$) cannot be causally connected to each other.

Why is that? Let's look at a simple example. Let x^μ and y^μ be space-like separated ($\Delta s^2 < 0$). For simplicity, we will assume that in initial frame $x^0 = y^0 = 0$. We have $\Delta s^2 = -(\mathbf{x} - \mathbf{y})^2$. Light cannot travel from x^μ to y^μ. Since they are space-like separated, there is no definite time ordering. For simplicity, we use $x^\mu = (0, \mathbf{0})$ and $y^\mu = (0, \mathbf{y})$. If we use a boost in x-direction, we can transform x^μ to $x'^\mu = (0, \mathbf{0})$ and

$$y^\mu \to y'^\mu = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} v y_x \\ \mathbf{y} \end{pmatrix},$$

which has a positive time component. If we used a boost in negative x-direction, we would get

$$y''^\mu = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} -v y_x \\ \mathbf{y} \end{pmatrix},$$

which has a negative time component. So y' happened after y and y'' happened after.

Chapter 2

The Lagrangian formalism

Think of a field as a limit of infinitely many coordinates $q_n(t)$. We can denote the position of the coordinate in the index $q_{n,\mathbf{x}}(t)$ and in the limit it becomes a field $\phi_n(t,\mathbf{x}) = \phi_n(x)$. The sum of the coordinates $\sum_{n,\mathbf{x}}$ becomes $\sum_n \int \mathrm{d}^3x$. The Lagrangian can be written as

$$L = \int \mathrm{d}^3x\ \mathcal{L}(\phi_n(x), \nabla\phi_n(x), \dot{\phi}_n(x)),$$

where $\mathcal{L}$ is called the Lagrangian density. The *Euler-Lagrange equations* require the variation of action to be zero:

$$\delta S = \delta \int \mathrm{d}^4x\ \mathcal{L} = 0.$$

We can expand the variation in Taylor series:

$$\delta S = \int \mathrm{d}^4x\ \left(\frac{\partial \mathcal{L}}{\partial \phi_n}\delta\phi_n + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial_\mu(\delta\phi_n) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_n)}\partial_\mu\partial_\nu(\delta\phi_n u) + \dots \right),$$

integrate by parts and use the fact that fields vanish at $|x| \to \infty$. We get

$$\delta S = \int \mathrm{d}^4x\ \left(\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_n)} + \partial_\mu\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_n)} + \dots \right) \delta\phi_n = 0.$$

Usually, we have $\mathcal{L} = \frac{1}{2}\partial_\mu\phi_n\partial^\mu\phi_n - V(\phi_n)$, so we can drop the second derivative. We get the Euler-Lagrange equation:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_n)} - \frac{\partial \mathcal{L}}{\partial \phi_n} = 0.$$

2.1 Canonical quantization

Now define the *canonically conjugate* (momentum) field

$$\pi_n(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi_n(x))}$$

and the Hamiltonian

$$H = \int \mathrm{d}^3x\ \mathcal{H}(\phi_n(x), \pi_n(x), \nabla\phi_n(x)) = \int \mathrm{d}^3x\ \pi_n(x)\dot{\phi}_n(x) - \mathcal{L}.$$

As in quantum mechanics, we will impose the (equal-time) *canonical commutation relations*

$$[\phi_n(t,\mathbf{x}),\pi_m(t,\mathbf{y})] = i\delta_{nm}\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

and

$$[\phi_n(t,\mathbf{x}),\phi_m(t,\mathbf{y})] = [\pi_n(t,\mathbf{x}),\pi_m(t,\mathbf{y})] = 0.$$

Time evolution is given by the *Heisenberg equation* of motion

$$\partial_t\phi_n(x) = i[H,\phi_n(x)], \quad \partial_t\pi_n(x) = i[H,\pi_n(x)].$$

We assume that H doesn't explicitly depend on time or equivalently, that we have a conserved system. Formal solution of the Heisenberg equation of motion is

$$\phi_n(t,\mathbf{x}) = e^{iHt}\phi_n(0,\mathbf{x})e^{-iHt}.$$

Comments:

1. Imposing canonical commutation relations at time t automatically preserves them at any other time $t_0 \neq t$:

$$\begin{aligned}[\phi_n(t,\mathbf{x}),\pi_m(t,\mathbf{y})] &= [e^{iH(t-t_0)}\phi_n(t_0,\mathbf{x})e^{-iH(t-t_0)}, e^{iH(t-t_0)}\pi_m(t_0,\mathbf{y})e^{-iH(t-t_0)}] \\ &= e^{iH(t-t_0)}[\phi_n(t_0,\mathbf{x}),\pi_m(t_0,\mathbf{y})]e^{-iH(t-t_0)} \\ &= e^{iH(t-t_0)}i\delta_{mn}\delta^{(3)}(\mathbf{x}-\mathbf{y})e^{-iH(t-t_0)} \\ &= i\delta_{mn}\delta^{(3)}(\mathbf{x}-\mathbf{y})\end{aligned}$$

2. We will use the *Heisenberg picture*, where operators evolve in time and states are time-independent.

2.2 Noether's theorem

In short, *Noether's theorem* states that for every continuous symmetry of the action, there is an associated conservation law. Even more – not only does the symmetry imply a *conserved charge*, but also a *conserved current* $\mathcal{J}^\mu(x)$ for each point x^μ.

Let's take a look at a generic symmetry transformation:

$$\phi_n'(x) = \phi_n(x) + \varepsilon F_n(\phi_n(x),\partial_\mu\phi_n(x)) + \mathcal{O}(\varepsilon^2),$$

which leaves the action $S[\phi_n(x)]$ invariant: $S[\phi_n(x)] = S[\phi_n'(x)]$.

Examples:

1. Translations: $x^\mu \to x^\mu + a^\mu$.

$$\phi_n'(x) = \phi_n(x+a) = \phi_n(x) + a^\mu\partial_\mu\phi_n(x) + \dots$$

2. Plane rotations:

$$\phi_n'(x) = e^{i\alpha}\phi_n(x) = \phi_n(\alpha) + i\alpha\phi_n(x) + \dots$$

3. Lorentz transformations: see exercises.

Noether's theorem: For any continuous symmetry of the action, there exists a conservation law ($\partial_\mu \mathcal{J}^\mu(x) = 0$) when equations of motion (e.o.m.) are satisfied.

The conserved current is given by

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n(x))} F_n(x) - K^\mu(x)$$

and the conservation law is $\partial_\mu j^\mu(x)\big|_{\text{EOM}} = 0$. The derivative of the K^μ is the variation of the Lagrangian:

$$\partial_\mu K^\mu = \delta \mathcal{L}.$$

The charge is given by

$$Q = \int \mathrm{d}^3x \; j^0(x),$$

and it is conserved $\mathrm{d}Q/\mathrm{d}t = 0$.

Derivation:

1. Action is invariant under the transformation $\phi_n(x) \to \phi'_n(x)$, so the variation of the action is zero:

$$0 = \delta_F S[\phi] = \int \mathrm{d}^4x \; \delta_F \mathcal{L} = \int \mathrm{d}^4x \; \left(\mathcal{L}(\phi', \partial_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi)\right) = \int \mathrm{d}^4x \; \varepsilon \partial_\mu K^\mu.$$

2. Further we have

$$\delta_F \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_n} \delta_F \phi_n + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \delta_F(\partial_\mu \phi_n) = \varepsilon \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} F_n \right).$$

We subtract the two equations:

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} F_n(\phi_n) - K^\mu(x) \right) = \partial_\mu j^\mu(x).$$

For the conserved charge we have

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int \mathrm{d}^3x \; \partial_0 j^0(x) = -\int_{\mathbb{R}^3} \mathrm{d}^3x \; \nabla \cdot \mathbf{j}(x) = -\int_{\partial \mathbb{R}^3} \mathrm{d}\mathbf{S} \cdot \mathbf{j}(x) = 0,$$

since current vanishes at infinity.

Examples:

1. *Translations:* $\mathcal{L}'(x) = \mathcal{L}(x + a) = \mathcal{L}(x) + a^\alpha \partial_\alpha \mathcal{L}(x) + \dots$ We have $F_{n,\alpha} = \partial_\alpha \phi_n(x)$ and $\partial_\alpha \mathcal{L} = \partial_\mu(\delta^\mu{}_\alpha \mathcal{L})$. So $K^\mu{}_\alpha = \delta^\mu{}_\alpha \mathcal{L}$. The conserved current is

$$T^\mu{}_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\alpha \phi_n - \delta^\mu_\alpha \mathcal{L}.$$

This is called the *energy-momentum tensor*. The index α is the index of the generator of the translation and μ is the Lorentz index. T^μ_α contains 4 conserved currents:

$$P^\alpha = \int \mathrm{d}^3x \; T^{\alpha 0} = \int \mathrm{d}^3x \; \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial^\alpha \phi_n - \eta^{\alpha 0} \mathcal{L} \right).$$

For $\alpha = 0$ we get the conservation of energy:

$$P^0 = \int \mathrm{d}^3x \ \left(\pi_n \dot{\phi}_n - \mathcal{L}\right) = H$$

and for $\alpha = i$ we get the conservation of momentum:

$$P^i = \int \mathrm{d}^3x \ \pi_n(x)\partial^i \phi_n(x).$$

It is proven that T^μ_α is the only conserved tensor.

2. *Pure rotations:* $\phi'_n(x) = e^{-i\alpha}\phi_n(x)$. We have $\mathcal{L}(\phi') = \mathcal{L}(\phi)$, thus $K^\mu = 0$. The field is now complex valued, so $\phi(x) = \varphi_1(x) + i\varphi_2(x)$ and $\varphi_{1,2} \in \mathbb{R}$. Further, $\delta\phi = -i\varepsilon\phi_n(x) = \varepsilon F_\phi$ and $\delta\phi^\dagger_n = i\varepsilon\phi^\dagger_n(x) = \varepsilon F_{\phi^\dagger}$. The conserved current is

$$j_\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}F_{\phi_n} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger_n)}F_{\phi^\dagger_n} = -i\left(\phi_n(x)\partial_\mu\phi^\dagger_n(x) - \phi^\dagger_n\partial_\mu\phi_n\right) = -i(\phi_n\overleftrightarrow{\partial_\mu}\phi^\dagger_n).$$

We assumed the Lagrangian has the form $\mathcal{L} = \partial_\mu\phi_n\partial^\mu\phi^*_n$. This charge corresponds to physical charge conservation and electric currents.

Chapter 3

Quantizing the free scalar field

We will use the natural units $\hbar = 1$ and $c = 1$. With that we can express many units just with some power of mass. Hence, we use the notation $[A]$ for the mass dimension of quantity A. Using equations $E^2 = m^2c^4 + p^2c^2$ and $\lambda = \hbar/p$, we find

$$[H] = [m] = [p] = [E] = [\partial] = 1, \quad [x] = -1, \quad [S] = [\hbar] = 0, \quad [\mathcal{L}] = 4.$$

We can also write

$$[\text{time}] = [\text{energy}^{-1}] = [\text{length}] = [\text{momentum}^{-1}].$$

Another important question is the dimension of the field $\phi(x)$. We can find it by looking at the kinetic term in the Lagrangian:

$$S = \int \mathrm{d}^d x \, \frac{1}{2}\partial_\mu\phi\partial^\mu\phi.$$

We get $[\phi] = (d-2)/2$. For $d = 4$, we have $[\phi] = 1$.

We will now examine the interaction terms in the Lagrangian. The kinetic term is

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi.$$

The interaction terms can be written as a Taylor series:

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \lambda_0 + \lambda_1\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{3!}\lambda_3\phi^3 + \frac{1}{4!}\lambda_4\phi^4 + \dots$$

The first constant λ_0 does not change the equations of motion and might only have the influence in the curved space-time as a cosmological constant (the contribution of type $\int \mathrm{d}^4x \, \sqrt{-g}\Lambda_{\text{cc}}$). We can get rid of λ_1 by substitution

$$\phi' = \phi - \frac{\lambda_1}{m^2} + \mathcal{O}(\lambda_3, \lambda_4).$$

For the remaining terms, we have $[m^2] = 2$, $[\lambda_3] = 1$ and $[\lambda_4] = 0$. All the following terms have negative *mass dimension* and are called irrelevant, since they contribute less and less to the physics low energies.

Our Lagrangian is then

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi),$$

where

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{3!}\lambda_3\phi^3 + \frac{1}{4!}\lambda_4\phi^4.$$

Solving the Euler-Lagrange equation, we get

$$(\partial^2 + m^2)\phi + \frac{\lambda_3}{2}\phi^2 + \frac{\lambda_4}{3!}\phi^3 = 0.$$

The first part of the equation is the *Klein-Gordon equation* which corresponds to the free theory. The other terms are non-linear and correspond to interactions.

Free scalar field: the equation of motion just the Klein-Gordon equation: $(\partial^2 + m^2)\phi = 0$. Using Fourier transformation, we get

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left(e^{-ipx}a_{\mathbf{p}} + e^{ipx}a^\dagger_{\mathbf{p}}\right),$$

where a and $a^\dagger$ are (for now) just the coefficients of Fourier transformation. The product of 4-vectors is implicit: $px = p_\mu x^\mu = E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x}$, where we demand that $E_{\mathbf{p}}$ is on-shell $p^0 = E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$. This corresponds to the fact that the Fourier transformation of Klein-Gordon equation requires $p^2 = m^2$. The factor of $2E_{\mathbf{p}}$ is there to ensure the Lorentz invariance of the measure:

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} = \int \frac{\mathrm{d}^4p}{(2\pi)^4}(2\pi)\delta(p^2 - m^2)\theta(p^0).$$

In order to quantize the free theory, we need to promote $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ to operators, which satisfy the canonical commutation relations:

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q})$$

and

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = [a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = 0.$$

The operators are called *annihilation* ($a_{\mathbf{p}}$) and *creation* ($a^\dagger_{\mathbf{p}}$) operators. A quick calculation shows

$$[\phi(x), \phi(y)] = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right) = \Delta(x - y).$$

The object $\Delta(x - y)$ is Lorentz invariant since both the integrand and the measure are Lorentz invariant. Setting the times $x^0 = y^0 = t$ to be equal, we get

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] = \Delta(x-y)\Big|_{x^0=y^0} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i}{2}\left(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right) = 0 \tag{3.1}$$

as we would expect from equal-time commutation relations. The interesting commutator is

$$\begin{aligned}
[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] &= \left[\phi(x^0,\mathbf{x}), \frac{\partial}{\partial y_0}\phi(y^0,\mathbf{y})\right] = \left(\frac{\partial}{\partial y_0}\Delta(x-y)\right)\Big|_{x^0=y^0} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} iE_{\mathbf{p}}\left(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right) \\
&= i\int \frac{\mathrm{d}^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\
&= i\delta^{(3)}(\mathbf{x} - \mathbf{y}).
\end{aligned}$$

Similarly, we can calculate $[\pi(x), \pi(y)]\big|_{x^0=y^0} = 0$.

Next, we will show that $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$. We can choose $x^0 = y^0 = 0$ and $(x-y)^2 = -(\mathbf{x}-\mathbf{y})^2$. The statement follows from (3.1). For time-like distances $\Delta(x-y) \neq 0$. For example $x^\mu - y^\mu = (t, \mathbf{0})$. We have

$$\Delta(t) = [\phi(x), \phi(y)] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} (-2i) \sin(E_{\mathbf{p}} t) \neq 0.$$

What about light-like distance? We will see this later on. The Hamilton operator becomes

$$H_0 = \int \mathrm{d}^3 x \, (\pi(x)\dot{\phi}(x) - \mathcal{L}_0) = \frac{1}{2} \int \mathrm{d}^3 x \, \left((\partial_t \phi)^2 + (\nabla\phi)^2 + m^2\phi^2 \right).$$

3.1 Particle states and quanta of the field

We shall construct a Hilbert space of particle states. We define the *vacuum state* $|0\rangle$ as the state, that gives zero for every annihilation operator: $a_{\mathbf{p}} |0\rangle = 0$ for all $\mathbf{p}$ and also impose normalization $\langle 0|0\rangle = 1$. The single particle states are $|\mathbf{p}\rangle = a^\dagger_{\mathbf{p}} |0\rangle$. Two states are related as

$$\langle \mathbf{p}|\mathbf{k}\rangle = \langle 0|a_{\mathbf{p}} a^\dagger_{\mathbf{k}}|0\rangle = \langle 0|0\rangle \, 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{k}) + \langle 0|a^\dagger_{\mathbf{k}} a_{\mathbf{p}}|0\rangle = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{k}).$$

State $|\mathbf{p}\rangle$ has $E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$. In the lecture notes we can find the proof of the following relation

$$\mathbf{P} = \int \mathrm{d}^3 x \, \pi(x) \nabla\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \mathbf{p} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}.$$

With that, we can evaluate the following:

$$\begin{aligned}
\mathbf{P}\,|\mathbf{p}\rangle &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2E_{\mathbf{k}}} \mathbf{k} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{p}} \,|0\rangle \\
&= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2E_{\mathbf{k}}} \mathbf{k} a_{\mathbf{k}} \left(a^\dagger_{\mathbf{p}} a_{\mathbf{k}} + 2Ep[k](2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{p}) \right) |0\rangle \\
&= \mathbf{p} a^\dagger_{\mathbf{p}} \,|0\rangle = \mathbf{p}\,|\mathbf{p}\rangle \,.
\end{aligned}$$

We can also show $H_0\,|\mathbf{p}\rangle = E_{\mathbf{p}}\,|\mathbf{p}\rangle$, where $E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$. ...

$$\mathcal{J}_i = Q_i = \epsilon_{ijk} \int \mathrm{d}^3 x \, (x^j T^{0k} - x^k T^{0i}).$$

For the low-energy limit $\mathcal{J}_i\,|\mathbf{p}=0\rangle = 0$, which corresponds to the fact that ϕ is a scalar field.

3.2 Multiple states

We want all n-particle Hilbert spaces, so we use a *Fock space*. A single particle state is created by a single creation operator on the vacuum state:

$$a^\dagger_{\mathbf{p}} \,|0\rangle = \frac{1}{2E_{\mathbf{p}}} \,|\mathbf{p}\rangle \,.$$

The vacuum state is a special state, for which $\langle 0|0\rangle = 1$. Multi-particle states are created by consecutive applications of creation operators on the vacuum state $|0\rangle$:

$$a^\dagger_{\mathbf{p_1}} a^\dagger_{\mathbf{p_2}} \dots a^\dagger_{\mathbf{p_n}} |0\rangle = \frac{1}{2E_{\mathbf{p_1}} 2E_{\mathbf{p_2}} \dots 2E_{\mathbf{p_n}}} |\mathbf{p_1}, \mathbf{p_2}, \dots, \mathbf{p_n}\rangle .$$

There is a special operator that counts the number of particles – a *number operator*

$$N = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}.$$

It works on a multi-particle state as

$$N |\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle = n |\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle .$$

In the free theory the number of particles is conserved $[N, H_0] = 0$. However, in the interaction theory $[H, N] \neq 0$. If we apply $\phi(x)$ to the vacuum state, we get

$$\phi(x) |0\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{ipx} |\mathbf{p}\rangle .$$

...

$$\langle 0|\phi(x)|\mathbf{p}\rangle = \langle 0| \int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \frac{1}{2E_{\mathbf{p'}}} \left(a_{\mathbf{p'}} e^{-ip'x} + a^\dagger_{\mathbf{p'}} e^{ip'x} \right) a^\dagger_{\mathbf{p}} |0\rangle = e^{-ipx}.$$

When $t = 0$ this is analogous to QM: $\langle \mathbf{x}|\mathbf{p}\rangle = e^{i\mathbf{x}\cdot\mathbf{p}}$.

We will briefly comment on the light-like separation. It turns out that $[\phi(x), \phi(y)] \neq 0$, but only if $m^2 = 0$.

3.3 First encounter with infinities in QFT

There are multiple types of infinities in QFT:

1. *IR divergences* are caused by assuming infinitely large scales,
2. *UV divergences* are caused by extrapolating to infinitely small scales (or infinite energies).

Let's calculate the energy of the vacuum state:

$$\begin{aligned} H_0 |0\rangle &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} E_{\mathbf{p}} a_{\mathbf{p}} a^\dagger_{\mathbf{p}} \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} E_{\mathbf{p}} \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(0) \right) |0\rangle \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} E^2_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(0) |0\rangle . \end{aligned}$$

This is an infinite quantity. Actually, it is more like ∞^2, as we integrate infinity over infinite range. We can remove the IR divergence by constraining ourselves to a finite volume V:

$$(2\pi)^3 \delta^{(3)}(\mathbf{p} = 0) = \lim_{L\to\infty} \int_{-L/2}^{L/2} \mathrm{d}x \int_{-L/2}^{L/2} \mathrm{d}y \int_{-L/2}^{L/2} \mathrm{d}z \, e^{i\mathbf{p}\cdot\mathbf{x}} \Big|_{\mathbf{p}=\mathbf{0}} = V.$$

Now we can calculate the energy density of the vacuum state:

$$\mathcal{E}_0 = \frac{E}{V} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} = \frac{4\pi}{2(2\pi)^2} \int_0^\infty \mathrm{d}|\mathbf{p}|\, |\mathbf{p}|^2 \sqrt{m^2 + \mathbf{p}^2} \to \infty.$$

The integral is *still* divergent. This is due to the fact that we integrated to infinite energy where we don't know our theory at all. We can regulate the integral by introducing an ultraviolet cut-off Λ:

$$\mathcal{E}_0(\Lambda) = \frac{1}{4\pi^2} \int_0^\Lambda \mathrm{d}|\mathbf{p}|\, |\mathbf{p}|^2 \sqrt{m^2 + \mathbf{p}^2} = a\Lambda^4 + \dots$$

This procedure is called *regularization*.

We can renormalize the theory by subtracting a constant parameter V_0 from the Lagrangian: $\mathcal{L}_{\text{new}} = \mathcal{L} - V - 0$. The parameter V_0 is called a bare parameter, and it cancels the infinity as $V_0 = -\mathcal{E}_0(\Lambda) + \chi$. The observed energy density is then

$$\frac{1}{V} H \left|0\right\rangle = (\mathcal{E}_0 + V_0) \left|0\right\rangle = (\mathcal{E}_0 - \mathcal{E}_0 + \chi) \left|0\right\rangle = \chi \left|0\right\rangle .$$

The value χ is the one that we actually measure and is finite.

3.4 Complex scalar field

Assume $\phi \in \mathbb{C}$ and hence $\phi^\dagger \neq \phi$. As always, we can expand ϕ as a linear combination of two real fields ϕ_1 and ϕ_2:

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)).$$

The Lagrangian is then

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi, \phi^\dagger),$$

where

$$V(\phi, \phi^\dagger) = m^2 \phi^\dagger \phi + \frac{\lambda}{4!}(\phi^\dagger \phi)^2 + [\lambda_4 \phi^2 + \phi_{31}\phi^3 + \lambda_{32}\phi^2\phi^\dagger + \lambda_{41}\phi^\dagger + \lambda_{42}\phi^3\phi^\dagger + \text{h.c.}].$$

The constants m^2 and λ are real, others are complex. If the terms in square brackets is equal to zero, the Lagrangian has a $\mathcal{U}(1)$ symmetry, i.e. $\phi \to e^{i\alpha}\phi$ for $\alpha \in \mathbb{R}$. If the terms are not zero, there is a linear combination of ϕ_1 and ϕ_2 with two eigenstates with different masses $m_1 \neq m_2$, so it is a theory of two massive, real scalar fields interacting with each other. In the case of the brackets being zero, the equation of motion is

$$\Box\phi \frac{\partial V}{\partial \phi^\dagger} = (\Box + m^2)\phi + \lambda(\phi^\dagger, \phi)\phi = 0.$$

The invariance of Lagrangian under $\mathcal{U}(1)$ symmetry does *not* imply the invariance of the equation of motion, but rather *covariance* under the symmetry. Because it is difficult to figure out if the equation is covariant, we prefer the Lagrangian description.

The Hamiltonian can be written as

$$H = \int \mathrm{d}^3 x \left(\frac{\partial \phi^\dagger}{\partial t} \frac{\partial \phi}{\partial t} + (\nabla \phi^\dagger)(\nabla \phi) + m^2 \phi^\dagger \phi + \frac{\lambda}{4}(\phi \phi^\dagger)^2 \right)$$

and the conjugate momentum is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \dot{\phi}^\dagger.$$

The free theory demands $(\Box + m^2)\phi = 0$. The wrong ansatz would be to write

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} a_{\mathbf{p}} e^{-ipx},$$

which gives $p^0 = E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$. However, since $\phi \neq \phi^\dagger$, the causality relation

$$[\phi^\dagger(x), \phi(y)] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \neq 0$$

is not equal zero! To fix this, we use the insight from the real field ϕ, which also used $a^\dagger_{\mathbf{p}}$. So we will add two new operators $b_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$ with the same commutation relations as $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$:

$$[b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$$

and

$$[a_{\mathbf{p}}, b^\dagger_{\mathbf{p}}] = [a^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{p}}] = [b_{\mathbf{p}}, b_{\mathbf{q}}] = 0.$$

The complex field ϕ can be written as

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(a_{\mathbf{p}} e^{-ipx} + \lambda b^\dagger_{\mathbf{p}} e^{ipx} \right)$$

for $\lambda \in \mathbb{C}$. Requiring the commutation relations to be satisfied $[\phi(x)^\dagger, \phi(y)] = 0$ for $(x-y)^2 < 0$, it follows $|\lambda| = 1$ or $\lambda = e^{i\beta}$. With $\pi(x) = \dot{\phi}^\dagger(x)$ it holds

$$[\phi(x), \pi(y)]\big|_{x^0 = y^0} = i\delta^{(3)}(\mathbf{x} - \mathbf{y}).$$

Interpretation: a complex field ϕ with $\mathcal{U}(1)$ symmetry describes two particles with equal mass: particle and anti-particle. The Hamiltonian is

$$H_0 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} E_{\mathbf{p}} \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}} \right)$$

with $E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$. The conserved current j^μ associated with the $\mathcal{U}(1)$ symmetry is

$$j_\mu(x) = -i \left(\phi \partial_\mu \phi^\dagger - \phi^\dagger \partial_\mu \phi \right)$$

and the conserved charge Q is

$$Q = \int \mathrm{d}^3 x \; j^0(x) = \cdots = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} (a^\dagger_{\mathbf{p}} a_{\mathbf{p}} - b^\dagger_{\mathbf{p}} b_{\mathbf{p}}) = N_a - N_b.$$

The operator $a^\dagger_{\mathbf{p}}$ creates particle with $q = +1$ and $b^\dagger_{\mathbf{p}}$ creates particle with $q = -1$.

Generally, relativistic causality requires for any particle with non-zero charge to exist a corresponding particle with the same mass. Since charge is conserved $\partial_t Q = 0$, it holds

$$\partial_t [H_0, Q] = 0$$

This is also true for the interaction Hamiltonian H^{int}, so the charge is conserved in the interaction theory as well $[H, Q] = 0$ for $H = H_0 + H^{\mathrm{int}}$.

Chapter 4

Interacting fields, interaction picture and perturbation theory

4.1 Time-dependence of interacting fields

In the free case, we distinguish between the *Heisenberg picture* ϕ_{H}, where operators evolve in time and states are time-independent, and the *Schrödinger picture* ϕ_{S}, where operators are time-independent and states evolve in time. They are related by

$$\phi_{\mathrm{H}}(x) = \phi_{\mathrm{H}}(t, \mathbf{x}) = e^{iHt}\phi_{\mathrm{S}}(\mathbf{x})e^{-iHt}.$$

When the interaction is added the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4.$$

The last two terms are the interaction terms $\mathcal{L}^{\mathrm{int}} = -\frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4$. The equation of motion is

$$(\Box + m^2)\phi = -\frac{\lambda_3}{2}\phi^2 - \frac{\lambda_4}{6}\phi^3.$$

We could include the time dependence of λ_3 and λ_4, we would find that in the interacting Lagrangian the $a_{\mathbf{p}}(t)$ and $a^\dagger_{\mathbf{p}}(t)$ are time dependent, hence the definition of a particle is time dependent.

From now on, we will assume that the interaction is weak, so that we can solve the equations of motion in perturbative expansion. The best picture most useful for this approach is called the *interaction picture*. It is a combination of Schrödinger and Heisenberg picture:

$$|\psi(t)\rangle_{\mathrm{I}} = e^{iH_0t}\,|\psi(t)\rangle_{\mathrm{S}}\,.$$

The operator in the interaction picture is

$$\mathcal{O}_{\mathrm{I}}(t) = e^{iH_0t}\mathcal{O}_{\mathrm{S}}e^{-iH_0t}.$$

Hence, the free Hamiltonian H_0 behaves as a Heisenberg operator and the interaction Hamiltonian H^{int} behaves as a Schrödinger operator:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(t)\rangle_{\mathrm{S}} = H_{\mathrm{S}}\,|\psi(t)\rangle_{\mathrm{S}}\,.$$

Furthermore,

$$i\partial_t \left(e^{-iH_0 t}\left|\psi\right\rangle_{\mathrm{I}}\right) = (H_0 + H^{\mathrm{int}})_{\mathrm{S}} e^{-iH_0 t}\left|\psi\right\rangle_{\mathrm{I}}$$

and

$$ie^{-iH_0 t}\left(-iH_0^{\mathrm{S}}\left|\psi\right\rangle_{\mathrm{I}} + \partial_t\left|\psi\right\rangle_{\mathrm{I}}\right) = (H_0 + H^{\mathrm{int}})_{\mathrm{S}}\left|\psi\right\rangle_{\mathrm{I}}.$$

The free Hamiltonian acting on a state is zero, so we get

$$\partial_t\left|\psi\right\rangle_{\mathrm{I}} = e^{iH_0 t}H_{\mathrm{S}}^{\mathrm{int}}e^{-iH_0 t}\left|\psi\right\rangle_{\mathrm{I}} = H_{\mathrm{I}}^{\mathrm{int}}\left|\psi\right\rangle_{\mathrm{I}}.$$

The transition between Heisenberg and interaction picture is given by

$$\phi_{\mathrm{I}}(0,\mathbf{x}) = \phi_{\mathrm{H}}(0,\mathbf{x})$$

and

$$H = H_{\mathrm{H}}^0 + H_{\mathrm{H}}^{\mathrm{int}} = H_{\mathrm{I}}^0 + H_{\mathrm{I}}^{\mathrm{int}}.$$

Define

$$\phi_{\mathrm{I}}(t,\mathbf{x}) = e^{iH_0 t}\phi_{\mathrm{I}}(0,\mathbf{x})e^{-iH_0 t}$$

with $H_0 = H_{\mathrm{H}}^0(t=0)$. The transformation is thus

$$\partial_t\phi_{\mathrm{I}}(t,\mathbf{x}) = i[H_{\mathrm{I}}^0, \phi(t,\mathbf{x})]$$

and

$$\left|\psi(t)\right\rangle_{\mathrm{I}} = e^{iH_0 t}e^{-iHt}\left|\psi(t)\right\rangle_{\mathrm{H}}.$$

The two pictures are equivalent:

$${}_{\mathrm{H}}\langle\psi|\mathcal{O}_{\mathrm{H}}(t)|\psi\rangle_{\mathrm{H}} = {}_{\mathrm{I}}\langle\psi|e^{iHt}e^{-iH_0 t}\,e^{iH_0 t}e^{iHt}\mathcal{O}_{\mathrm{I}}(t)e^{-iHt}e^{-iH_0 t}\,e^{iH_0 t}e^{-iHt}|\psi\rangle_{\mathrm{I}} = {}_{\mathrm{I}}\langle\psi|\mathcal{O}_{\mathrm{I}}(t)|\psi\rangle_{\mathrm{I}}.$$

Now we need the interaction picture interaction Hamiltonian. At $t = 0$ we have $H^{\mathrm{int}}(\phi,\pi) = H_{\mathrm{I}}^{\mathrm{int}}(\phi_{\mathrm{I}},\pi_{\mathrm{I}})$, where

$$\pi_{\mathrm{I}} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_{\mathrm{I}})} \quad \text{and} \quad \dot{\phi}_{\mathrm{I}} = i[H_{\mathrm{I}}^0, \phi_{\mathrm{I}}].$$

There is a subtlety: $\dot{\phi}_{\mathrm{I}} = \dot{\phi}_{\mathrm{I}}(\phi_{\mathrm{I}},\pi_{\mathrm{I}})$ might be different from $\dot{\phi} = \dot{\phi}(\phi,\pi)$ if H^{int} contains terms $\partial_t\phi$.

Example: let $\mathcal{L}^{\mathrm{int}} = -\lambda_3\phi^3/3!$. The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) - \mathcal{L}^{\mathrm{int}},$$

where $\mathcal{H}^{\mathrm{int}} = -\mathcal{L}^{\mathrm{int}}$. We have

$$\dot{\phi}_{\mathrm{I}} = i[H_{\mathrm{I}}^0, \phi_{\mathrm{I}}(t,\mathbf{x})] = \int \mathrm{d}^y x\; i[\mathcal{H}_{\mathrm{I}}^0(t,\mathbf{x}), \phi_{\mathrm{I}}(t,\mathbf{x})] = \pi_{\mathrm{I}}(t,\mathbf{x}),$$

where we used $[A^2, B] = A[A,B] + [A,B]A$. The Legendre transformation works as before:

$$H_{\mathrm{I}}^{\mathrm{int}} = -\int \mathrm{d}^3x\; \mathcal{L}^{\mathrm{int}}(\phi_{\mathrm{I}}).$$

4.2 Perturbative expansion (Dyson series)

At $t = 0$ it holds

$$\phi(0, \mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} (e^{i\mathbf{p}\cdot\mathbf{x}} a_{\mathbf{p}} + \text{h.c.}).$$

Time translation in Heisenberg picture is given by

$$\phi_{\mathrm{H}}(t, \mathbf{x}) = e^{iH(t-t_0)} \phi_{\mathrm{H}}(t_0, \mathbf{x}) e^{-iH(t-t_0)}.$$

For $H^{\mathrm{int}} = 0$ it holds

$$\phi_{\mathrm{H}}(t, \mathbf{x}) = e^{iH_0(t-t_0)} \phi_{\mathrm{H}}(t_0, \mathbf{x}) e^{-iH_0(t-t_0)} = \phi_{\mathrm{I}}(t, \mathbf{x}).$$

Thus, for weak interaction $\lambda_3 \ll 1$, we can write the interacting field as

$$\phi_{\mathrm{I}}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} (e^{-ipx} a_{\mathbf{p}} + \text{h.c.}),$$

which is the most important time-evolution. The transformation between interaction and Heisenberg picture is

$$\phi_{\mathrm{H}}(t, \mathbf{x}) = e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \phi_{\mathrm{I}}(t, \mathbf{x}) e^{iH_0(t-t_0)} e^{-iH(t-t_0)} = \mathcal{U}^\dagger(t, t_0) \phi_{\mathrm{I}}(t, \mathbf{x}) \mathcal{U}(t, t_0),$$

where we introduced the interaction picture propagator

$$\mathcal{U}(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}.$$

States in interaction picture are $|\psi(t)\rangle_{\mathrm{I}} = \mathcal{U}(t, t_0) |\psi(t_0)\rangle_{\mathrm{I}}$ with

$$i \frac{\partial}{\partial t} \mathcal{U}(t, 0) = H_{\mathrm{I}}^{\mathrm{int}} \mathcal{U}(t, 0)$$

and the Hamiltonian evolves in time as

$$H_{\mathrm{I}}^{\mathrm{int}} = e^{iH_{\mathrm{S}}^{=} t} H_{\mathrm{I}}^{\mathrm{int}}(0) e^{-iH_{\mathrm{S}}^{0} t}.$$

Naive solution would for the interaction picture propagator would be

$$\mathcal{U}(t) \sim \exp\left(-i \int_0^t \mathrm{d}t' \; H_{\mathrm{I}}^{\mathrm{int}}(t') \right),$$

but $[H_{\mathrm{I}}^{\mathrm{int}}, H_{\mathrm{H}}^{\mathrm{int}}] \neq 0$. The correct solution is the *Dyson series*

$$\mathcal{U}(t) = 1 - i \int_{t_0=0}^{t} \mathrm{d}t_1' \; H_{\mathrm{I}}^{\mathrm{int}}(t_1) + (-i)^2 \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \; H_{\mathrm{I}}^{\mathrm{int}}(t_1) H_{\mathrm{I}}^{\mathrm{int}}(t_2) + \dots$$

One of the best aspects of canonical quantization approach is the fact that $H_{\mathrm{I}}^{\mathrm{int}}$ is unitary. Unfortunately, the expression is not manifestly Lorentz invariant. In the path integral formalism the Lorentz invariance is manifest.

The Dyson series as written above is dependent on the time ordering. In order to simplify the expression, we define a *time ordered product*:

$$T(A(t_1)B(t_2)) = \begin{cases} A(t_1)B(t_2); & t_1 > t_2, \\ B(t_2)A(t_1); & t_2 > t_1. \end{cases}$$

At this point, we don't care about the case $t_1 = t_2$. The second part can be rewritten as

$$\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, H(t_1)H(t_2) = \frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 T\{H(t_1)H(t_2)\}.$$

The integration goes over the whole rectangle $[0,t]\times[0,t]$ instead just over the triangle. To fix the difference, we use the time ordering and a one half in front of the integral. More generally,

$$\int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n\, H(t_1)\dots H(t_n) = \frac{1}{n!}\int_0^t \mathrm{d}t_1 \cdots \int_0^t \mathrm{d}t_n T\{H(t_1)\dots H(t_n)\}.$$

We can write the whole operator more compactly as

$$\mathcal{U}(t,t_0) = T\left\{\exp\left(-i\int_{t_0}^t \mathrm{d}t'\; H_{\mathrm{I}}^{\mathrm{int}}(t')\right)\right\}.$$

This is called *Dyson's formula*. When we expand the exponential, we can calculate the perturbative expansion order by order in $H_{\mathrm{I}}^{\mathrm{int}}$.

4.3 Intuitive S-matrix

Our initial state $|i\rangle$ at $t_i \to -\infty$ with two particles with initial momenta $\mathbf{p}_1^i$ and $\mathbf{p}_2^i$. Our initial state $|f\rangle$ at $t_f \to \infty$ with two particles with initial momenta $\mathbf{p}_1^f$ and $\mathbf{p}_2^f$. The *probability amplitude* is

$$S_{fi} =_I \langle f|\mathcal{U}(t_f,t_i)|i\rangle_I\,.$$

Example 1. Let $|i\rangle = |\mathbf{p}_1^i,\mathbf{p}_2^i\rangle$ and $|f\rangle = |\mathbf{p}_1^f,\mathbf{p}_2^f\mathbf{p}_3^f,\mathbf{p}_4^f\rangle$ with $\mathbf{p}_i^a \neq \mathbf{p}_j^b$ and $H_{\mathrm{I}}^{\mathrm{int}} = \frac{\lambda}{4!}\phi_{\mathrm{I}}^4$. Then

$$S_{fi} = \langle f|T\left\{\exp\left(-i\int_{-\infty}^{\infty}\mathrm{d}t'\; H_{\mathrm{I}}^{\mathrm{int}}(t')\right)\right\}|i\rangle = \langle f|T\left\{\exp\left(-i\int \mathrm{d}^4x\;\; \mathcal{H}_{\mathrm{I}}^{\mathrm{int}}\right)\right\}|i\rangle$$

$$S_{fi} = \langle 0|\, a_{\mathbf{p}_1^f} a_{\mathbf{p}_2^f} a_{\mathbf{p}_3^f} a_{\mathbf{p}_4^f}\,[\pi -$$
$$-i\int \mathrm{d}^4x\; \frac{\lambda\phi_{\mathrm{I}}^4(x)}{4!} + \frac{(-i)^2}{2}\int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2\; T\left\{\frac{\lambda}{4!}\phi_{\mathrm{I}}^4(x_1)\frac{\lambda}{4!}\phi_{\mathrm{I}}^4(x_2)\right\}\Big]\, a^\dagger_{\mathbf{p}_1^i} a^\dagger_{\mathbf{p}_2^i}\,|0\rangle\,.$$

For a non-vanishing result avoid $a_{\mathbf{p}}\,|0\rangle = 0$ and $\langle 0|\, a^\dagger_{\mathbf{p}} = 0$.

Chapter 5

Path integral representation of QFT

When we do QFT of gauge theories, the canonical formalism becomes extremely unwieldy, resorting to negative amplitudes and other weird methods. It is also much easier to prove the consistency of the *Standard Model* (SM) using path integral formulation. Ultimately, this formulation is the preferred choice of physicists because it explains non-perturbative effects.

5.1 Motivation using QM

In the double slit experiment the amplitude at the point O on the screen is the sum of amplitudes through both slits:

$$A(\text{detected at } O) = \sum_{i=1}^{2} A(A \to A_i \to O).$$

Now take infinitely many slits and infinitely many plates, and we get

$$a(S \to O \text{ in } T) = \sum_{\text{paths}} A(S \to O \text{ any path in } T).$$

5.2 Hamiltonian path integral

Let Q_n and P_n be Schrödinger operators with regular commutation relations $[Q_n, P_m] = i\delta_{mn}$ and $[Q_m, Q_n] = [P_m, P_n] = 0$. The eigenstates are $Q_a |q\rangle = q_a |q\rangle$. These should be orthonormal

$$\langle q'|q\rangle = \prod_a \delta(q'_a - q_a) = \delta(q' - q)$$

and complete

$$\mathbb{1} = \int \left(\prod_a \mathrm{d}q_a \right) |q\rangle \langle q| .$$

Almost the same holds for P_a:

$$P_a |p\rangle = p_a |p\rangle , \quad \langle p'|p\rangle = \prod_a (2\pi)\delta(p'_a - p_a) = (2\pi)^N \delta(p - q) \quad \text{and} \quad \mathbb{1} = \int \left(\prod_a \frac{\mathrm{d}p_a}{2\pi} \right) |p\rangle \langle p| .$$

The scalar product is

$$\langle q|p\rangle = \prod_a \exp(ip_a q_a) = e^{ipq}.$$

We can check the normalization:

$$\langle q|q'\rangle = \langle q| \int \left(\prod_a \frac{\mathrm{d}p_a}{2\pi}\right) |p\rangle \langle p|q'\rangle = \int \left(\prod_a \frac{\mathrm{d}p_a}{2\pi}\right) e^{ip_a(q_a - q'_a)} = \delta(q-q').$$

The commutation relations imply:

$$P_m \langle q|p\rangle = \langle q|P_m|p\rangle = \frac{\partial}{\partial p_m} \langle q|p\rangle .$$

Now transition to Heisenberg picture:

$$Q^n_{\mathrm{H}}(t) = e^{iHt} Q^n_{\mathrm{S}} e^{-iHt} \quad \text{and} \quad P^n_{\mathrm{H}}(t) = e^{iHt} P^n_{\mathrm{S}} e^{-iHt}.$$

The eigenstates change slightly

$$Q^{a(t)}_{\mathrm{H}} |q;t\rangle_{\mathrm{H}} = q_a |q;t\rangle \quad \text{and} \quad P^{a(t)}_{\mathrm{H}} |p;t\rangle_{\mathrm{H}} = p_a |p;t\rangle$$

(there is no time dependence with q_a and p_a) and the transition to Schrödinger picture is

$$|q;t\rangle_{\mathrm{H}} = e^{iHt} |q\rangle_{\mathrm{S}} \quad \text{and} \quad |p;t\rangle_{\mathrm{H}} = e^{iHt} |p\rangle_{\mathrm{S}} .$$

Heisenberg eigenstates are also orthonormal and complete as Schrödinger states. (a tiny proof missing).

After time $\Delta t = t - t'$, how likely has state $|q,t\rangle_{\mathrm{H}}$ transformed into $|q',t'\rangle_{\mathrm{H}}$, i.e. what is the value of ${}_{\mathrm{H}}\langle q';t'|q;t\rangle_{\mathrm{H}}$. We can go straight forward to infinitesimal $t' = \tau + \mathrm{d}\tau$, so

$$\langle q';\tau|e^{-iH\mathrm{d}\tau}|q;t\rangle = {}_{\mathrm{S}}\langle q'|e^{-iH\tau} e^{-iH\mathrm{d}\tau} e^{iH\tau}|q\rangle_{\mathrm{S}} =_S \langle q'|e^{-iH\mathrm{d}\tau}|q\rangle .$$

The next question is the right ordering of components in the Hamiltonian. We adopt *Weyl-ordering*, where all Q's go to the left and all P's go to the right. After Weyl-ordering, we have

$$\begin{aligned}
\langle q';\tau|e^{-iH(Q(t),P(t))\mathrm{d}\tau}|q;\tau\rangle &= \langle q';\tau|e^{-iH(q'_n,P(t))\mathrm{d}\tau}|q;\tau\rangle \\
&= \int \prod_a \left(\frac{\mathrm{d}P_a}{2\pi}\right) \langle q';\tau|e^{-iH(q'_n,P(t))\mathrm{d}\tau}|p;\tau\rangle \langle p;\tau|q;\tau\rangle \\
&= \int_{-\infty}^{\infty} \prod_a \left(\frac{\mathrm{d}p_a}{2\pi}\right) e^{-iH(q'_m,p_n)\mathrm{d}\tau + i(q'-q)p}.
\end{aligned}$$

Thus, we showed

$$e^{-iH\mathrm{d}\tau} = \pi - i\mathrm{d}\tau H(Q,P) + \frac{(-i)^2}{2}(\mathrm{d}\tau)^2 H(Q,P)H(Q,P) + \dots$$

Now we will split the interval $[t_i,t_f]$ into $N+1$ intervals $t_i = t_0 < t_1 < \cdots < t_N < t_{N+1} = t_f$ and denote $\mathrm{d}\tau = (t_f - t_i)/(N+1)$. The notation can be slightly simplified $\int \prod_n \mathrm{d}q_n = \int \mathrm{d}q$ and

$\int \prod_n p_n/(2\pi) = \int \mathrm{d}\tilde{p}$. Next

$$\begin{aligned}\langle q_f; t_f | q_i; t_i\rangle &= \int \mathrm{d}q^1 \dots \mathrm{d}q^n \\ &= \langle q_f; t_f | q^N; t^N\rangle \langle q^N; t_N | q^{N-1}; t_{N-1}\rangle \dots \langle q^1; t_1 | q_i; t_i\rangle \\ &= \int \prod_{k=1}^{N} \mathrm{d}q^k \prod_{l=0}^{N} \mathrm{d}\tilde{p}^l \exp\left(i \sum_{m=1}^{N+1} (q^m - q^{m-1}p^{m-1}) - H(q^m, p^{m-1})\mathrm{d}\tau \right).\end{aligned}$$

When we send $N \to \infty$ we make the following substitutions:

$$\begin{aligned} q_n^k &\to q_n(t) \\ p_n^k &\to p_n(t) \\ q_n^k - q_n^{k-1} &\to \dot{q}_n(t)\mathrm{d}t \\ \sum_{m=1}^{N+1} \mathrm{d}\tau &\to \int_{t_i}^{t_f} \mathrm{d}t. \end{aligned}$$

The path integral measure is then

$$\int \prod_{k=1}^{N} \prod_n \mathrm{d}q_n^k \prod_{l=0}^{N} \prod_b \frac{\mathrm{d}p_b^l}{2\pi} \xrightarrow{N\to\infty} \int_{q(t_i)=q_i}^{q(t_f)=g_f} \prod_t \prod_a \mathrm{d}q_a(t) \prod_t \prod_b \frac{\mathrm{d}p_b(t)}{2\pi} = \int_{q_i}^{q_f} \mathcal{D}[q_n(t)]\mathcal{D}[p_n(t)].$$

Hence, the full Hamiltonian path integral is

$$\langle q_f; t_f | q_i; t_i\rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}[q_n(t)]\mathcal{D}[p_n(t)] \exp\left(i \int_{t_i}^{t_f} \sum_n \dot{q}_n(t)p_n(t) - H(q_n(t), p_n(t))\,\mathrm{d}t \right).$$

Here we are integrating over all paths from q_i to q_f with the weight of complex phase. Note that boundary conditions for p_i and p_f are unconstrained and p_k and q_k don't satisfy the classical equations of motion:

$$\dot{q}_k \neq \frac{\partial H}{\partial p_k}, \quad \dot{p}_k \neq -\frac{\partial H}{\partial q_k}.$$

When we generalize the equation above to fields, we have to do the transformation

$$\begin{aligned} H &= \int \mathrm{d}^3x\ \mathcal{H}(\phi(x), \Pi(x)), \\ |q, t\rangle &\to |\varphi, t\rangle\,. \end{aligned}$$

Note that $\phi(\mathbf{x})$ is a c-function eigenvalue of the operator $\phi(x)$ and the same holds for operator $\Pi(x)$ with eigenfunction $\pi(\mathbf{x})$. The *Hamiltonian path integral* is then

$$\langle \varphi_f; t_f | \varphi_i; t_i\rangle = \int_{\varphi(t_i,\mathbf{x})=\varphi_i(\mathbf{x})}^{\varphi(t_f,\mathbf{x})=\varphi_f(\mathbf{x})} \mathcal{D}[\varphi(x)]\mathcal{D}[\pi(x)] \exp\left(i \int_{t_i}^{t_f} \mathrm{d}t\,(\pi(x)\dot{\varphi}(x) - \mathcal{H}(\varphi, \pi)) \right).$$

5.3 Path integral for Green functions

Since we want to calculate scattering amplitudes and not fields themselves we will generalize the path integral formula to

$$\langle q_f; t_f | \mathcal{O}_a(t_a)\mathcal{O}_b(t_b) \dots | q_i; t_i\rangle$$

using the same strategy as before. This would only work if operators O_i are time ordered: $t_f > t_a > t_b > \cdots > t_i$. We want $\langle\Omega|T\{\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\dots\}|\Omega\rangle$, where $|\Omega\rangle$ is the real vacuum of the full interacting theory. The generalization to fields is $H = \int \mathrm{d}^3x\ \mathcal{H}(\phi(x),\pi(x))$. We do the substitution $\sum_n \to \int \mathrm{d}^3x$ and $|q_n;t\rangle \to |\phi;t\rangle$. The *two point correlation function* is

$$\langle\phi_f,t_f;\phi_i,t_i\rangle = \int_{\phi(t_i,\mathbf{x})=\phi_i(\mathbf{x})}^{\phi(t_f,\mathbf{x}_f)=\phi_f(\mathbf{x})} \mathcal{D}[\phi(x)]\mathcal{D}[\pi(x)]\exp\left(i\int_{t_i}^{t_f}\mathrm{d}t\int\mathrm{d}^3x\ (\pi(x)\dot{\phi}(x)-\mathcal{H}(\phi,\pi))\right).$$

The result that we want to show is

$$\langle\Omega|T\{\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\dots\}|\Omega\rangle = |\mathcal{N}|^2\int\mathcal{D}\phi\mathcal{D}\pi\exp\left(i\int\mathrm{d}^4x\ (\pi\dot{\phi}-\mathcal{H})\right)\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\dots$$

To show that, write

$$\begin{aligned}
&\langle\Omega|T\{\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\dots\}|\Omega\rangle = \\
&= \lim_{\substack{t_i\to-\infty\\ t_f\to\infty}}\int\prod_{\mathbf{x}}\mathrm{d}\phi_i(\mathbf{x})\prod_{\mathbf{x}}\mathrm{d}\dot{\phi}(\mathbf{x})\,\langle\Omega|\phi_f,t_f\rangle\,\langle\phi_f,t_f|T\{\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\dots\}|\phi_i,t_i\rangle\,\langle\phi_i,t_i|\Omega\rangle \\
&= \int\mathcal{D}\phi\mathcal{D}\pi\,\langle\Omega|\phi,t_f=+\infty\rangle\,\langle\phi,t_i=-\infty|\Omega\rangle\exp\left(i\int\mathrm{d}^4x\ (\pi\dot{\phi}-\mathcal{H})\right)\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\dots
\end{aligned}$$

The operators $\mathcal{D}\phi(\mathbf{x})$ are unconstrained. We need to find $\langle\phi,\pm\infty|\Omega\rangle$ the wave function of the vacuum at $t\pm\infty$. In order to continue the derivation, the assumption must be made that at $t\to\pm\infty$ the filed behaves as a free field:

$$\phi(\mathbf{x},-\infty) = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ipx}a_{\mathbf{p},\mathrm{in}}+e^{ipx}a^\dagger_{\mathbf{p},\mathrm{in}}\right),$$

$$\phi(\mathbf{x},-\infty) = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ipx}a_{\mathbf{p},\mathrm{out}}+e^{ipx}a^\dagger_{\mathbf{p},\mathrm{out}}\right).$$

More on that later. In the exercise sheet, we proved that

$$a_{\mathbf{p}} = e^{iE_{\mathbf{p}}t}\left(E_{\mathbf{p}}\tilde{\phi}_{\mathbf{p}}(t)+i\tilde{\pi}_{\mathbf{p}}(t)\right),$$

with

$$\tilde{\phi}_{\mathbf{p}}(t) = \int\mathrm{d}^3x\ e^{-i\mathbf{p}\cdot\mathbf{x}}\phi(x).$$

Now

$$a_{\mathbf{p},\mathrm{in/out}} = \lim_{t\to\pm\infty}\int\mathrm{d}^3x\ e^{ipx}E_{\mathbf{p}}\left(\phi(x)+\frac{i}{E_{\mathbf{p}}}\pi(x)\right).$$

Since $a_{\mathbf{p},\mathrm{in/out}}\,|\Omega\rangle = 0$ and since we assumed we can treat $\phi(x)$ as a free field at $t\to\pm\infty$,

$$\begin{aligned}
0 &= \langle\phi,\pm\infty|a_{\mathbf{p},\mathrm{in/out}}|\Omega\rangle \\
&= \lim_{t\to\pm\infty}\int\mathrm{d}^3x\ e^{ipx}E_{\mathbf{p}}\,\langle\phi,\pm\infty|\phi(x)+i\pi(x)/E_{\mathbf{p}}|\Omega\rangle \\
&= \lim_{t\to\pm\infty}\int\mathrm{d}^3x\ e^{ipx}E_{\mathbf{p}}\left(\phi(\mathbf{x})+\frac{1}{E_{\mathbf{p}}}\frac{\delta}{\delta\phi(\mathbf{x})}\right)\langle\phi,\pm\infty|\Omega\rangle,
\end{aligned}$$

where we used a *functional derivative*

$$\langle q;t|\,\hat{P}_n(t) = -i\partial q_n\,\langle q,t|$$

and consequently

$$\langle\phi,t|\pi(t,\mathbf{x})|\psi\rangle = -i\frac{\delta}{\delta\phi(\mathbf{x})}\,\langle\phi,t|\psi\rangle = -i\frac{\delta}{\delta\phi(\mathbf{x})}\psi[\phi(\mathbf{x})].$$

We have a first order functional differential equation

$$\left(\phi(\mathbf{x}) + \frac{1}{E_\mathbf{p}}\frac{\delta}{\delta\phi(\mathbf{x})}\right)\langle\phi,\pm\infty|\Omega\rangle = 0$$

which we will approximate with the ordinary differential equation:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + ax\right) f(x) = 0.$$

The solution is $f(x) \propto \exp(-ax^2/2)$. For the functional differential equation, we use the ansatz

$$\langle\phi,\pm\infty\rangle = N\exp\left(-\frac{1}{2}\int \mathrm{d}^3x\;\mathrm{d}^3y\,K(\mathbf{x},\mathbf{y})\phi(\mathbf{x})\phi(\mathbf{y})\right).$$

Plugging the ansatz in the original equation and considering

$$\frac{\delta}{\delta\phi(\mathbf{x})}\phi(\mathbf{y}) = \delta^{(3)}(\mathbf{x}-\mathbf{y})$$

we get

$$0 = \int \mathrm{d}^3x\;e^{i,\mathbf{p}\cdot\mathbf{x}}\left(-\int \mathrm{d}^3y\;K(\mathbf{x},\mathbf{y})\phi(\mathbf{y}) + E_\mathbf{p}\phi(\mathbf{x})\right)\exp\left(-\frac{1}{2}\int \mathrm{d}^3x\;\mathrm{d}^3y\,K(\mathbf{x},\mathbf{y})\phi(\mathbf{x})\phi(\mathbf{y})\right).$$

From that we see

$$\int \mathrm{d}^3x\;e^{-i\mathbf{p}\cdot\mathbf{x}}K(\mathbf{x},\mathbf{y}) = E_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{y}}$$

with the kernel

$$K(\mathbf{x},\mathbf{y}) = \int\frac{\mathrm{d}^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}E_\mathbf{p}.$$

The overlap is

$$\begin{aligned}\langle\Omega|\phi,t_f=+\infty\rangle\,\langle\phi,t_i=-\infty|\Omega\rangle = \\ = |N|^2\exp\left(-\frac{1}{2}\int \mathrm{d}^3x\;\mathrm{d}^3y\,K(\mathbf{x},\mathbf{y})\left[\phi(\infty,\mathbf{x})\phi(\infty,\mathbf{y}) + \phi(-\infty,\mathbf{x})\phi(-\infty,\mathbf{y})\right]\right).\end{aligned}$$

Now we use

$$f(\infty) + f(-\infty) = \lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}t\,f(t)e^{-\epsilon|t|}.$$

Why is that? Because

$$\lim_{\epsilon\to0^+}\epsilon\int_0^\infty \mathrm{d}t f(t)e^{-\epsilon t} = \lim_{\epsilon\to0^+}\epsilon\int_0^\infty\frac{\mathrm{d}\tilde{t}}{\epsilon}f(\tilde{t}/\epsilon)e^{-\tilde{t}} = f(+\infty)\int_0^\infty\mathrm{d}t\,e^{-t} = f(+\infty)$$

and the same for $f(-\infty)$. Next we write

$$\langle\Omega|\phi, t_f = +\infty\rangle \langle\phi, t_i = -\infty|\Omega\rangle = |N|^2 \exp\left(-\frac{1}{2}\epsilon \int \mathrm{d}^3x\, \mathrm{d}^3y \int_{-\infty}^{\infty} \mathrm{d}t\, K(\mathbf{x},\mathbf{y})\phi(t,\mathbf{x})\phi(t,\mathbf{y})\right).$$

We finally derive the Hamilton path integral for *Green functions*:

$$\langle\Omega|T\{\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\dots\}|\Omega\rangle = |\mathcal{N}|^2 \int \mathcal{D}\phi\mathcal{D}\pi \exp\left(i\int \mathrm{d}^4x\, (\pi\dot{\phi} - \mathcal{H} + i\epsilon)\right)\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\dots$$

When we require normalization $\langle\Omega|\Omega\rangle = 1$, we get

$$|N|^{-2} = \int \mathcal{D}\phi\mathcal{D}\pi \exp\left(i\int \mathrm{d}^4x\, (\pi\dot{\phi} - \mathcal{H} + i\epsilon)\right).$$

The $i\epsilon$ with the Hamiltonian is the same $i\epsilon$ that appears in the Feynman propagator $i/(p^2 - m^2 + i\epsilon)$, and it is implied that we always take the limit $\epsilon \to 0^+$.

Let's get some quantum-mechanical intuition. Writing a state as

$$|q,t\rangle = \exp(iHt)\,|q\rangle = \sum_{n=0}^{\infty} e^{iHt}\,|n\rangle\,\langle n|q\rangle = \sum_{n=0}^{\infty} e^{iE_nt}\psi_n^*(q)\,|n\rangle\,,$$

where $|n\rangle$ are eigenstates $H\,|n\rangle = E_n\,|n\rangle$. How to find the coefficients of $|0\rangle$. Replace H with $(1 - i\epsilon)H = \tilde{H}$. Then take the limit $t \to -\infty$ to get

$$|q,n\rangle = \sum me^{iE_nt}e^{E_n\epsilon t}\psi_n^*(q)\,|n\rangle\,.$$

So to find the vacuum wave-function, we take $\psi_0^*(q)\,|0\rangle$.

5.4 Lagrangian path integral

To formulate *Lagrangian path integral*, we have to assume a few conditions:

- $\mathcal{H}$ should be at most quadratic in $\pi(x)$ and
- $\mathcal{O}_i(\pi(x), \phi(x))$ doesn't depend on $\pi(x)$.

The only path integrals that we can fully solve are the Gaussian integrals:

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(-\frac{ax^2}{2}\right) = \sqrt{\frac{2\pi}{a}}.$$

For a quadratic form $-\frac{1}{2}x_iA_{ij}x_j$ with A being symmetric ($A_{ij} = A_{ji}$) we get the integral

$$\int_{-\infty}^{\infty} \mathrm{d}x_1\mathrm{d}x_2\dots\mathrm{d}x_n\, \exp\left(-\frac{1}{2}x_iA_{ij}x_j\right) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}.$$

A slightly modified Gaussian is

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(-\frac{ax^2}{2} - bx\right) = \sqrt{\frac{2\pi}{a}}\exp\left(\frac{b^2}{2a}\right)$$

and in more dimensions for a general quadratic form $Q(x) = x_i A_{ij} x_j + B_k x_k + C$

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \, \exp\left(-\frac{1}{2} x_i A_{ij} x_j - B_k x_k - C\right) = \left(\det\left(\frac{A}{2\pi}\right)\right)^{-\frac{1}{2}} \exp\left(\frac{1}{2} B_m (A^{-1})_{mn} B_n - C\right)$$
$$= \left(\prod_{k=1}^{n} \sqrt{\frac{2\pi}{a_k}}\right) \exp(-Q(\overline{x})),$$

where a_k are the eigenvalues of A and $\overline{x} = -A^{-1}B$ is a stationary point, i.e. $\nabla Q|_{\overline{x}} = 0$. To generalize this to functional integrals we substitute

$$x_k \to x(t),$$
$$\int \left(\prod_{k=1}^{n} \mathrm{d}x_k\right) \to \int \mathcal{D}x(t),$$
$$A_{kl} \to A(t,t'),$$
$$B_k \to B(t).$$

We get

$$\int \mathcal{D}x(t) \exp\left(-\frac{1}{2}\int \mathrm{d}t\mathrm{d}t' \, A(t,t')x(t)x(t') - \int \mathrm{d}t \, B(t)x(t) - C\right) =$$
$$= \det\left(\frac{A}{2\pi}\right)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int \mathrm{d}t\mathrm{d}t' \, B(t)A^{-1}(t,t')B(t') - C\right).$$

We can again introduce the stationary function

$$\overline{x}(t) = -\int \mathrm{d}t' A^{-1}(t,t')B(t)$$

and eigenvalues

$$\lambda x(t) = \int \mathrm{d}t' \, A(t,t')x(t').$$

Trace and determinant can also be defined:

$$\mathrm{tr}(A) = \sum_n \lambda_n, \qquad \det(A) = \prod_n \lambda_n.$$

Note that there can be (uncountably) infinite eigenvalues. In this course, we won't be very interested in these two operations, since they (specifically determinant) will be trivial for most of the time. Interesting things happen when the determinant is not trivial, and it is the topic of QFT 2. There is also the inverse of the operator $A(t,t')$, which is defined such that

$$\int \mathrm{d}t\mathrm{d}t' \, A^{-1}(t,t')A(t,t') = \delta(t-t').$$

We will apply these results to the Hamiltonian path integral:

$$\int \mathcal{D}\pi \exp\left(i\int \mathrm{d}^4x \, (\pi(x)\dot{\phi}(x) - \mathcal{H}(x))\right).$$

From here on, we will assume $\mathcal{H}$ is at most quadratic in π. We rewrite the Hamiltonian into quadratic canonical form

$$\mathcal{H} = -\frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, i\pi(x)A[\phi(x)]\pi(y)$$

with $A[\phi(x)] \propto \delta^{(4)}(x-y)$. It follows

$$\int \mathcal{D}\pi \exp\left(i\int \mathrm{d}^4x\, (\pi(x)\dot{\phi}(x) - \mathcal{H}(x))\right) = \\ = (\det(2\pi i A[\phi]))^{-1/2} \exp\left(i\int \mathrm{d}^4x\, (\overline{\pi}(x)\phi(x) - \mathcal{H}(\overline{\pi}(x),\phi(x)))\right),$$

where $\overline{\pi}(x)$ is the stationary point (maximum) of the quadratic form:

$$\frac{\delta}{\delta\pi(x)}\int \mathrm{d}^4y\, (\pi(y)\dot{\phi}(y) - \mathcal{H}(\pi(y),\phi(y))) = 0.$$

It follows

$$\boxed{\dot{\phi}(x) - \frac{\partial\mathcal{H}}{\partial\pi(x)} = 0}$$

which is exactly the first Hamiltonian equation. The result is

$$\boxed{\overline{\pi}(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)}}$$

which is canonically conjugate momentum. Until now in our derivation π and ϕ were unrelated. We have seen that Gaussian path integral in $\mathcal{D}\pi$ performs a *Legendre transformation*. Hence,

$$\boxed{\begin{aligned}&\langle\Omega|T\{\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\dots\}|\Omega\rangle = \\ &= |\mathcal{N}|^2\int \mathcal{D}\phi \frac{1}{\sqrt{\det A[\phi(x)]}} \exp\left(i\int \mathrm{d}^4x\, \mathcal{L}(\phi(x),\partial_\mu\phi(x))\right)\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\dots.\end{aligned}}$$

The normalization $\langle\Omega|\Omega\rangle = 1$ requires

$$|N|^{-2} = \int \mathcal{D}\phi(\det(A[\phi(x)]))^{-1/2}\exp\left(i\int \mathrm{d}^4x\, \mathcal{L}(\phi(x),\partial_\mu\phi(x))\right).$$

Note that the $2\pi i$ factors were cancelled with the normalization. If $A[\phi(x)]$ is independent of $\phi(x)$, then we can pull the $\det(A[\phi])$ in front of the integral and also cancels with the normalization. For example, if Lagrangian is $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - \mathcal{L}^{\mathrm{int}}(\phi)$ the Hamiltonian is $\mathcal{H} = \frac{1}{2}(\pi^2 + (\nabla\phi)^2 + m^2\phi^2) - \mathcal{L}^{\mathrm{int}}(\phi)$ and $A[\phi] = \delta^{(4)}(x-y)$. A famous counter example is a non-linear σ-model

$$\mathcal{L} = \frac{1}{2(1+\dot{\phi}^a\phi^a)}\partial_\mu\phi_a\partial^\mu\phi_a + \dots$$

and this gives additional terms in the path integral. When deriving that it helps to know that

$$\det(A) = \exp(\operatorname{tr} \ln A),$$

but this will be the topic in QFT2.

Classical limit: for classical trajectories it holds $\delta S = 0$ and $[Q, P] = i\hbar \to 0$. The path integral is

$$\langle q_f, t_f | q_i, t_i \rangle \sim \int \mathcal{D}q \, \exp\left(\frac{i}{\hbar} S[q(t)]\right) \sim F^0 + \hbar F^1 + \hbar^2 F^2 + \dots$$

In the limit $\hbar \to 0$ we integrate over wildly fluctuating paths which gives us only the path, where $\delta S = 0$. We could also analytically continue the integral to the real axis to get

$$\int \mathcal{D}q \, \exp(-S/\hbar) \sim e^{-S[\overline{q}]/\hbar},$$

where $\overline{q}$ is the extremal point of the action $\delta S|_{\overline{q}} = 0$.

5.5 Perturbative solution of Green functions

We have two options.

- We can discretize space-time and sum over all configurations. This is called the lattice field theory (done by Wilson). The main trick is to analytically continue the path integral to Euclidean space, and it works for any $\mathcal{L}^{\text{int}}(\phi)$.
- We can find a perturbative expansion for $\mathcal{L}^{\text{int}}(\phi)$. This is what we will do in this lecture.

Our master formula for quadratic Hamiltonians is

$$\langle \Omega | T\{\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\dots\} | \Omega \rangle = \int \mathcal{D}\phi \, e^{i \int \mathrm{d}^4x \, \mathcal{L}_0} \sum_{N=0}^{\infty} \frac{1}{N!} \left(i \int \mathrm{d}^4x \, \mathcal{L}^{\text{int}}(\phi) \right)^N \mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\dots$$

It can be calculated since

$$\int_{-\infty}^{\infty} \mathrm{d}q \, \exp\left(-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4\right) = \int_{-\infty}^{\infty} \mathrm{d}q \, \exp(-mq^2/2) \left[1 - \frac{\lambda}{4!}q^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 q^8 + \dots\right].$$

We use the trick – *generating functionals*:

$$Z[\mathcal{J}] = \int_{-\infty}^{\infty} \mathrm{d}q \, \exp\left(-\frac{m^2}{2}q^2 - \frac{\lambda}{4!}q^4 + \mathcal{J}q\right)$$

and

$$Z_0[\mathcal{J}] = \int_{-\infty}^{\infty} \mathrm{d}q \, \left(-\frac{m^2}{2}q^2 + \mathcal{J}q\right).$$

Namely,

$$\int_{-\infty}^{\infty} \mathrm{d}q \, \exp\left(-\frac{m^2}{2}q^2\right) q^{4n} = \left(\frac{\mathrm{d}}{\mathrm{d}\mathcal{J}}\right)^{4n} \int -\infty^{\infty} \mathrm{d}q \, \exp(-m^2q^2/2 + \mathcal{J}q)\Big|_{\mathcal{J}=0}.$$

We can write

$$\begin{aligned} Z[\mathcal{J}] &= \exp\left(-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}\mathcal{J}}\right)^4\right)\int_{-\infty}^{\infty}\mathrm{d}q\,\exp(-m^2q^2/2+\mathcal{J}q) \\ &= \sqrt{\frac{2\pi}{m^2}}\exp\left(-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}\mathcal{J}}\right)^4\right)\exp\left(\frac{\mathcal{J}^2}{2m^2}\right) \end{aligned}$$

because, using the previously mentioned Gaussian integral rules, it holds

$$Z_0[\mathcal{J}] = \sqrt{\frac{2\pi}{m^2}}\exp\left(\frac{\mathcal{J}^2}{2m^2}\right).$$

To generalize this trick to work on path integrals we write

$$Z[\mathcal{J}] = |N|^2 \in \mathcal{D}\phi\exp\left(i\int \mathrm{d}^4x\,\mathcal{L}_0 + \mathcal{L}^{\mathrm{int}} + \mathcal{J}(x)\phi(x)\right).$$

For the free theory $\mathcal{L}^{\mathrm{int}} = 0$ we have

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi) - \frac{m^2}{2}\phi^2.$$

We generate the *Green functions* (*n-point functions*) by

$$G_0(x_1,\ldots,x_n) = \langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle = \frac{\delta}{i\delta\phi(x_1)}\cdots\frac{\delta}{i\delta\phi(x_n)}Z_0[\mathcal{J}]\Big|_{\mathcal{J}=0}.$$

Using the fact that

$$\frac{1}{i}\frac{\delta}{\delta\mathcal{J}(y)}\exp\left('\int \mathrm{d}^4x\,\mathcal{J}(x)\varphi(x)\right) = \varphi(y)\exp\left('\int \mathrm{d}^4x\,\mathcal{J}(x)\varphi(x)\right)$$

we can write the closed form of expression for $Z_0[\mathcal{J}]$

$$\begin{aligned} Z_0[\mathcal{J}] &= |\mathcal{N}|^2\int\mathcal{D}\varphi\,\exp\left(-\frac{1}{2}\int\mathrm{d}^4x\,\mathrm{d}^4y\,\varphi(x)D(x,y)\varphi(y) - \int\mathrm{d}^4x\,(-i\mathcal{J}(x)\varphi(x))\right) \\ &= |\mathcal{N}|^2\sqrt{\det\left(\frac{D(x,y)}{2\pi}\right)}\exp\left(\frac{1}{2}\int\mathrm{d}^4x\,\mathrm{d}^4y\,(-i\mathcal{J}(x))D^{-1}(x,y)(-i\mathcal{J}(y))\right) \\ &= \exp\left(\frac{1}{2}\int\mathrm{d}^4x\,\mathrm{d}^4y\,(-i\mathcal{J}(x))D^{-1}(x,y)(-i\mathcal{J}(y))\right), \end{aligned}$$

where

$$D(x,y) = i\delta^{(4)}(x-y)(\partial_\mu\partial^\mu + m^2).$$

We require $Z_0[\mathcal{J}=0] = \langle\Omega|\Omega\rangle$ hence

$$|\mathcal{N}|^{-2} = \sqrt{\det\left(\frac{D(x,y)}{2\pi}\right)}.$$

Writing the expression above even shorter we get

$$Z_0[\mathcal{J}] = \exp\left(-\frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathcal{J}(x)\Delta_F(x-y)\mathcal{J}(y)\right),$$

where $\Delta_F(x-y) = D^{-1}(x-y)$ is called the *Feynman propagator*. The definition of the inverse operator is

$$\int \mathrm{d}^4x\, D(y,x)D^{-1}(x,z) = \delta^{(4)}(y-z).$$

Calculating this for our kernel, we get

$$\delta^{(4)}(y-z)\int \mathrm{d}^4x\, i\delta^{(4)}(y-x)(\partial_\mu\partial^\mu + m^2)\Delta_F(x,z) = (\Box_y + m^2)i\Delta_F(y-z).$$

Performing Fourier transformation and making substitution $(x,y,z) \to (z,x,y)$ we get

$$\int \mathrm{d}^4x\, e^{-ip(x-y)}(\Box_x + m^2)\Delta_F(x-y) = 1.$$

The Fourier transformation of $\Box_x$ is $-p^2$. Due to the diverging integral, we move the poles at $\pm E_{\mathbf{p}}$ slightly above and below the real line:

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{ie^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}.$$

The $i\epsilon$ terms allow us to invert $D(x,y)$ and define the contour around the singularities. In conclusion: change every occurrence of m^2 in the Lagrangian with $m^2 - i\epsilon$.

Let's now calculate the 1-point function in the free theory:

$$\begin{aligned}
G_0(z) &= \frac{1}{i}\frac{\delta}{\delta\mathcal{J}(z)}Z_0[\mathcal{J}]\bigg|_{\mathcal{J}=0} \\
&= \frac{i}{i}\left(-\frac{1}{2}\int \mathrm{d}^4y\, \Delta_F(z-y)\mathcal{J}(y) - \frac{1}{2}\int \mathrm{d}^4x\, \mathcal{J}(x)\Delta_F(x-z)\right)\cdot \\
&\qquad\qquad \cdot \exp\left(\frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, (-i\mathcal{J}(x))D^{-1}(x,y)(-i\mathcal{J}(y))\right) \\
&= 0
\end{aligned}$$

because $\mathcal{J} = 0$. This happens for all odd n-point functions. For $n = 2$ we have

$$\begin{aligned}
&\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \\
&= \frac{1}{i^2}\frac{\delta}{\delta\mathcal{J}(x)}\frac{\delta}{\delta\mathcal{J}(y)}\exp\left(-\frac{1}{2}\int \mathrm{d}^4z_1\, \mathrm{d}^4z_2\, \mathcal{J}(z_1)\Delta_F(z_1-z_2)\mathcal{J}(z_2)\right) \\
&= -\frac{\delta}{\delta\mathcal{J}(x)}\left(-\frac{1}{2}\Delta_F(y-z_2)\mathcal{J}(z_2) - \frac{1}{2}\mathcal{J}(z_1)\Delta_F(z_2,y)\right)\cdot \\
&\qquad \cdot \exp\left(-\frac{1}{2}\int \mathrm{d}^4z_1\, \mathrm{d}^4z_2\, \mathcal{J}(z_1)\Delta_F(z_1-z_2)\mathcal{J}(z_2)\right) \\
&= \frac{1}{2}\left(\Delta_F(x-y) + \Delta_F(y-x) + \mathcal{O}(\mathcal{J})\right)\exp\left(-\frac{1}{2}\int \mathrm{d}^4z_1\, \mathrm{d}^4z_2\, \mathcal{J}(z_1)\Delta_F(z_1-z_2)\mathcal{J}(z_2)\right)\bigg|_{\mathcal{J}=0} \\
&= \Delta_F(x-y).
\end{aligned}$$

We have seen that the two-point correlation function is equal to the Feynman propagator. The four-point correlation function gives us:

$$\begin{aligned}
&\langle\Omega|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|\Omega\rangle = \\
&= \frac{1}{i^4}\frac{\delta}{\delta\mathcal{J}(x_1)}\frac{\delta}{\delta\mathcal{J}(x_2)}\frac{\delta}{\delta\mathcal{J}(x_3)}\frac{\delta}{\delta\mathcal{J}(x_4)}\exp\left(-\frac{1}{2}\int \mathrm{d}^4z_1\,\mathrm{d}^4z_2\,\mathcal{J}(z_1)\Delta_F(z_1-z_2)\mathcal{J}(z_2)\right) \\
&= (\Delta_F(x_3-x_4)\Delta_F(x_1-x_2)+\Delta_F(x_2-x_4)\Delta_F(x_1-x_3)+\Delta_F(x_2-x_3)\Delta_F(x_1-x_4)) \\
&= \sum \text{all pairwise } \textit{contractions}
\end{aligned}$$

$$= \overbracket{\phi(x_1)\phi(x_2)}\overbracket{\phi(x_3)\phi(x_4)} + \overbracket{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)} + \overbracket{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)}.$$

The general result is

$$\langle\Omega|T\{\phi(x_1)\dots\phi(x_m)\}|\Omega\rangle = \begin{cases} 0 & n \text{ odd} \\ \sum_{\text{pairwise}} \Delta_F(x_{i_1},x_{i_2})\dots\Delta_F(x_{i_{n-1}},x_{i_n}) & n \text{ even.} \end{cases}$$

5.6 Free complex scalar

The Lagrangian for a free complex scalar field is

$$\mathcal{L} = \partial_\mu\phi^\dagger\mu^\mu\phi - m^2\phi^\dagger\phi$$

and is symmetric under $\mathcal{U}(1)$ symmetry. The generating functional is

$$\begin{aligned}
Z_0(\mathcal{J},\mathcal{J}^*) &= |\mathcal{N}|^2\int\mathcal{D}\varphi\mathcal{D}\varphi^\dagger\exp\left(i\int\mathrm{d}^4x\;\mathcal{L}_0+\mathcal{J}\varphi+\varphi^\dagger\mathcal{J}^*\right) \\
&= \exp\left(-\int\mathrm{d}^4x\,\mathrm{d}^4y\,\mathcal{J}(x)\Delta_F(x-y)\mathcal{J}^*(y)\right).
\end{aligned}$$

The substitution is the same as before

$$\varphi(x)\to\frac{1}{i}\frac{\delta}{\delta\mathcal{J}(x)},\quad \varphi^\dagger(x)\to\frac{1}{i}\frac{\delta}{\delta\mathcal{J}^*(x)}.$$

The two-point function is

$$\overbracket{\phi(x)\phi(y)} = \langle\Omega|T\{\phi(x)\phi(y)\}\Omega\rangle = 0 = \overbracket{\phi^\dagger(x)\phi^\dagger(y)}$$

and

$$\overbracket{\phi(x)\phi^\dagger(y)} = \langle\Omega|T\{\phi(x)\phi^\dagger(y)\}\Omega\rangle = \Delta_F(x-y) = \overbracket{\phi^\dagger(x)\phi(y)}.$$

In general, we need the same number of ϕ and $\phi^\dagger$ to find a non-vanishing n-point function.

Chapter 6

Interacting theories, Feynman diagrams and perturbative evaluation of Green functions

The interacting Lagrangian is

$$\mathcal{L} = \mathcal{L}^0 + \mathcal{L}^{\text{int}}$$

with

$$\mathcal{L}^{\text{int}} = -\frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4.$$

The generating functional is exactly the same as before:

$$Z[\mathcal{J}] = \frac{\int \mathcal{D}\varphi \exp\left(i\int \mathrm{d}^4x\, \mathcal{L} + \mathcal{J}\varphi\right)}{\int \mathcal{D}\varphi \exp\left(i\int \mathrm{d}^4x\, \mathcal{L}\right)}$$

and

$$G(x_1,\ldots,x_n) = \langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle = \frac{\delta}{i\delta\mathcal{J}(x_1)}\cdots\frac{\delta}{i\delta\mathcal{J}(x_n)}Z[\mathcal{J}]\bigg|_{\mathcal{J}=0}.$$

Now we expand interaction

$$\exp\left(i\int \mathrm{d}^4x\, \mathcal{L}^{\text{int}}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}(i\int \mathrm{d}^4x\, \mathcal{L}^{\text{int}})^n$$

and assuming the interaction is small, we can truncate the sum after some finite n. With $Z_0[\mathcal{J}]$ we can calculate $Z[\mathcal{J}]$ perturbatively:

$$\begin{aligned} Z[\mathcal{J}] &= |\mathcal{N}|^2 \int \mathcal{D}\varphi \exp\left(i\int \mathrm{d}^4x\, \mathcal{L}_0 + \mathcal{J}\varphi\right)\sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}^4z_1\ldots\mathrm{d}^4z_n\mathcal{L}^{\text{int}}(\varphi(z_1))\ldots\mathcal{L}^{\text{int}}(\varphi(z_n)) \\ &= |\mathcal{N}|^2 \int_{n=0}^{\infty}\frac{1}{n!}\left[i\int \mathrm{d}^4x\, \mathcal{L}^{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta\mathcal{J}(x)}\right)\right]^n Z_0[\mathcal{J}]. \end{aligned}$$

This is the central identity of QFT, the term $Z_0[\mathcal{J}]$ represents the propagators and the infinite sum represents the vertices.

$$\int \mathcal{D}\varphi \exp\left(i\int \mathrm{d}^4x\, \mathcal{L}_0\right) (\text{polynomial in } \varphi) = \text{“sum over pairwise contractions”}.$$

Therefore, the n-point function can be expressed as

$$\int \mathcal{D}\varphi\, \varphi(x_1)\dots\varphi(x_n) \exp\left(i\int \mathrm{d}^4x\, \mathcal{L}_0\right) \sum_{n=0}^{\infty} \frac{1}{m!} \left(i\int \mathrm{d}^4z\, \mathcal{L}^{\text{int}}\right)^m =$$

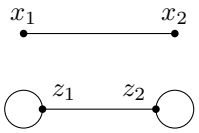
$$= \sum_{m=0}^{\infty} \mathrm{d}z_1\dots\mathrm{d}z_m \left[\text{all two particle contractions of } \varphi(x_1)\dots\varphi(x_n)\mathcal{L}^{\text{int}}(\varphi(z_1))\dots\mathcal{L}^{\text{int}}(\varphi(z_m))\right].$$

Example 2. Let's calculate a 2-point function in ϕ^3-theory. The interacting part of the Lagrangian is $\mathcal{L}^{\text{int}} = \frac{g}{3!}\phi^3$. In the zeroth order there is only the trivial one, the Feynman propagator $\Delta_F(x_1 - x_2)$. In the first and all odd orders there are no contributions. There are 5 different *Feynman diagrams* we can write for the first non-trivial (second) order in the expansion.

1. The first contraction goes as

$$\frac{1}{2}\left(-\frac{ig}{3!}\right)^2 \phi(x_1)\phi(x_2) \int \mathrm{d}^4z_1\, \phi(z_1)\phi(z_1)\phi(z_1) \int \mathrm{d}^4z_2\, \phi(z_2)\phi(z_2)\phi(z_2) =$$
$$= \frac{1}{2}\cdot 3\cdot 3\cdot 1\cdot \left(-\frac{ig}{3!}\right)^2 \Delta_F(x_1 - x_2) \int \mathrm{d}^4z_1\, \mathrm{d}^4z_2\, \Delta(z_1 - z_2)\Delta_F(0)^2.$$

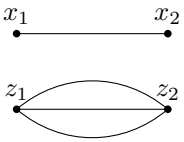
   If we number the legs in each vertex we would have $3\cdot 3$ options to connect them with the middle propagator and 1 option for the rest of the propagators. It is an example of *vacuum fluctuation*.



2. The second contraction goes as

$$\frac{1}{2}\left(-\frac{ig}{3!}\right)^2 \phi(x_1)\phi(x_2) \int \mathrm{d}^4z_1\, \phi(z_1)\phi(z_1)\phi(z_1) \int \mathrm{d}^4z_2\, \phi(z_2)\phi(z_2)\phi(z_2) =$$
$$= \frac{1}{2}\cdot 3\cdot 2\cdot \Delta_F(x_1 - x_2) \left(-\frac{ig}{3!}\right)^2 \int \mathrm{d}^4z_1\, \mathrm{d}^4z_2\, \Delta(z_1 - z_2)^3.$$

   If we number the legs in each vertex we would have 3 options to connect the first propagator between z_1 and z_2 and 2 options for the second propagator. This is also an example of vacuum fluctuation.

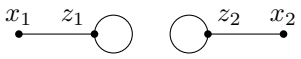


3. The third contraction goes as

$$\frac{1}{2}\left(-\frac{ig}{3!}\right)^2 \overbrace{\phi(x_1)\phi(x_2)\int \mathrm{d}^4z_1\ \phi(z_1)\phi(z_1)\phi(z_1)\int \mathrm{d}^4z_2\ \phi(z_2)\phi(z_2)\phi(z_2)}^{} =$$

$$= \frac{1}{2}\cdot 3\cdot 3\cdot 2\cdot\left(-\frac{ig}{3!}\right)^2 \int \mathrm{d}^4z_1\,\mathrm{d}^4z_2\,\Delta(y_1-z_1)\Delta_F(x_2-z_2)\Delta_F(0)^2 = G^{(1)}(x_1)G^{(1)}(x_2).$$

There are 3×3 options to connect each x to each y and the additional factor of 2 comes from the switch $z_1 \leftrightarrow z_2$.

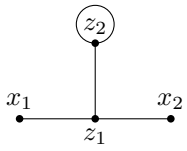


4. The fourth contraction goes as

$$\frac{1}{2}\left(-\frac{ig}{3!}\right)^2 \overbrace{\phi(x_1)\phi(x_2)\int \mathrm{d}^4z_1\ \phi(z_1)\phi(z_1)\phi(z_1)\int \mathrm{d}^4z_2\ \phi(z_2)\phi(z_2)\phi(z_2)}^{} =$$

$$= \frac{1}{2}\cdot 3\cdot 2\cdot 3\cdot 2\cdot\left(-\frac{ig}{3!}\right)^2 \int \mathrm{d}^4z_1\,\mathrm{d}^4z_2\,\Delta(x_1-z_2)\Delta_F(x_2-z_2)\Delta_F(z_1-z_2)\Delta_F(0).$$

Here we have 3 options to connect x_1 to z_1, and then 2 options to connect x_2 to z_1. There are three options to connect z_1 to z_2 and another factor of 2 for the switch $z_1 \leftrightarrow z_2$.

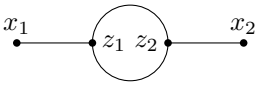


5. The fifth contraction goes as

$$\frac{1}{2}\left(-\frac{ig}{3!}\right)^2 \overbrace{\phi(x_1)\phi(x_2)\int \mathrm{d}^4z_1\ \phi(z_1)\phi(z_1)\phi(z_1)\int \mathrm{d}^4z_2\ \phi(z_2)\phi(z_2)\phi(z_2)}^{} =$$

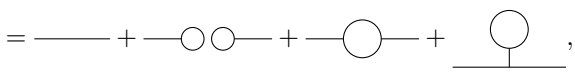
$$= \frac{1}{2}\cdot 3\cdot 3\cdot 2\cdot 2\cdot\left(-\frac{ig}{3!}\right)^2 \int \mathrm{d}^4z_1\,\mathrm{d}^4z_2\,\Delta(x_1-z_1)\Delta_F(x_2-z_2)\Delta_F(z_1-z_2)^2.$$

There are 3 was to connect x_1 to z_1 and 3 options to connect x_2 to z_2. There are 2 ways to connect z_1 to z_2 and another factor of 2 for the switch $z_1 \leftrightarrow z_2$.



Vacuum fluctuation sub-diagrams don't contribute to the total scattering amplitude, because they are already included in the denominator. The second order in the expansion in g is given by

$$G^{(2)}(x_1,x_2) = G^{(1)}(x_1,x_2) + \underbrace{G^{(1)}(x_1)G^{(1)}(x_2)}_{\text{disconnected}} + \text{both connected diagrams}$$



where $G^{(1)}(x_1, x_2) = \Delta_F(x_1 - x_2)$ is just the Feynman propagator.

6.1 Feynman rules

Let's write down the Feynman rules for position space Green functions. In general, we could have multiple interactions, so

$$\mathcal{L}^{\text{int}} = g_{l_1 \dots l_k} \phi_{l_1} \dots \phi_{l_k}.$$

Perturbative expansion of $\langle \Omega | T\{\phi(x_1) \dots \phi(x_n)\} | \Omega \rangle$ is computed as

1. Determine all non-vanishing contractions $\phi_l \overbrace{(x) \phi_{l'}}(y) = \Delta_F(x - y)$ and vertex factors $i g_{l_1 \dots l_k}$.
2. Start with external point x_i and draw a line from each point.
3. Calculate $\mathcal{O}(g^m)$ contribution and draw m internal vertices.
4. Draw lines between vertices and external points.
5. Leave out vacuum polarization diagrams.
6. Calculate *symmetry factors* by counting contractions and integrate over all internal points.

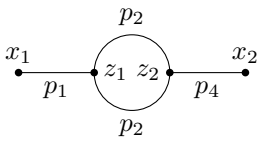
6.2 Green functions in momentum space

The Fourier transform of the Feynman propagator is

$$\int \mathrm{d}^4 x_1 \dots \mathrm{d}^4 x_n \exp(i p_1 x_1 + \dots i p_n x_n) G(x_1, \dots, x_n) = (2\pi)^4 \delta^{(4)}(p_1 + \dots + p_n) \tilde{G}(p_1, \dots, p_n).$$

It is a bit unusual with the δ-function on the right-hand side, but this is how it works.

Example 3. Let's consider the following diagram in ϕ^3 theory.



The momentum space representation of this diagram is

$$I = \frac{1}{2} 2 \cdot 3 \cdot 2 \cdot 3 \cdot \left(-\frac{ig}{3!} \right)^2 \int \mathrm{d}^4 z_1 \int \mathrm{d}^4 z_2 \, \Delta_F(x_1 - z_1) \Delta_F(x_2 - z_2) \Delta_F(z_1 - z_2)^2$$

$$= -\frac{g^2}{2} \int \mathrm{d}^4 z_1 \int \mathrm{d}^4 z_2 \int \prod_{j=1}^{4} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p_j^2 - i\epsilon} e^{-ip_1(x_1 - z_1)} e^{-ip_2(z_2 - x_2)} e^{-ip_3(z_1 - z_2)} e^{-ip_4(z_1 - z_2)}.$$

Now we use

$$\int \mathrm{d}^4 z_1 \, e^{i z_1 (p_1 - p_3 - p_4)} = (2\pi)^4 \delta^{(4)}(p_1 - p_3 - p_4)$$

$$\int \mathrm{d}^4 z_2 \; e^{iz_2(-p_2+p_3+p_4)} = (2\pi)^4 \delta^{(4)}(-p_2+p_3+p_4),$$

which impose the 4-momentum conservation in each vertex. After performing a Fourier transformation we get

$$-\frac{g}{2}\int \mathrm{d}^4 x_1 \, \mathrm{d}^4 x_2 \prod_{i=1}^{4} \frac{\mathrm{d}^4 p_i}{(2\pi)^4}(2\pi)^8 \delta^{(4)}(p_1-p_3-p_4)\delta^{(4)}(-p_2+p_3+p_4)\frac{ie^{-ip_1x_1}}{p_1^2+i\epsilon}\frac{ie^{ip_2x_2}}{p_2^2+i\epsilon}\frac{ie^{ik_1x_1}}{p_3^2+i\epsilon}\frac{ie^{ik_2x_2}}{p_4^2+i\epsilon}.$$

The incoming momentum is represented by e^{-ikx} and the outgoing momentum with e^{ikx}. Integrating over x_1 and x_2 gives $\delta^{(4)}(p_1-k_1)\delta^{(4)}(p_2+k_2)$ and simplifying even further:

$$G(k_1,k_2) = -\frac{g^2}{2}\frac{i}{k_1^2+i\epsilon}\frac{i}{k_2^2+i\epsilon}(2\pi)^4\delta^{(4)}(k_1+k_2)\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}\frac{i}{(k_1-p)^2+i\epsilon}.$$

Claim 6.1. *Overall δ-function is a result of translation invariance of the Green function in position space,*

$$G(x_1,\ldots,x_n) = G(x_1+a,\ldots,x_n+a), \quad \forall a^\mu = const.$$

Proof. We set $\alpha^\nu = -x_1^\nu$:

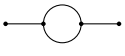
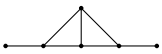
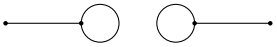
$$\begin{aligned}
&\int \mathrm{d}^4 x_1 \; \ldots \mathrm{d}^4 x_n e^{ip_1x_1+\ldots ip_nx_n} G(x_1,\ldots,x_n) = \\
&= \int \mathrm{d}^4 x_1 \; \ldots \mathrm{d}^4 x_n e^{ip_1x_1+\ldots ip_nx_n} G(0,x_2-x_1,\ldots,x_n-x_1) \\
&= \int \mathrm{d}^4 x_1 \; e^{ip_1x_1+\cdots+ip_nx_1} \int \mathrm{d}^4 y_2 \; \ldots \mathrm{d}^4 y_n e^{ip_2y_2+\cdots+ip_ny_n} \tilde{G}(0,y_2,\ldots,y_n) \\
&= (2\pi)^4\delta^{(4)}(p_1+\cdots+p_n) I(p_2,\ldots,p_n).
\end{aligned}$$

□

An important question is, how many $\int \mathrm{d}^4 p_i$ integrals are left after using δ-function at each vertex. We want to eliminate as many p_i as possible using δ-functions. It holds that the number of loops L is related to the number of propagators P the number of internal and external vertices V and the number of connected components n by

$$L = P - V + n.$$

Example 4. For each propagator we have $\int [4][p_i]$, for each vertex we have $\delta(p_i+\ldots)$ and for each component we have $\delta(k_1+\cdots+k_n)$.

1. For the number of loops is $L = 1-2+1 = 0$.
2. For  the number of loops is $L = 4-4+1 = 1$.
3. For  the number of loops is $L = 7-6+1 = 2$.
4. For  the number of loops is $L = 4-4+2 = 2$.

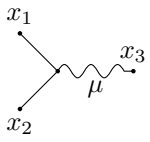
6.3 Feynman diagrams in momentum space

The external momenta will be denoted with k_i and internal momenta with p_i.

1. For each field $\langle\Omega|T\{\phi_1 \dots \phi_n\}|\Omega\rangle$ draw external point add vertices up to given order in truncated expansion.
2. Each external line gets a factor $\tilde{\Delta}_F(p_i) = \frac{i}{p_i^2 - m^2 + i\epsilon}$. Each internal line is $\tilde{\Delta}_F(k_i)$.
3. Momentum is conserved in each vertex.
4. Integrate over all undetermined 4-momenta.
5. Apply symmetry factor.
6. Sum over all possible diagrams.

The next question is how does vertex look in momentum space? Let's consider $\mathcal{L} = \mathcal{L}_0 + g\phi(\partial_\mu\phi)A^\mu$. The method is as follows. Choose n-point function with the same fields as in the vertex:

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)A^\mu(x_3)\}|\Omega\rangle =$$



$$+ \mathcal{O}(g^2)$$

$$= \sum_{\text{all contractions}} ig \int \mathrm{d}^4 z \; \phi(x_1)\phi(x_2)A_\mu(x_3)\phi(z)\partial_\rho^z\phi(z)A^\rho(z)$$

$$= ig \int \mathrm{d}^4 z \; \left(\Delta_F^{\mu\rho}(x_3 - z)(\Delta_F(x_1 - z)\partial_\rho^z\Delta_F(x_2 - z) + \Delta_F(x_2 - z)\partial_\rho^z\Delta_F(x_1 - z))\right).$$

After the Fourier transformation we get

$$(2\pi)^4\delta(k_1 + k_2 + k_3)\tilde{G}(k_1, k_2, k_3) =$$

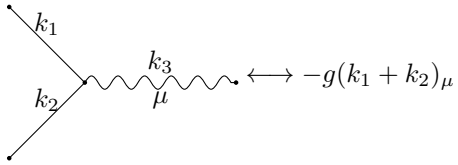
$$= \int \mathrm{d}^4 z \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \mathrm{d}^4 x_3 e^{ik_1x_1 + ik_2x_2 + ik_3x_3} \sum_{\text{contractions}} ig \int \mathrm{d}^4 z \; \phi(x_1)\phi(x_2)A_\mu(x_3)\phi(z)\partial_\rho^z\phi(z)A_\mu(z)$$

$$= \int \mathrm{d}^4 z \int \prod_{j=1}^{3} \left[\frac{\mathrm{d}^4 p_j}{(2\pi)^4}\mathrm{d}^4 x_j e^{ik_jx_j}\right] \Big\{e^{-ip_3(x_3 - z)}\tilde{\Delta}_F^{\mu\rho}(p_3)\cdot$$

$$\cdot \left[e^{-ip_1(x_1 - z)}\tilde{\Delta}_F(p_1)ig\partial_\rho^z\left(e^{-ip_2(x_2 - z)}\tilde{\Delta}_F(p_2)\right) + e^{-ip_1(x_2 - z)}\tilde{\Delta}_F(p_1)ig\partial_\rho^z\left(e^{-ip_2(x_1 - z)}\tilde{\Delta}_F(p_2)\right)\right]\Big\}$$

$$= (2\pi)^4\delta^{(4)}(k_1 + k_2 + k_3)\tilde{\Delta}_F(k_1)\tilde{\Delta}_F(k_2)\Delta_F^{\mu\rho}(p_3)(-g(p_1 + p_2)_\rho).$$

So we have seen that we can replace the vertex



6.4 Composite diagram Green functions

Sometimes we have to calculate Green functions with composite operators, such as $\mathcal{O}(x_1) = \phi(x_1)^3$. In this case we just do the straightforward substitution $\phi(x) \mapsto \mathcal{O}(x)$.

6.5 T^* product for path integral n-point functions

So far we required our operators $\mathcal{O}_a(x_a)$ in the path integral to contain $\phi(x)^m$ and not $\pi(x) = \dot{\phi}(x)$. So we define a *T^* time ordering* for path integrals as

$$\langle\Omega|T^*(\partial_0\phi(x)\dots)|\Omega\rangle = \partial_0^{(*)}\langle\Omega|T\phi(x)\dots|\Omega\rangle = \lim_{\epsilon\to 0}\frac{1}{\epsilon t}\left(\langle\Omega|T\phi(x+\epsilon t)\dots|\Omega\rangle - \langle\Omega|T\phi(x)\dots|\Omega\rangle\right).$$

T^* is different from T, since

$$T\{A(x)B(x)\} = \theta(x^0-y^0)A(x)B(y) + \theta(y^0-x^0)B(y)A(x).$$

Then for $A(x) = \partial_0\phi(x)$,

$$T\{\partial_0\phi(x)B(x)\} = \theta(x^0-y^0)\partial_0\phi(x)B(y) + \theta(y^0-x^0)B(y)\partial_0\phi(x).$$

For T^* we have

$$\begin{aligned} T^*\{\partial_0\phi(x)B(y)\} &= \partial_0 T\{\phi(x)B(y)\} \\ &= T\{\partial_0\phi(x)B(y)\} + \delta(x^0-y^0)\phi(x)B(y) - \delta(y^0-x^0)B(y)\phi(x) \\ &= T\{\partial_0\phi(x)B(x)\} + \delta(x^0-y^0)[\phi(x),B(y)]. \end{aligned}$$

From this point further we will use T^* ordering for path integrals (even if we write only T).

6.6 Connected and 1PI Green functions

We want a functional like $Z[\mathcal{J}]$ for only connected diagrams. Let's define

$$\begin{aligned} G(x) &= G^c(x), \\ G(x_1,x_2) &= G^c(x_1,x_2) + G^c(x_1)G^c(x_2), \\ G(x_1,x_2,x_3) &= G^c(x_1,x_2,x_3) + G^c(x_1,x_2)G^c(x_3) + G^c(x_1,x_3)G^c(x_2) + \\ &\quad + G^c(x_2,x_3)G^c(x_1) + G^c(x_1)G^c(x_2)G^c(x_3). \end{aligned}$$

Writing the general formula gives

$$G(x_1,\dots,x_n) = \sum_{\text{partitions}}\prod_{\alpha} G^c(\{x_{I_\alpha}\}),$$

where I_α are all partitions of $\{1,\dots,n\}$. We can also define $Z[\mathcal{J}] = \exp(iW[\mathcal{J}])$ or $W[\mathcal{J}] = -i\ln Z[\mathcal{J}]$. We claim that $iW[\mathcal{J}]$ generates connected Green functions:

Claim 6.2.

$$G^c(x_1,\dots,x_n) = \frac{\delta}{i\delta\mathcal{J}(x_1)}\dots\frac{\delta}{i\delta\mathcal{J}(x_n)}iW[\mathcal{J}].$$

Proof. The proof is by induction. For $n=1$ we have

$$G(x) = \frac{1}{i}\frac{\delta}{\delta\mathcal{J}(x)}Z[\mathcal{J}]\bigg|_{\mathcal{J}=0} = \frac{1}{i}\frac{\delta(iW)}{\delta\mathcal{J}(x)}e^{iW[\mathcal{J}]}\bigg|_{\mathcal{J}=0} = \frac{\delta W[\mathcal{J}]}{\delta\mathcal{J}(x)}\bigg|_{\mathcal{J}=0} = G^c(x).$$

For $n = 2$ it holds

$$G(x_1, x_2) = \left(\frac{i\delta}{\delta \mathcal{J}(x_1)} \frac{\delta(iW[\mathcal{J}])}{i\delta \mathcal{J}(x_2)} + \frac{\delta(iW)}{i\delta \mathcal{J}(x_1)} \frac{\delta(iW)}{i\delta \mathcal{J}(x_2)} \right) e^{iW} \Big|_{\mathcal{J}=0} = G^c(x_1, x_2) + G^c(x_1) G^c(x_2).$$

The case for $n = 3$ is left as an exercise to the reader. □

For free scalar field theory we have

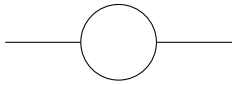
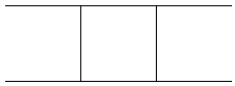
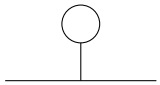
$$Z_0[\mathcal{J}] = \exp\left(-\frac{1}{2} \int \mathrm{d}^4x \, \mathrm{d}^4y \, \mathcal{J}(x) \Delta_F(x-y) \mathcal{J}(y) \right)$$

and so

$$iW[\mathcal{J}] = -\frac{1}{2} \int \mathrm{d}^4x \, \mathrm{d}^4y \, \mathcal{J}(x) \Delta_F(x-y) \mathcal{J}(y).$$

Hence, only the two-point function is non-vanishing. We want something stronger from connected diagrams – we are actually interested in diagrams that cannot be disconnected by cutting a single internal line. This defines *one particle irreducible* diagrams (denoted as 1PI):

Example 5.

-  is 1PI;
-  is also 1PI;
-  is not 1PI.

6.7 Path integral master formula

The Master formula enables us to

- find all order relationships between n-point functions and
- see explicit property of $S[\phi]$: defines the equation of motion, gives us quantum Noether theorem (Ward identities) and also quantum non-conservation of classical symmetries (discussed in QFT2).

Consider a field transformation

$$\phi_n(x) \to \phi'_n(x) = \phi_n(x) + \epsilon F_n(\phi_n, x).$$

Let's write down the generating functional:

$$\begin{aligned} Z[\mathcal{J}_n] &= \int \mathcal{D}\phi_n \exp\left(iS[\phi_n] + i \int \mathrm{d}^4x \, \mathcal{J}_n(x) \phi_n(x) \right) \\ &= \int \mathcal{D}\phi_n \exp\left(iS[\phi'_n] + i \int \mathrm{d}^4x \, \mathcal{J}_n(x) \phi'_n \right) \\ &= \int \mathcal{D}\phi_n \left| \det\left(\frac{\delta \phi'_n}{\delta \phi_n} \right) \right| \exp\left(iS[\phi_n] + \int \mathrm{d}^4x \, \mathcal{J}_n(x) \phi_n(x) + \right. \\ &\qquad \left. + \int \mathrm{d}^4x \, \left(i \frac{\delta S[\phi_n]}{\delta \phi_n(x)} \delta\phi_n(x) + i \mathcal{J}_n(x) \delta\phi_n(x) \right) \right). \end{aligned}$$

Writing Jacobian in the exponent $\ln\det A = \operatorname{tr}\ln A$ we get

$$\left|\det\left(\frac{\delta\phi_{n'}(y)}{\delta\phi_n(x)}\right)\right| = \exp\operatorname{tr}\ln\left(1+\epsilon\frac{\delta F_{n'}[\phi_n(y)]}{\delta\phi_n(x)}\right) \approx 1+\epsilon\operatorname{tr}\left(\frac{\delta_{n'}(\phi(y))}{\delta\phi_n(x)}\right).$$

We insert the transformed measure the generating functional

$$\begin{aligned} Z[\mathcal{J}_n] &= \int \mathcal{D}\phi_n \exp\left(iS[\phi_n] + i\int \mathrm{d}^4x\ \mathcal{J}_n(x)\phi_n(x)\right)\left[1+\epsilon\int \mathrm{d}^4x\cdot\right. \\ &\qquad \left.\cdot\left\{\sum_n \frac{\delta F_n[\phi_n(x)]}{\delta\phi_n(x)} + i\left(\frac{\delta S[\phi]}{\delta\phi_n[x]} + \mathcal{J}_n(x)\right)F_n(\phi_n(x))\right\}\right] \\ &= \int \mathcal{D}\phi_n \exp\left(iS[\phi_n] + i\int \mathrm{d}^4x\ \mathcal{J}_n(x)\phi_n(x)\right). \end{aligned}$$

If we subtract the lines, we get the One particle irreducible (1PI) *Master formula*:

$$\begin{aligned} 0 = \int \mathcal{D}\phi_n \exp\left(iS[\phi_n] + i\int \mathrm{d}^4x\ \mathcal{J}_n(x)\phi_n(x)\right)\cdot \\ \cdot\int \mathrm{d}^4x\ \left\{\sum_n \frac{\delta f_n(\phi_n(x))}{\delta\phi_n(x)} + i\left(\frac{\delta S[\phi_n]}{\delta\phi_n(x)} + \mathcal{J}_n(x)\right)F_n(\phi_n(x))\right\}. \end{aligned}$$

6.7.1 Equations of motion in QFT

In classical theory it holds

$$0 = \frac{\delta S}{\delta\phi_n} = \frac{\partial\mathcal{L}}{\partial\phi_n} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_n}.$$

How do the Euler-Lagrange equations look like for QFT? For example for the ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}((\partial_\mu\phi)^2 - m^2\phi^2) - \frac{\lambda}{4}\phi^4 \leftrightarrow (\Box + m^2)\phi + \frac{\lambda}{3!}\phi^3 = 0.$$

Using Master formula with the transformation

$$\phi'_n(x) = \phi_n(x) + \epsilon_n(x)$$

we find that the measure is invariant under the transformation $\delta F[\phi_n(x)]/\delta\phi_n(x) = 0$. Since it holds for arbitrary $\epsilon(x)$, we can choose $\epsilon_n(x) = \delta^{(4)}(x-y)$. This gives us the *Schwinger-Dyson equation*:

$$0 = \int \mathcal{D}\phi_n \exp\left(iS[\phi_n] + i\int \mathrm{d}^4x\ \mathcal{J}_n(x)\phi_n(x)\right)\left(\frac{\delta S[\phi_n]}{\delta\phi_n(x)} + \mathcal{J}_n(x)\right).$$

Setting $\mathcal{J}_n(x) = 0$ gives

$$\langle\Omega|T\frac{\delta S[\phi_n]}{\delta\phi_n(x)}|\Omega\rangle\,.$$

More generally, using multiple $\mathcal{J}_{n_i}(x_i)$ derivatives in a row, we can show

$$\langle\Omega|T\{\frac{\delta S[\phi_n]}{\delta\phi_n(x)}\phi_{n_1}(x_1)\dots\phi_{n_N}(x_N)\}|\Omega\rangle =$$
$$= \sum_{i=1}^{N} i\delta^{(4)}(x-x_i)\delta_{nn_i}\langle\Omega|T\phi_{n_1}(x_1)\dots\phi_{n_{i-1}}(x_{i-1})\phi_{n_{i+1}}(x_{i+1}\dots\phi_{N_n}(x_N)|\Omega\rangle\,.$$

With this we can find exact relations between n-point functions:

$$\langle\Omega|T\{(-(\Box+m^2)\phi(x)-\lambda\phi^3(x)/3!)\phi(y)\}|\Omega\rangle = i\delta^{(4)}(x-y)$$

and hence

$$(\Box_x+m^2)\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle + \frac{\lambda}{3!}\langle\Omega|T\{\phi^3(x)\phi(y)\}|\Omega\rangle = -i\delta^{(4)}(x-y).$$

In the limit $\lambda\to 0$, $\mathcal{L}\to\mathcal{L}_0$ the equation simplifies to

$$(\Box_x+m^2)\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = i\delta^{(4)}(x-y).$$

6.8 Internal symmetries: Ward identities

Let's assume we have an internal symmetry $\phi\to e^{i\alpha}\phi\approx\phi+i\alpha\phi$, which leaves $S[\phi]$ invariant. The transformation is generally

$$\phi'_n(x) = \phi_n(x) + \epsilon\theta^a F^a_n(\phi_{n'}).$$

Specifically

$$\phi'_n(x) = D_{nn'}(\theta^a)\phi_{n'}(x),$$

where $a=1,\dots,\dim(G)$ with which we parametrize transformation in a representation of the group G.

Example 6. Let's consider two complex ϕ-fileds in fundamental representation of $SU(2)$. Then

$$\begin{bmatrix}\phi'_1(x)\\ \phi'_2(x)\end{bmatrix}_n = \exp(i\epsilon\theta^a T^a_{nn'})\begin{bmatrix}\phi_1(x)\\ \phi_2(x)\end{bmatrix}_{n'}.$$

Objects T^a are called generators of the group G. For SU(2) there are three generators $T^a=\sigma^a/2$, which are the Pauli matrices. Writing that out gives

$$\phi'_n(x) = (\mathbb{1}_{2\times 2}+\frac{i\epsilon}{2}\theta^a\sigma^a_{nn'})\phi_{n'}(x).$$

We can think about generators as independent symmetry transformations.

If we assume that θ^a is independent on x, it is called global symmetry, and we can write

$$\epsilon\theta^a F^a_n(\phi_{n'}) = \epsilon\theta^a T^a_{nn'}\phi_{n'}(x).$$

If θ^a is dependent on x (local symmetry), the symmetries become the part of the gauge freedom, and are not physical.

Continue now with the invariant action:

$$\frac{\partial}{\partial\theta^a}S[\phi_n+\epsilon\theta^a F_n]=0.$$

We take the chain rule:

$$0=\int \mathrm{d}^4x\ \frac{\delta S[\phi_n]}{\delta\phi_n}\frac{\delta\phi_n}{\delta\theta^a}=\int \mathrm{d}^4x\ \frac{\delta S[phi_n]}{\delta\phi_n}\epsilon F^a(\phi_n),$$

where again $a=1,\dots,\dim G$. Here we have to make an important distinction:

1. path integral measure is not invariant under G-transformation, i.e.

$$\left|\frac{\delta\phi_n'(x)}{\delta\phi_n(y)}\right|$$

is non-trivial (anomalous symmetry). In this case the QFT does not share classical symmetries – the classical action is invariant but $Z[\mathcal{J}]$ is not invariant.

2. Path integral measure is invariant/trivial under G-transformation. In this case the Master formula gives us

$$\int\mathcal{D}\phi_n\exp\left(iS[\phi_n]+i\int \mathrm{d}^4x\ \mathcal{J}_n(x)\phi_n(x)\right)\int \mathrm{d}^4x\ \mathcal{J}_n(x)F_n^a(\phi_{n'}(x))=0.$$

With this we can derive *Ward-Takahashi identities* by taking derivatives with respect to $\mathcal{J}_n(x)$:

$$\sum_{k=1}^{m}\langle\Omega|T\{\phi_{n_1}(x_1)\dots\phi_{n_{k-1}}(x_{k-1})F_{n_k}(\phi_l(x_k))\phi_{n_{k+1}}(x_{k+1})\dots\phi_{n_m}(x_m)\}|\Omega\rangle=0.$$

To derive this formula for $m=2$, we write

$$\begin{aligned}
&\frac{1}{i}\frac{\delta}{\delta\mathcal{J}_1}\frac{1}{i}\frac{\delta}{\delta\mathcal{J}_2}\left(\int\mathcal{D}\phi_n e^{iS+i\int \mathrm{d}^4x\ \mathcal{J}_n(x)\phi_n(x)}\int \mathrm{d}^4x\ F_n(\phi_{n'}(x))\right)=\\
&=\frac{1}{i}\frac{\delta}{\delta\mathcal{J}_1}\left(\int\mathcal{D}\phi_n e^{iS+i\int \mathrm{d}^4x\ \mathcal{J}_n(x)\phi_n(x)}\left(\phi_2(x_2)\int \mathrm{d}^4z\ J_n(z_n)F_n(\phi_{n'}(z))+\frac{1}{2}F_2(\phi_2(x_2))\right)\right)\\
&=\int\mathcal{D}\phi_n e^{iS+i\int \mathrm{d}^4x\ \mathcal{J}_n(x)\phi_n(x)}\left(\phi_1(x_1)\phi_2(x_2)\cdot\right.\\
&\qquad\left.\cdot\int \mathrm{d}^4z\ \mathcal{J}_n(z)F_n(\phi_{n'}(z))+\frac{1}{i}\phi_2(x_2)F_1(\phi_1(x_1))+\frac{1}{i}\phi_1(x_1)F_2(\phi_2(x_2))\right)\\
&=\langle\Omega|T\{\phi_1(x_1)F_2(\phi_2(x_2))\}|\Omega\rangle+\langle\Omega|T\{F_1(\phi_1(x_1))\phi_2(x_2)\}|\Omega\rangle\,.
\end{aligned}$$

Example 7. Let's consider $\mathcal{U}(1)$ symmetry $\phi\to e^{i\theta}\phi$. The transformation is $\delta\phi=i\theta\phi$ and $\delta\phi^\dagger=-i\theta\phi^\dagger$ and $F(\phi)=i\phi$ and $F(\phi^\dagger)=-i\phi^\dagger$. For $m=1$ we get

$$\langle\Omega|T\{i\phi(x_1)\phi^\dagger(x_2)\}|\Omega\rangle+\langle\Omega|T\{\phi^\dagger(x_1)\phi(x_2)\}|\Omega\rangle=0,$$

which is trivially satisfied. For $m=2$ we get

$$\langle\Omega|T\{\phi_1(x_1)\phi_2(x_2)\phi_3^\dagger(x_3)\}|\Omega\rangle=0$$

which is not so trivial. More generally, any n-point function with unequal number of ϕ and $\phi^\dagger$ vanishes. This is one example of a selection rule – charge carried by ϕ is conserved. Even if the measure is invariant under the transformation, we can construct the anomalous Ward identities by including $\int \mathrm{d}^4x \sum_n \frac{\delta F_n}{\delta \phi_n}$.

6.9 Local transformations and Noether current

What is the QFT equivalent of $\partial_\mu j^\mu(x) = 0$? Assume we have $\theta^a = \theta^a(x)$. The action is no longer invariant under local transformations (in general).

Claim 6.3.

$$\int \mathrm{d}^4x \, \frac{\delta S}{\delta \phi_n(x)} F_n^a(\phi_n(x))\theta^a(x) = -\int \mathrm{d}^4x \, (\partial^\mu j_\mu^a(x))\theta^a(x),$$

where

$$j_\mu^a(x) = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_n(x)} F_n^a(\phi_n(x)) - K_\mu^a$$

is the classical Noether current.

Proof. Take

$$\begin{aligned}
\int \mathrm{d}^4x \, \frac{\delta S}{\delta \phi_n(x)} F_n^a(\phi_n(x))\theta^a(x) &= \int \mathrm{d}^4x \, \left(\frac{\partial \mathcal{L}}{\partial \phi_n(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_n(x)} \right) F_n^a(\phi_n(x))\theta^a(x) \\
&= \int \mathrm{d}^4x \, \left(\frac{\partial \mathcal{L}}{\partial \phi_n} F_n^a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_n} \partial^\mu F_n^a - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_n} F_n^a \right) \right) \theta^a(x) \\
&= \int \mathrm{d}^4x \, \partial_\mu \left(K_\mu^n - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_n} F_n^a \right) \theta^a(x) \\
&= -\int \mathrm{d}^4x \, \partial^\mu j_\mu^a(x) \theta^a(x).
\end{aligned}$$

We used

$$\frac{1}{\epsilon} \frac{\partial \mathcal{L}(\phi_n + \epsilon\theta^a F^a, \partial_\mu(\phi_n + \epsilon\theta^a F^a))}{\partial \theta^a} = \frac{1}{\epsilon}(\mathcal{L}(\phi_n', \partial_\mu \phi_n') - \mathcal{L}(\phi_m, \partial_\mu \phi_n)) = \frac{1}{\epsilon} \frac{\delta \mathcal{L}}{\delta \theta^a(x)} = \partial^\mu K_\mu^a.$$

□

Using this claim, we see that if ϕ_n solves the equations of motion, then

$$\frac{\delta S}{\delta \phi_n^{\mathrm{EOM}}} = 0$$

and for every $\theta^a(x)$ we have a conserved current $\partial_\mu j_\mu^a(x) = 0$. We now insert this into the Master formula and assume that the measure is invariant under the transformation. We get

$$0 = \int \mathcal{D}\phi_n \exp\left(iS[\phi_n] + i \int \mathrm{d}^4x \, \mathcal{J}_n(x)\phi_n(x) \right) \left(-\partial^\mu j_\mu^a(\phi_n) + \mathcal{J}_n(x) F_n(\phi_{n'}(x)) \right).$$

Technically we should get an integral in x, but since it holds for all $\theta^a(x)$, we can choose $\delta(x)$ and the integral vanishes. Taking $\mathcal{J}_{n_i}(x_i)$ derivatives we get the Ward identities:

$$\begin{aligned}&\partial_\mu^{(x)} \langle\Omega|T\{j_a^\mu(\phi_n(x))\phi_{n_1}(x_1)\dots\phi_{n_m}(x_m)\}|\Omega\rangle = \\ &= (-i)\sum_{k=1}^{m}\delta^{(4)}(x-x_k)\langle\Omega|T\{\phi_{n_1}(x_1)\dots\phi_{n_{k-1}}(x_{k-1})F_{n_k}(\phi_l(x_k))\phi_{n_{k+1}}(x_{k+1})\dots\phi_{n_m}(x_m)\}|\Omega\rangle\,.\end{aligned}$$

Keep in mind that the time ordering is always T^*. Terms on the right are called *contact terms*. Inserting the QFT Noether current we get

$$\partial^\mu \langle\Omega|T\{j_\mu^a(\phi_n(x))\}|\Omega\rangle = 0.$$

For the $\mathcal{U}(1)$ symmetry it holds

$$\begin{aligned}&\partial_\mu \langle\Omega|T\{j_\mu\phi(x_1)\phi^\dagger(x_2)\}|\Omega\rangle = \\ &= (-i)\delta(x-x_1)\langle\Omega|T\{i\phi(x_1)\phi^\dagger(x_2)\}|\Omega\rangle + (-i)\delta(x-x_2)\langle\Omega|T\{\phi(x_1)(-i\phi^\dagger(x_2))\}|\Omega\rangle \\ &= (\delta(x-x_1)-\delta(x-x_2))\langle\Omega|T\{\phi_1(x_1)\phi^\dagger(x_2)\}|\Omega\rangle\,.\end{aligned}$$

Chapter 7

Scattering theory

First we will prepare initial states which are well separated. Then the states will interact and at the end the final states will again be well separated. We will also assume that the interaction has a finite range. The initial state should be almost momentum eigenstates but still without overlap in the positions space. Define initial state as

$$|\psi_i\rangle = \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \frac{1}{2E_{\mathbf{p_1}}} \int \frac{\mathrm{d}^3 p_2}{(2\pi)^3} \frac{1}{2E_{\mathbf{p_2}}} f(p_1) f(p_2) \, |p_1, p_2, \mathrm{in}\rangle \, .$$

The final state should fulfil the same conditions. The asymptotic states are defined in the interacting picture. We define the 4-momenum operator P_μ from $\mathcal{L} = \mathcal{L}^0 \mathcal{L}^{\mathrm{int}}$ and set $E_0 = 0$. The Poincaré invariance of the vacuum gives $\mathcal{U}(\Lambda, a) \, |\Omega\rangle = |\Omega\rangle$ and hence $P_\mu \, |\Omega\rangle = 0$.

7.1 One-particle states

Assume that eigenvalues of $P_\mu P^\mu = P^2 \geq 0$, one particle states are part of discrete spectrum of P^2. The basis states for P^μ operator are $|p, n\rangle$, with corresponding eigenvalues $P^0 = H = \sqrt{m'^2 - \mathbf{p}^2}$. Index n denotes other properties such as spin and charge. The problem is that one particle states of the full theory are not the same as one particle states of the free theory.

Example 8. Let

$$\mathcal{L} = \frac{1}{2}((\partial_\mu \phi)^2 - m^2 \phi^2) + \mathcal{L}^{\mathrm{int}}(\phi).$$

If $\mathcal{L}^{\mathrm{int}} = 0$, P^2 has one discrete eigenvalue m^2. For $\mathcal{L}^{\mathrm{int}} \neq 0$ the spectrum of P^2 is a shifted function of mass: $M^2 = M^2(m^2, \lambda)$. In this case the spectrum contains only one particle states (hyperbola at m), of multiple particle states (which is a continuous spectrum starting at a hyperbola at $2m$) and of bound two-particle states (also a hyperbola slightly below $2m$).

7.2 Multi-particle states

The multi-particle states are defined as

$$|\phi_\alpha\rangle = |p_1, n_1, p_2, n_2, \ldots\rangle \, .$$

They are created from full theory single particle states. Define

$$H = \tilde{H}_0 + \tilde{H}^{\text{int}},$$

where $\tilde{H}_0$ only contains free-field terms of single-particle states of the full Hamiltonian H (including e.g. bound states, if they exist). States $|\phi_\alpha\rangle$ are eigenstates of $\tilde{H}_0$ with

$$\sum_i p_i^0 = \sum_i \sqrt{M_i^2 + \mathbf{p}_i^2}.$$

The spectrum contains single particle states but also the continuum of multi-particle states. For two particles $(M, \mathbf{p}_1)$ and $(M, \mathbf{p}_2)$

$$P^2 |\phi_\alpha\rangle = \left[\left(\sqrt{M^2 + \mathbf{p}_1^2} + \sqrt{M^2 + \mathbf{p}_1^2}\right) - (\mathbf{p}_1 + \mathbf{p}_2)^2\right] |\phi_\alpha\rangle .$$

Since H^{int} has finite range, we expect H to have full set of eignestates $|\psi_\alpha\rangle$ with $H|\psi_\alpha\rangle = E_\alpha |\psi_\alpha\rangle$. Each eigentstate is a well-separated wave-packet. Next we define the initial and final states as $|\psi_\alpha^{\text{in}}\rangle$ and $|\psi_\alpha^{\text{out}}\rangle$ and the scattering matrix as

$$S_{\beta\alpha} = \langle \psi_\beta^{\text{out}} | \psi_\alpha^{\text{in}} \rangle .$$

Later we will assume these are momentum eigenstates.

7.3 Lippmann-Schwinger equation

The *Lippmann-Schwinger equation* is sometimes called the old-fashioned on-shell perturbation theory. Initially, we have $\tilde{H}_0 |\phi_S\rangle = E |\phi_S\rangle$. If energies are continuous we can write $(\tilde{H}_0 + \tilde{H}^{\text{int}}) |\psi_S\rangle = E |\psi_S\rangle$. Combining those two equations gives

$$|\psi_S\rangle = |\phi_S\rangle + \frac{1}{-\tilde{H}_0} \tilde{H}^{\text{int}} |\psi_S\rangle .$$

For the old-fashioned perturbation theory we have $\tilde{H}^{\text{int}} |\psi_S\rangle = T |\phi_S\rangle$, so

$$|\psi_S\rangle = |\phi_S\rangle + \frac{1}{E - \tilde{H}_0} T |\phi_S\rangle .$$

Multiplying the equation by $\tilde{H}^{\text{int}}$ and $\langle \phi_j|$ we get

$$\langle \phi_j | \tilde{H}^{\text{int}} | \psi_S \rangle = \langle \phi_j | T | \phi_S \rangle = \langle \phi_j | \tilde{H}^{\text{int}} | \phi_S \rangle + \langle \phi_j | \tilde{H}^{\text{int}} \frac{1}{E - \tilde{H}_0} | \phi_S \rangle .$$

For each ϕ_j and ϕ_S we have

$$T = \tilde{H}^{\text{int}} + \tilde{H}^{\text{int}} \frac{1}{E - \tilde{H}_0} T,$$

and it can be easily solved perturbatively:

$$T = \tilde{H}^{\text{int}} + \tilde{H}^{\text{int}} \frac{1}{E - \tilde{H}_0} + \tilde{H}^{\text{int}} \frac{1}{E - \tilde{H}_0} \tilde{H}^{\text{int}} \frac{1}{E - \tilde{H}_0} + \dots$$

Since it is a perturbative theory with only on-shell states it is not manifest relativistic invariant. Formal solution is $1/(E-\tilde{H}_0)$ which is singular for on-state particles. We take care of the singularity by switching off the interaction: $H^{\text{int}} \to H^{\text{int}} e^{\epsilon t}$. The solution becomes

$$|\psi_\alpha^{\text{in}}\rangle = |\phi_\alpha\rangle + \frac{1}{E - \tilde{H}_0 + i\epsilon} \tilde{H}^{\text{int}} |\psi_\alpha^{\text{in}}\rangle .$$

Lippmann-Schwinger equation implies that $|\psi_\alpha^{\text{in}}\rangle = |\psi_\alpha^{\text{out}}\rangle = |\phi_\alpha\rangle$ and $H|\psi_\alpha^{\text{in}}\rangle = E_\alpha |\psi_\alpha^{\text{in}}\rangle$. Next we have

$$e^{-iHt} \int \mathrm{d}x\, f(x) |\psi_\alpha^{\text{in/out}}\rangle \xrightarrow{t\to\mp\infty} e^{-iH_0 t} \int \mathrm{d}x\, f(x) |\phi_\alpha\rangle .$$

We define *Møller operators* as

$$|\psi_\alpha^{\text{in/out}}\rangle = \Omega^{\text{in/out}} |\phi_\alpha\rangle ,$$

in particular

$$\Omega^{\text{in/out}} = \lim_{t\to\pm\infty} e^{iHt} e^{-i\tilde{H}_0 t}.$$

Now we can define the scattering matrix as

$$S_{\alpha\beta} \langle\psi_\beta^{\text{out}}|\psi_\alpha^{\text{in}}\rangle = \langle\phi_\beta|S|\phi_\alpha\rangle$$

with

$$S = (\Omega^{\text{out}})^\dagger \Omega^{\text{in}} = \lim_{t_2\to\infty} \lim_{t_1\to-\infty} e^{i\tilde{H}_0 t_2} e^{-iH(t_2-t_1)} e^{-i\tilde{H}_0 t_1}.$$

Some points need to be clarified:

1. S is unitary since $\Omega_{\text{in/out}}$ are unitary, $SS^\dagger = \mathbb{1} = S^\dagger S$.
2. Poincaré invariance of S: if $|\phi'_\alpha\rangle = \mathcal{U}(\Lambda, a)|\phi_\alpha\rangle$, then $\langle\phi'_\alpha|S|\phi'_\beta\rangle = \langle\phi_\alpha|S|\phi_\beta\rangle$ and $\mathcal{U}^\dagger S \mathcal{U} = S$ and $[\mathcal{U}, S] = 0$. We can show that from Poincaré invariance of the action.
3. $S|\psi_\alpha^{\text{out}}\rangle = |\alpha\rangle^{\text{in}}$.

To prove the last point we write

$$\begin{aligned}
\langle\psi_\beta^{\text{out}}|S|\psi_\alpha^{\text{out}}\rangle &= \langle\phi_\beta|\Omega^{\text{out}} S \Omega^{\text{out}}|\phi_\alpha\rangle \\
&= \lim_{t\to\infty} \langle\phi_\beta| e^{i\tilde{H}_0 t} (e^{-iHt} S e^{iHt}) e^{-\tilde{H}_0 t} |\phi_\alpha\rangle \\
&= \lim_{t\to\infty} e^{i(E_\beta - E_\alpha)t} \langle\phi_\beta|S|\phi_\alpha\rangle \\
&= \langle\psi_\beta^{\text{in}}|\psi_\alpha^{\text{in}}\rangle .
\end{aligned}$$

We assume $|\phi_\alpha\rangle$ (or $|\psi_\alpha^{\text{in}}\rangle$ or $|\psi_\beta^{\text{out}}\rangle$) form basis of the Hilbert space. This is called *asymptotic completeness*. With this we can also assume that $|\phi_\alpha\rangle$ are orthonormal and complete:

$$\langle\phi_\beta|\phi_\alpha\rangle = \delta(\beta - \alpha) = (2\pi)^3 2E_{\mathbf{p_\alpha}} \delta^{(3)}(\mathbf{p_\beta} - \mathbf{p_\alpha}).$$

7.4 Asymptotic fields

The Fock space of $|\psi_\alpha^{\text{in/out}}\rangle$ contains non-interacting multi-particle states. The corresponding creation and annihilation operators are $(a_{\mathbf{p}}^\dagger)^{\text{in}}$, $a_{\mathbf{p}}^{\text{in}}$, $(a_{\mathbf{p}}^\dagger)^{\text{out}}$ and $a_{\mathbf{p}}^{\text{out}}$. The single particle states are defined as $(a_{\mathbf{p},n}^\dagger)^{\text{in/out}}|\Omega\rangle$. The asymptotic fields are defined as

$$\phi_n^{\text{in/out}} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} (e^{-ipx} a_{\mathbf{p},b}^{\text{in/out}} + (e^{ipx} a_{\mathbf{p},n}^\dagger)^{\text{in/out}}).$$

The time evolution of the asymptotic fields is according to the free Hamiltonian $\tilde{H}_0$. Now we will consider a weakly coupled QFT. We have

$$\langle\psi_1|\phi_n|\psi_2\rangle \xrightarrow{t=\mp\infty} \sqrt{Z_n}\,\langle\psi_1|\phi_n^{\text{in/out}}|\psi_2\rangle\,,$$

where ϕ_n belongs to the full theory and $\phi_n^{\text{in/out}}$ is a free theory state with mass $M = m + \mathcal{O}(\lambda)$. In the limit $\lambda \to 0$ it must continuously approach $\phi_n^{\text{in/out}}$ and the characteristics of the full theory (spin, charge) stay the same. Note that this does not hold for operator products. The asymptotic fields obey the Klein-Gordon equation $(\Box + M^2)\phi_n^{\text{in/out}} = 0$.

7.5 Scattering cross-section

We will look at the process $\alpha \to \beta$:

$$(p_1, n_1), \ldots, (p_i, n_i) \to (p_1', n_1'), \ldots, (p_j', n_j').$$

First assume only one nucleus in the target. The *cross-section* is then

$$\sigma = \frac{\text{umber of particles scattered}}{\text{time} \cdot \text{number density} \cdot \text{velocity of the particles in a beam}} = \frac{N}{T\Phi}.$$

Classically it is equal to the geometric size of object as the beam sees it. We can also define the *differential cross-section* as $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$, where $\mathrm{d}\Omega$ is the infinitesimal solid angle. The *transition rate* is defined as

$$R_{\alpha_\beta} = (\text{transition rate})_{\alpha\beta} = (\text{incoming flux})(number of target particles)\sigma_{\alpha\beta}.$$

More generally, the quantum mechanical probability of scattering is

$$P = \frac{(N_{\text{scattered}})_\alpha}{(N_{\text{incoming}})_\beta} \quad \Longrightarrow \quad \mathrm{d}\sigma_{\alpha\beta} = \frac{1}{T}\frac{dP_\beta}{\Phi_\alpha}.$$

The differential number of scattering events is

$$\mathrm{d}N_{\alpha\beta} = L \cdot \mathrm{d}\sigma_{\alpha\beta},$$

where L is called *luminosity*.

For the decay we consider the process $\alpha \to \beta$

$$(p_1, n_1) \to (p_1', n_1'), \ldots, (p_j', n_j').$$

The differential *decay rate* $\Gamma_{\alpha\beta}$ is the probability that an unstable particle decays in a time interval:

$$\mathrm{d}\Gamma_{\alpha\beta} = \frac{1}{T}\mathrm{d}P_{\alpha\beta}.$$

The lifetime of the particle is $\tau = 1/\Gamma_{\text{tot}}$, where

$$\Gamma_{\text{tot}} = \int \mathrm{d}\beta\, \mathrm{d}\Gamma_{\alpha\beta}$$

is the *total decay rate*.

Naively it seems impossible to us the S-matrix for this, however, we can formally switch off the interaction $H \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}}$ for $t \to -\infty$ using hierarchy of the time scales. The decay rate is then calculated as $1 \to m$ scattering process. We want to calculate $\mathrm{d}\sigma_{\alpha\beta}$ and $\mathrm{d}\Gamma_{\alpha\beta}$ in terms of the S-matrix. We have

$$\delta_{\beta\alpha} = \langle \phi_\beta | S | \phi_\alpha \rangle = \delta(\beta - \alpha) + i(2\pi)^4 \delta^{(4)}(p_\beta - p_\alpha) T_{\alpha\beta}.$$

The last term is called the *T-matrix*. The expression is guaranteed to be true since $[S, P_\mu] = 0$. We need to figure the factor N in

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = N |S_{\alpha\beta}|^2.$$

We will change the normalization by putting our system in a box V and consider finite time T. The new orthogonality relation is

$$\langle p', n' | p, n \rangle = N_p \delta^{(3)}(\mathbf{p} - \mathbf{p'}),$$

where $N_p = (2\pi)^3 / V$. To check this, write

$$\langle p, n | p, n \rangle = N_p \delta^{(3)}(0) = N_p \int \frac{\mathrm{d}^3 x}{(2\pi)^3} e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p})} = N_p \frac{V}{(2\pi)^3} = 1.$$

The transition rate is then

$$\begin{aligned} R_{\beta\alpha} &= \frac{\langle \psi_\beta^{\mathrm{out}} | \psi_\alpha^{\mathrm{in}} \rangle}{T} \\ &= \lim_{T\to\infty} \frac{1}{T} \left| i(2\pi)^4 \delta^{(4)}(p_\alpha - p_\beta) T^V_{\beta\alpha} \right|^2 \\ &= V(2\pi)^4 \delta^{(4)}(p_\beta - p_\alpha) \prod_{i=1}^{2} \prod_{j=1}^{n} \left(\frac{N_i}{(2\pi)^3 2E_{\mathbf{p}_i}} \right) \left(\frac{N_j}{(2\pi)^3 2E_{\mathbf{p}_j}} \right) |T_{\beta\alpha}|^2, \end{aligned}$$

with

$$T^V_{\beta\alpha} = \prod_{i=1}^{2} \prod_{j=1}^{n} \sqrt{\frac{N_i}{(2\pi)^3 2E_{\mathbf{p}_i}}} \sqrt{\frac{N_j}{(2\pi)^3 2E_{\mathbf{p}_j}}} T_{\beta\alpha}.$$

The new commutation relation is

$$T^V_{\alpha\beta} = [a, a^\dagger]_V = N_p \delta(p_1 - p_2).$$

The delta function can be expressed as

$$|\delta^{(4)}(p_\alpha - p_\beta)|^2 = \frac{1}{(2\pi)^4} \int \mathrm{d}^3 x \; e^{ix(p_\alpha - p_\beta)} \delta^{(4)}(p_\alpha - p_\beta) = \frac{VT}{(2\pi)^4} \delta^{(4)}(p_\beta - p_\alpha).$$

For the collision $p_1, p_2 \top'_1, p'_2, \ldots, p'_n$ we have

$$\mathrm{d}R_{\alpha\beta} = \prod_{i=1}^{2} \frac{N_j}{(2\pi)^3 2E_{\mathbf{p}_i}} \prod_{j=1}^{n} \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_{\mathbf{p}_i}} V(2\pi)^4 \delta^{(4)}(p_\beta - p_\alpha) |T_{\beta\alpha}|^2.$$

One can check the following dimensions $[R] = 1$, $[a] = [a^\dagger] = 1$ and $[T] = 2 - n$. In order to get the cross-section we need to:

1. divide by the flux of incoming particles:
$$\frac{1}{V}|\mathbf{v}_1 - \mathbf{v_2}| = \frac{N_1}{(2\pi)^3}|\mathbf{v}_1 - \mathbf{v}_2|.$$
2. Divide by the number of particles: $1 = \frac{N_2 V}{(2\pi)^3}$. This is the same as replacing N_2 by its definition.

So we get
$$\mathrm{d}\sigma_{\beta\alpha} = \frac{1}{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}|\mathbf{v}_1 - \mathbf{v}_2|}\prod_{j=1}^{n'}\frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_{\mathbf{p}_j}}(2\pi)^4\delta^{(4)}(p_\alpha - p_\beta)|T_{\beta\alpha}|^2.$$
The pre-factor can be written also more generally as
$$E_{\mathbf{p}_1}E_{\mathbf{p}_2}|\mathbf{v}_1 - \mathbf{v}_2| = \sqrt{(p_1 p_2)^2 - (m_1 m_2)^2}.$$
The final result is
$$\mathrm{d}\sigma_{\beta\alpha} = \frac{1}{(2E_{\mathbf{p}_1})(2E_{\mathbf{p}_2})}\frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|}|T_{\beta\alpha}|^2 \mathrm{d}\Pi_{\mathrm{LIPS}},$$
where LIPS stands for "*Lorentz invariant phase space*" and is defined as
$$\mathrm{d}\Pi_{\mathrm{LIPS}} = \prod_{j=1}^{n'}\frac{\mathrm{d}^3 p_j}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}_i}}(2\pi)^4\delta^{(4)}(p_\beta - p_\alpha).$$

Remark.

1. If there are multiple identical particles in final state, we need to divide by the symmetry factor:
$$\frac{1}{S} = \frac{1}{n_1'! n_2'! \dots}$$
because we have to count the states that are indistinguishable only once.
2. Generally particles with spin (or any other internal quantum number) $|p_i, s_i, n_i\rangle$: massive spin J particles have $2J+1$ states. If we don't measure spin, we have to trace over it:
$$\sum_{s_1', s_2', \dots}(\dots) = \text{final state spin sum.}$$
If the spin of initial state is not prepared, we have to take the spin average:
$$\mathrm{d}\sigma_{\beta\alpha} \to \frac{1}{(2J_1+1)(2J_2+1)\dots}\sum_{s_1', s_2', \dots}\mathrm{d}\sigma_{\beta\alpha} = \text{initial spin average.}$$
3. The total cross-section is
$$\sigma_{\beta\alpha} = \int \mathrm{d}\sigma_{\beta\alpha}$$
integrating over entire LIPS.

7.6 Differential decay width

The differential decay width is

$$d\Gamma_{\beta\alpha} = \frac{1}{T}\mathrm{d}P_{\beta\alpha}.$$

To get the result, we repeat all steps from the previous section and get

$$\mathrm{d}\Gamma_{\beta\alpha} = \frac{1}{2E_1}\mathrm{d}\Pi_{\mathrm{LIPS}}|T_{\beta\alpha}|^2.$$

After time t, the probability that the particle has not decayed is $P = e^{-\Gamma t}$. This is also called the width of a particle: when we want to determine the mass of the particle, we can only do it in time $\tau = 1/\Gamma$, so the final mass has some uncertainty.

7.7 Field strength renormalization and spectral representation

We will show that we can write

$$\langle\Omega|T\phi(x_1)\phi(x_2)|\Omega\rangle \overset{F.T.}{=} \int \mathrm{d}M^2\,\rho(M^2)\frac{i}{p^2 - M^2 + i\epsilon},$$

where $\rho(M^2) \geq 0$ and

$$\int \mathrm{d}M^2\,\rho(M^2) = 1.$$

Consider a complex scalar field, which is Poincaré invariant

$$\mathcal{U}(\Lambda, a)\phi(x)\mathcal{U}^\dagger(\Lambda, a) = \phi(x + a).$$

Basis of Hilbert space using eigenstates of P_μ (the total 4-momentum operator) is

$$\mathbb{1} = \sum\!\!\!\!\!\!\int |\lambda\rangle\langle\lambda|.$$

Each $|\lambda\rangle$ is described by total 4-momentum $|\lambda\rangle = |\lambda_{\mathbf{p}}\rangle$ and the eigenvalue of P_μ is p_μ. The orthogonality reads

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_{\lambda}\!\!\!\!\!\!\int \frac{\mathrm{d}^3p}{(2\pi)^3 2e_{\mathbf{p}}}|\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}|,$$

where $|\lambda_{\mathbf{p}}\rangle$ are single and multi-particle states. Particularly

$$\langle\lambda_{\mathbf{p}}|\lambda_{\mathbf{p}'}\rangle = 2E_{\mathbf{p}(\lambda)}(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p} - \mathbf{p}'),$$

where

$$E_{\mathbf{p}(\lambda)} = \sqrt{m_\lambda^2 + \mathbf{p}_\lambda^2}, \quad \text{i.e.} \quad \mathbf{P}^2|\lambda_{\mathbf{p}}\rangle = m_\lambda^2|\lambda_{\mathbf{p}}\rangle.$$

At $\mathbf{p} = 0$ we have $|\lambda_0\rangle$ and $H|\lambda_0\rangle = m_\lambda|\lambda_0\rangle$. For single particle states $m_\lambda = M =$ mass of the particle. If $|\lambda_0\rangle$ is a two-particle states, then

$$m_\lambda^2 = \left(\sqrt{M^2 + \mathbf{p}_2^2} + \sqrt{M^2 + \mathbf{p}_2^2}\right)^2 - (\mathbf{p}_1 - \mathbf{p}_2)^2 = 4(M^2 + \mathbf{p}_1^2),$$

because $\mathbf{p}_1 = -\mathbf{p}_2$.

Next, we squeeze an identity into the two-point function:

$$\langle\Omega|\phi(x)\mathbb{1}\phi^\dagger(y)|\Omega\rangle = \langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi^\dagger(y)|\Omega\rangle + \sum\!\!\!\!\!\!\int_\lambda \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}(\lambda)}}\langle\Omega|\phi(x)|\lambda_\mathbf{p}\rangle\langle\lambda_\mathbf{p}|\phi^\dagger(y)|\Omega\rangle\,.$$

Consider just $\langle\Omega|\phi(x)|\lambda_\mathbf{p}\rangle$:

$$\begin{aligned}\langle\Omega|\mathcal{U}(\mathbb{1},x)\phi(0)\mathcal{U}^\dagger(\mathbb{1},x)|\lambda_\mathbf{p}\rangle &= \langle\Omega|e^{iPx}\phi(0)e^{-iPx}|\lambda_\mathbf{p}\rangle\\ &= e^{-ipx}\langle\Omega|\phi(0)|\lambda_\mathbf{p}\rangle\\ &= e^{-ipx}\langle\Omega|\mathcal{U}(\Lambda_\mathbf{p},0)\phi(0)\mathcal{U}^\dagger(\Lambda_\mathbf{p},0)|\lambda_\mathbf{p}\rangle\\ &= e^{-ipx}\langle\Omega|\phi(0)|\lambda_0\rangle\\ &= e^{-ipx}Z(\lambda).\end{aligned}$$

We defined $Z(\lambda) = \langle\Omega|\phi(0)|\lambda_0\rangle$, which is called the *spectral function*. Next we look at $\langle\Omega|\phi(x)|\Omega\rangle = \langle\Omega|\phi(0)|\Omega\rangle = v$, which is a constant called the vacuum expectation value. We redefine the field to absorb the vacuum expectation value $\phi(x) \to \phi(x) - v$ or just assume $V = 0$. We have

$$\begin{aligned}\langle\Omega|\phi(x)\phi^\dagger(y)|\Omega\rangle &= \sum\!\!\!\!\!\!\int_\lambda \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}(\lambda)}}e^{-ip(x-y)}|Z(\lambda)|^2\\ &= \int_0^\infty \mathrm{d}M^2\,\rho(M^2)\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}e^{-ip(x-y)}\\ &= \int_0^\infty \mathrm{d}M^2\,\rho(M^2)\Delta(x-y).\end{aligned}$$

The function f has a prescription

$$\rho(M^2) = \sum\!\!\!\!\!\!\int_\lambda \delta(M^2 - m_\lambda^2)|Z(\lambda)|^2.$$

The inner integral can be more explicitly denoted by

$$\Delta(x-y, M^2) = \langle 0|\phi_0(x)\phi_0(y)|0\rangle\,,$$

where ϕ_0 satisfy $(\Box + M^2)\phi_0 = 0$ with mass M and energy $p^0 = \sqrt{M^2 - \mathbf{p}^2}$. Now redo the whole procedure for the time ordered product:

$$\langle\Omega|T\phi(x)\phi^\dagger(y)|\Omega\rangle = \theta(x^0-y^0)\langle\Omega|\phi(x)\phi^\dagger(y)|\Omega\rangle + \theta(y^0-x^0)\langle\Omega|\phi^\dagger(y)\phi(x)|\Omega\rangle\,.$$

From the previous chapter we know, that the time ordering requires only the substitution $\Delta(x-y, M^2) \to \Delta_F(x-y, M^2)$. The result is

$$\langle\Omega|T\{\phi(x)\phi^\dagger(y)\}|\Omega\rangle = \int_0^M \mathrm{d}M^2\,\rho(M^2)\Delta_F(x-y, M^2)$$

with

$$\boxed{\rho(M^2) = \sum\!\!\!\!\!\!\int_\lambda \delta(M^2 - m_\lambda^2)|Z(\lambda)|^2.}$$

It is called the *Källén-Lehmann spectral representation of the full two point function*. The spectral function is normalized. We can show that using canonical commutators:

$$\begin{aligned} i\delta^{(3)}(\mathbf{x}-\mathbf{y}) &= \langle\Omega|[\phi(t,\mathbf{x}),\phi^\dagger(t,\mathbf{y})]|\Omega\rangle \\ &= \int_0^\infty \mathrm{d}M^2\,\rho(M^2)\,\langle\Omega|[\phi_0(t,\mathbf{x}),\phi_0^\dagger(t,\mathbf{y})]|\Omega\rangle \\ &= i\delta^{(3)}(\mathbf{x}-\mathbf{y})\int_0^\infty \mathrm{d}M^2\,\rho(M^2). \end{aligned}$$

Assume single one particle state. Define

$$\langle\Omega|\phi(0)|\lambda_0\rangle = \sqrt{Z}$$

and here Z doesn't depend on λ. For free field we have

$$\langle\Omega|\phi(0)|\lambda_0\rangle = 1$$

or $\sqrt{Z_{\text{free}}} = 1$. Generally we can write

$$\rho(M^2) = Z\delta(M^2-m^2) + (\text{terms from multi-particle states}).$$

Since $\rho(M^2) \geq 0$ and it is normalized, we know that $0 \leq Z \leq 1$. Furthermore, $Z = 1$ only if $\phi(x)$ is free since then for multi-particle states it holds $\langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\rangle = 0$. Using the Poincaré transformation from before we can show

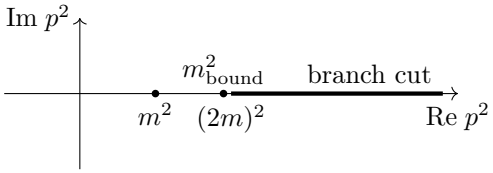
$$\langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\rangle = \sqrt{Z}e^{-ipx}$$

and this is exactly the same thing we already assumed before:

$$\phi(x) \xrightarrow{t\to\pm\infty} \sqrt{Z}\phi^{\text{in/out}}(x).$$

The value $\sqrt{Z}$ is called the *field strength renormalization constant*. Next, let's look at the Fourier decomposition of the two-point function:

$$\int \mathrm{d}^4(x-y)e^{ip(x-y)}\,\langle\Omega|T\{\phi(x)\phi^\dagger(y)\}|\Omega\rangle = \frac{iZ}{p^2-M^2+i\epsilon} + \int_0^\infty \mathrm{d}M^2\,\rho(M^2)\frac{i}{p^2-M^2+i\epsilon}.$$



The location of the poles represents the physical mass of the particle. The residue gives us the field strength renormalization. Note that $\rho(M^2)$ is smooth function, so the branch cut is equivalent to the one in complex logarithm.

Example 9. Let

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m_0^2\phi^\dagger\phi + \mathcal{L}^{\text{int}}.$$

We have

$$\begin{aligned}\int \mathrm{d}^4(x-y)e^{ip(x-y)}\,\langle\Omega|T\{\phi(x)\phi^\dagger(y)\}|\Omega\rangle &= \\ &= \sum_{n=0}^{\infty}\{n \text{ 1PI connected in a row}\} \\ &= \frac{i}{p^2-m_0^2+i\epsilon}+\frac{i}{p^2-m_0^2+i\epsilon}(-i\Pi(p^2,m_0^2))\frac{i}{p^2-m_0^2+i\epsilon}+\dots \\ &= \frac{i}{p^2-m_0^2-\Pi(p^2,m_0^2)+i\epsilon}.\end{aligned}$$

The residue will give us the implicit definition of the physical mass:

$$p^2-m_0^2-\Pi(p^2,m_0^2)|_{p^2=m^2}=0.$$

We can compute this perturbatively:

$$\Pi(p^2,m_0^2)=\lambda\Pi^{(1)}+\lambda^2\Pi^{(2)}+\dots$$

and the physical mass is $m^2=m_0^2+\delta m^2$, where $\delta m^2=\lambda\delta m^2_{(1)}+\dots$. It is called *mass renormalization*. This is familiar to certain QM effects such as the Stark effect and the spin-orbit coupling. Next, we can expand the pole at $m^2=p^2$:

$$p^2-m_0^2-\Pi(p^2,m_0^2)=(m^2-m_0^2-\Pi(m^2,m_0^2))+\left(1-\frac{\partial\Pi}{\partial p^2}\right)_{p^2=m^2}(p^2-m^2)+\dots$$

The first term is zero by definition and therefore

$$\frac{i}{p^2-m_0^2-\Pi(p^2,m_0^2)+i\epsilon}\xrightarrow{p^2\to m^2}\frac{i}{p^2-m^2+i\epsilon}\cdot\frac{1}{1-\frac{\partial\Pi}{\partial p^2}|_{p^2=m^2}}+(\text{non-singular terms}).$$

The field strength renormalization is then

$$Z^{-1}=1-\left.\frac{\partial\Pi}{\partial p^2}\right|_{p^2=m^2}.$$

What if physical states are composite objects (e.g. protons and neutrons – in low energy QCD there is no free quarks and gluons)? The analysis of $\langle\Omega|T\{\phi(x)\phi^\dagger(y)\}|\Omega\rangle$ is valid if we replace $\phi(x)$ with $\mathcal{O}(x)$:

$$\langle\Omega|T\{\mathcal{O}(x)\mathcal{O}(y)\}|\Omega\rangle$$

and $\langle\Omega|\mathcal{O}(0)|\text{hadron}\rangle=\sqrt{Z_0}$.

7.8 Constraints on the spectral function

Assume we have the following Lagrangian:

$$\mathcal{L}=-\phi^\dagger\left(\Box+c\frac{\Box^2}{\Lambda^2}+m^2\right)\phi+\mathcal{L}^{\text{int}}(\phi^\dagger,\phi).$$

The propagator is

$$\frac{i}{p^2 - m^2 - cp^4/\Lambda^2} \xrightarrow{p^2 \to \infty} -\frac{i\Lambda^2}{cp^4}.$$

We will use the spectral decomposition to show that this violates the unitarity. We have

$$\Pi(p^2) = \int_0^\infty \mathrm{d}M^2\, \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon}.$$

The analytic continuation $p^0 \to ip^0$ is a Wick rotation $p^2 = -p_\mathrm{E}^2$. We will show that $\Pi(-p_\mathrm{E}^2) \propto p_\mathrm{E}^4$ is forbidden by unitarity:

$$|\Pi(-p_\mathrm{E}^2)| = \left| \int_0^\infty \mathrm{d}M^2\, \frac{\rho(m^2)}{p_\mathrm{E}^2 + M^2} \right| \geq \left| \int_0^{M_0^2} \mathrm{d}M^2\, \frac{\rho(M^2)}{p_\mathrm{E}^2 + M^2} \right|.$$

Then

$$\lim_{p_\mathrm{E}^2 \to \infty} p_\mathrm{E}^2 |\Pi(-p_\mathrm{E}^2)| \geq \lim_{p_\mathrm{E}^2 \to \infty} p_\mathrm{E}^2 \left| \int_0^{M_0^2} \mathrm{d}M^2\, \frac{\rho(M^2)}{p_\mathrm{E}^2 + M^2} \right| \geq \frac{A}{2}.$$

This violates our previous assumption, so the two-point function cannot decrease faster than $1/p^2$ for $p^2 \to \infty$. But if $E < \lambda$ our Lagrangian can make sense as an effective field theory.

7.9 LSZ reduction formula

The idea of the *Lehmann-Symanzik-Zimmermann reduction formula* is to relate S-matrix elements to the asymptotic eigenstates. We want to convert

$$\langle k_1, \ldots, k_m; \mathrm{out} | p_1, \ldots, p_n; \mathrm{in} \rangle$$

to an expression involving Green functions. We will show

$$\begin{aligned}\langle k_1, \ldots, k_m; \mathrm{out} | p_1, \ldots, p_n; \mathrm{in} \rangle = \\ = \frac{1}{\sqrt{Z}^{n+m}} \prod_{i=1}^m \left(i \int \mathrm{d}^4 x_i\, e^{-ik_i x_i} (\Box_{x_i} + m^2) \right) \langle \Omega | \phi(x_1) \ldots \phi(x_m) \phi^\dagger(y_1) \ldots \phi^\dagger(y_n) | \Omega \rangle \\ \prod_{j=1}^n \left(i \int \mathrm{d}^3 y_j\, (\overleftarrow{\Box}_{y_j} + m^2) e^{-ip_j y_j} \right).\end{aligned}$$

In the momentum space, we have

$$\sqrt{Z}^{m+n} \tilde{G}_{\text{amputated}}(k_1, \ldots, k_m; -p_1, \ldots, -p_n)_{\text{on shell}} = iT_{k_1, \ldots, k_m; p_1, \ldots, p_n}.$$

The factors $(\Box + m^2) \xrightarrow{\text{F.T.}} (-p^2 + m^2)$ project onto the S-matrix, and they vanish for the asymptotic states. This is equivalent to truncating all the external propagators and removing all the non-single particle contributions from the external legs. Only those with poles of type $1/(p^2 - m^2)$ will remain.

Assume a complex scalar field $\phi(x)$, which can be expressed as

$$\sqrt{Z}\phi^{\mathrm{in/out}}(x) = \sqrt{Z} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} (e^{-ipx} a_\mathbf{p}^{\mathrm{in/out}} + e^{ipx} (b_\mathbf{p}^\dagger)^{\mathrm{in/out}}).$$

The conjugate momentum is

$$i\partial_0\phi^{\text{in/out}}(x) = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(e^{-ipx}a_{\mathbf{p}}^{\text{in/out}} - e^{ipx}(b_{\mathbf{p}}^\dagger)^{\text{in/out}}).$$

With this we can write

$$a_{\mathbf{p}}^{\text{in/out}} = \int \mathrm{d}^3x\, e^{ipx}(E_{\mathbf{p}}\phi^{\text{in(out)}}\phi(x) + i\partial_0\phi^{\text{in/out}}(x))$$

and

$$(b_{\mathbf{p}}^\dagger)^{\text{in/out}} = \int \mathrm{d}^3x\, e^{ipx}\partial_0\phi^{\text{in/out}}(x),$$

where $\partial_0 = \vec{\partial}_0 - \overleftarrow{\partial}_0$. Consider $\mathcal{O}(y_1,\ldots,y_n)$ which is a product of $\psi(y_1)$, …, $\psi(y_n)$ with $\psi(x_i)$ being either $\phi(x_i)$ or $\phi^\dagger(x_i)$. We claim the next relation:

$$\begin{aligned} a_{\mathbf{p}}^{\text{out}}T\{\mathcal{O}(y_1,\ldots,y_n)\} - T\{\mathcal{O}(y_1,\ldots,y_n)\}a_{\mathbf{p}}^{\text{in}} = \\ = \frac{1}{\sqrt{Z}}\int_{-\infty}^{\infty}\mathrm{d}x^0\,\partial_0\left[\int \mathrm{d}^3x\, e^{ipx}T\{p^0\phi(x) - i\partial_0\phi(x)\mathcal{O}(y_1,\ldots,y_n)\}\right]. \end{aligned}$$

To show this we use that $\phi(x) \xrightarrow{x^0\to\pm\infty} \sqrt{Z}\phi^{\text{in/out}}(x)$ and that

$$\lim_{x^0\to\infty}\phi(x)T\{\mathcal{O}(y_1,\ldots,y_n)\} = \lim_{x^0\to\infty}T\{\phi(x)\mathcal{O}(y_1,\ldots,y_n)\}$$

and

$$\lim_{x^0\to-\infty}T\{\mathcal{O}(y_1,\ldots,y_n)\}\phi(x) = \lim_{x^0\to-\infty}T\{\mathcal{O}(y_1,\ldots,y_n)\phi(x)\},$$

assuming $x^0 > y_i^0$ for all i. Lastly, we use the total derivative

$$\int_{-\infty}^{\infty}\mathrm{d}x^0\,\partial_0F(x) = F(x^0=\infty,\mathbf{x}) - F(x^0=-\infty,\mathbf{x}).$$

Continuing where we left with the previous expression:

$$\begin{aligned} a_{\mathbf{p}}^{\text{out}}T\{\mathcal{O}(y_1,\ldots,y_n)\} - T\{\mathcal{O}(y_1,\ldots,y_n)\}a_{\mathbf{p}}^{\text{in}} = \\ &= \frac{1}{\sqrt{Z}}\int_{-\infty}^{\infty}\mathrm{d}x^0\,\partial_0\left[\int \mathrm{d}^3x\, e^{ipx}T\{p^0\phi(x) - i\partial_0\phi(x)\mathcal{O}(y_1,\ldots,y_n)\}\right] \\ &= \frac{1}{\sqrt{Z}}\int \mathrm{d}^4x\, e^{ipx}i(p^0 - i\partial_0)(p^0 + i\partial_0)T\{\phi(x)\mathcal{O}(y_1,\ldots,y_n)\} \\ &= \frac{i}{\sqrt{Z}}\int \mathrm{d}^4x\, e^{ipx}(\Box_x + m^2)T\{\phi(x)\mathcal{O}(y_1,\ldots,y_n)\}. \end{aligned}$$

Now replace $\mathcal{O}$ with $\mathcal{O}^\dagger$ and take the adjoint of the previous expression: $a_{\mathbf{p}} \to a_{\mathbf{p}}^\dagger$, $b_{\mathbf{p}} \to b_{\mathbf{p}}^\dagger$,

$\pm ie^{\pm ipx} = \mp e^{\mp ipx}$ and $\phi(x) \to \phi^\dagger(x)$. We have

$$\begin{aligned}\langle k_1,\dots,k_m;\text{out}|p_1,\dots,p_n;\text{in}\rangle = \\ = \langle\Omega|\underbrace{a^{\text{out}}_{\mathbf{k}_m}\dots a^{\text{out}}_{\mathbf{k}_2}}_{\mathcal{O}(y_1,\dots y_n)} a^{\text{out}}_{\mathbf{k}_1} a^{\dagger\ \text{in}}_{\mathbf{p}_1}\dots a^{\dagger\ \text{in}}_{\mathbf{p}_n}|\Omega\rangle\cdot \\ \cdot\frac{1}{\sqrt{Z}}\left(i\int \mathrm{d}^4x_1\, e^{ik_1x_1}(\Box_{x_1}+m^2)\,\langle\Omega|a^{\text{out}}_{\mathbf{k}_m}\dots a^{\text{out}}_{\mathbf{k}_2}\phi(x_1)a^{\dagger\ \text{in}}_{\mathbf{p}_1}\dots a^{\dagger\ \text{in}}_{\mathbf{p}_n}|\Omega\rangle + \right. \\ \left. + \underbrace{\langle\Omega|a^{\text{out}}_{\mathbf{k}_m}\dots a^{\text{out}}_{\mathbf{k}_2}a^{\text{in}}_{\mathbf{k}_1}a^{\dagger\ \text{in}}_{\mathbf{p}_1}\dots a^{\dagger\ \text{in}}_{\mathbf{p}_n}|\Omega\rangle}_{=0\text{ for non-trivial scattering}}\right).\end{aligned}$$

To prove that the last term in the last equation vanishes for non-trivial scattering, we use $[a^{\text{in}}_{\mathbf{k}_1}, a^{\dagger\ \text{in}}_{\mathbf{p}_i}] = 0$ for $\mathbf{k}_1 \neq \mathbf{p}_i$. Assuming that $k_1 \neq p_i$ for all j we can write

$$\langle\Omega|a^{\text{out}}_{\mathbf{k}_m}\dots a^{\text{out}}_{\mathbf{k}_2}a^{\text{in}}_{\mathbf{k}_1}a^{\dagger\ \text{in}}_{\mathbf{p}_1}\dots a^{\dagger\ \text{in}}_{\mathbf{p}_n}|\Omega\rangle = \langle\Omega|a^{\text{out}}_{\mathbf{k}_m}\dots a^{\text{out}}_{\mathbf{k}_2}a^{\dagger\ \text{in}}_{\mathbf{p}_1}\dots a^{\dagger\ \text{in}}_{\mathbf{p}_n}a^{\text{in}}_{\mathbf{k}_1}|\Omega\rangle = 0.$$

Now repeat this process for all $a^{\text{in}}_{\mathbf{k}_i} \to \phi(x_i)$ and $a^{\dagger\ \text{out}}_{\mathbf{p}_l} \to \phi^\dagger(x_l)$ to get the main intermediate result:

$$\begin{aligned}&\langle k_1,\dots,k_m;\text{out}|p_1,\dots,p_n;\text{in}\rangle = \\ &= \frac{1}{\sqrt{Z}^{n+m}}\prod_{i=1}^{m}\left(i\int \mathrm{d}^4x_i\, e^{-ik_ix_i}(\Box_{x_i}+m^2)\right)\langle\Omega|T\{\phi(x_1)\dots\phi(x_m)\phi^\dagger(y_1)\dots\phi^\dagger(y_n)\}|\Omega\rangle \\ &\qquad\prod_{j=1}^{n}\left(i\int \mathrm{d}^3y_j\,(\overleftarrow{\Box}_{y_j}+m^2)e^{-ip_jy_j}\right).\end{aligned}$$

Integrating by parts we get

$$\begin{aligned}&\langle k_1,\dots,k_m;\text{out}|p_1,\dots,p_n;\text{in}\rangle = \\ &= \frac{(-i)^{m+n}}{\sqrt{Z}^{m+n}}\prod_{i=1}^{m}(k_i^2-m^2)\prod_{j=1}^{n}(p_j^2-m^2)\int \mathrm{d}^4x_1\dots\mathrm{d}^4x_m\mathrm{d}^4y_1\dots\mathrm{d}^4y_n \\ &\exp\left(i\sum_{i=1}^{m}k_ix_i - i\sum_{j=1}^{n}p_jy_j\right)\langle\Omega|T\{\phi(x_1)\dots\phi(x_m)\phi^\dagger(y_1)\dots\phi^\dagger(y_n)\}|\Omega\rangle\,.\end{aligned}$$

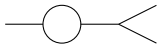
Naively, the prefactors for the on-shell asymptotic states reduce the expression to 0 ($k_i^2 = p_j^2 = m^2$). However, we have to consider the poles in the n-point function that perfectly counteract the zeros.

7.9.1 Amputated Green function

Define the amputated Green function $\tilde{G}^{\text{amp}}$ as

$$\tilde{G}^c(q_1,\dots,q_k) = \left(\prod_{l=1}^{k}\tilde{G}(q_l)\right)\tilde{G}^{\text{amp}}(q_1,\dots,q_n).$$

Example 10.

- The diagram  is connected but not amputated.
- The diagram  is connected and amputated 3-point function.

From our discussion of the spectral representation we know that

$$\tilde{G}(k) \xrightarrow{k^0 \to E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}} \frac{iZ}{k^2 - m^2} + (\text{non-singula terms}).$$

Putting that into LSZ reduction, we get

$$\frac{-i}{\sqrt{Z}}(k_i^2 - m^2)\tilde{G}(k_i) \xrightarrow{k^0 \to E_{\mathbf{k}}} \sqrt{Z}.$$

We repeat this for all external legs and get

$$\begin{aligned}
&\langle k_1, \ldots, k_m; \text{out} | p_1, \ldots, p_n; \text{in} \rangle = \\
&\quad = (2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^{m} k_i + \sum_{j=1}^{n} p_j \right) \sqrt{Z}^{m+n} \tilde{G}^{\text{amp}}(k_1, \ldots, k_m, -p_1, \ldots, -p_n)\big|_{\text{on-shell}} \\
&\quad = i(2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^{m} k_i + \sum_{j=1}^{n} p_j \right) T_{k_1 \ldots k_m; p_1 \ldots p_n}.
\end{aligned}$$

The on-shell condition means $k_i^0 = \sqrt{\mathbf{k}_i^2 + m^2}$ and $p_j^0 = \sqrt{\mathbf{p}_j^2 + m^2}$.

Comments:

- rules for computing the amputated (connected) Green functions:

	particle	anti-particle
outgoing	ϕ	$\phi^\dagger$
incoming	$\phi^\dagger$	ϕ

- Particle/antiparticle differ by momentum assignment $\tilde{G}^{\text{amp}} : p \to -p$.
- Different species of particles have different field strength renormalization constants: $\phi_1, \phi_2 \to \sqrt{Z_1}, \sqrt{Z_2}$.
- Particle with intrinsic spin also carry polarization factors (e.g. $\epsilon_\mu^*(k_1)$).

7.10 Example of computation of scattering cross-section

Let the Lagrangian be

$$\mathcal{L} = \frac{1}{2}(\partial_\mu H \partial^\mu H - m_H^2 H^2) + \partial_\mu \phi \partial^\mu \phi^\dagger - m_\phi^2 \phi^\dagger \phi + \lambda \phi^\dagger \phi H + \frac{g_3}{3!} H^3 - \frac{g_4}{2} \phi^\dagger \phi H^2.$$

We assume that all coupling constants are much smaller than 1 and that $g_4 \sim \lambda^2 \sim g_3^2$. Our Lagrangian describes one real and one complex scalar field, so in total the interaction between

three particles: scalar h, scalar ϕ^+ and anti-scalar ϕ^-. The goal is to compute the scattering cross-section: $\phi^+\phi^- \to hh$. The object we are interested in is

$$\langle\Omega|T\{H(x_1)H(x_2)\phi(x_3)\phi^\dagger(x_4)\}|\Omega\rangle\,.$$

Using the momentum Feynman rules we have three different vertices with factors

- $i\lambda$ for $\phi^+\phi^- \to h$,
- ig_3 for $hh \to h$,
- $-ig_4$ for $\phi^+\phi^- \to hh$.

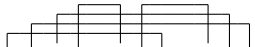
The propagators are $\frac{i}{p^2-m_H^2+i\epsilon}$ for h and $\frac{i}{p^2-m_\phi^2+i\epsilon}$ for ϕ. The simplest contributions for $\phi\phi^\dagger \to hh$ with $p_1+p_2 \to k_1+k_2$ are the following:

- All going through g_4 vertex. The contractions are



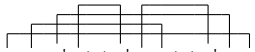
$$-\frac{ig_4}{2}(HH\phi\phi^\dagger)(\phi^\dagger\phi HH)\cdot 2 = -ig_4.$$

- t channel with the propagator p_1-k_1:



$$2\frac{1}{2!}(i\lambda)^2(HH\phi^\dagger\phi)(\phi^\dagger\phi H)(\phi^\dagger\phi H).$$

- u channel with the propagator p_1-k_2:



$$2\frac{1}{2!}(i\lambda)^2(HH\phi^\dagger\phi)(\phi^\dagger\phi H)(\phi^\dagger\phi H).$$

- s channel with propagator p_1+p_2:



$$3!\frac{1}{2!}2(i\lambda)\frac{g_3}{3!}(HH\phi^\dagger\phi)(\phi^\dagger\phi H)(HHH).$$

The T-matrix is

$$\begin{aligned} T &= (-i)(\sqrt{Z_H})^2(\sqrt{Z_\phi})^2\tilde{G}^{\text{amp}}(k_1,k_2;-p_1,-p_2)\big|_{\text{on-shell}} \\ &= (-1)\left(-ig_4-\lambda^2\left[\frac{i}{(p_1-k_1)^2-m_\phi^2}+\frac{i}{(p_1-k_2)^2-m_\phi^2}\right]-\lambda g_3\frac{i}{(p_1+p_2)^2-m_H^2}\right). \end{aligned}$$

The field normalization constants are of order $1+\lambda \approx 1$. Next, we introduce the *Mandelstam variables*:

$$t=(p_1+p_2)^2, \quad t=(p_1-k_1)^2, \quad u=(p_1-k_2)^2.$$

This simplifies our expression for T-matrix:

$$T=-\left(g_4+\frac{\lambda^2}{t-m_\phi^2}+\frac{\lambda^2}{u-m_\phi^2}+\frac{\lambda g_3}{s-m_H^2}\right).$$

In the centre of mass system (CMS), we have

$$\mathbf{p}_1 + \mathbf{p_2} = \mathbf{k}_1 + \mathbf{k}_2 = 0.$$

Thus, we choose

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ |\mathbf{p}| \end{pmatrix}, \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -|\mathbf{p}| \end{pmatrix}, \quad k_1 = \begin{pmatrix} E \\ |\mathbf{k}|\sin\theta \\ 0 \\ |\mathbf{k}|\cos\theta \end{pmatrix}, \qquad k_2 = \begin{pmatrix} E \\ -|\mathbf{k}|\sin\theta \\ 0 \\ -|\mathbf{k}|\cos\theta \end{pmatrix},$$

where $|\mathbf{p}| = \sqrt{E^2 - m_\phi^2}$ and $|\mathbf{k}| = \sqrt{E^2 - m_H^2}$. The Mandelstam variables are

$$\begin{aligned} s &= 4E^2 \\ \hat{t} = t - m_\phi^2 &= -2E^2 + m_H^2 + 2\sqrt{E^2 - m_\phi^2}\sqrt{E^2 - m_H^2}\cos\theta \\ \hat{u} = u - m_\phi^2 &= -2E^2 + m_H^2 - 2\sqrt{E^2 - m_\phi^2}\sqrt{E^2 - m_H^2}\cos\theta \end{aligned}$$

We can show that $\hat{t}, \hat{u} \leq m_\phi^2$ and so $t, u \leq 0$. That means that the propagators will never have on-shell singularity. The differential cross-section is

$$\mathrm{d}\sigma = \frac{1}{2}\frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_{\mathbf{k}_1}}\frac{\mathrm{d}^3k_2}{(2\pi)^3 2E_{\mathbf{k}_2}}\frac{1}{4\sqrt{(p_1p_2)^2 - m_\phi^4}}(2\pi)^4\delta^{(4)}(k_1 + k_2 - p_1 - p_2)|T|^2.$$

The 1/2 in front counts the fact that the final state consists of two identical particles. We transform the measure

$$\mathrm{d}^3k_1 = \mathrm{d}|\mathbf{k}_1|\mathrm{d}\Omega|\mathbf{k}_1|^2, \quad \mathrm{d}|\mathbf{k}_1| = \mathrm{d}E_{\mathbf{k}_1}\frac{E_{\mathbf{k}_1}}{|\mathbf{k}_1|}.$$

The energy part of delta function becomes $\frac{1}{2}\delta(E_{\mathbf{k}_1} - E_{\mathbf{k}_2})$. Considering all that, we get

$$\mathrm{d}\sigma = \frac{1}{2}\frac{1}{4\pi^2}\frac{E_{\mathbf{k}_1}|\mathbf{k}_1|^2}{|\mathbf{k}_1|}\mathrm{d}\Omega\frac{1}{4E_{\mathbf{k}_1}E_{\mathbf{k}_2}}\frac{1}{4\sqrt{(p_1p_2)^2 - m_\phi^4}}\frac{1}{2}|T|^2.$$

Using

$$(2E^2 - m_\phi)^2 - m_\phi^4 = 4E^2(E^2 - m_\phi^2)$$

we get

$$\mathrm{d}\sigma = \frac{1}{512\pi^2E^2}\sqrt{\frac{E^2 - m_H^2}{E^2 - m_\phi^2}}|T|^2\mathrm{d}\Omega.$$

The T-matrix doesn't depend on φ, so we can write $\mathrm{d}\Omega = 2\pi\mathrm{d}\theta$. Therefore,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = 2\pi\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$$

and the *total cross-section* is

$$\sigma_{\mathrm{tot}} = \int_{-1}^{1}\mathrm{d}\cos\theta\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}.$$

Interesting limits:

- **High energies**: $E \gg m_\phi, m_H$ and $|\cos\theta| \not\approx 1$. The T-matrix is

$$T = -g_4 + \mathcal{O}(E^{-2}).$$

  The cross-section is isotropic and dominated by four-point interaction.

- High energy **forward/backward-scattering**: $|\cos\theta| \approx 1$, $\hat{t} \approx -2E^2(1-\cos\theta)$ and $\hat{u} \approx -2E^2(1+\cos\theta)$. The T-matrix is

$$T \approx -\left(g_4 - \frac{\lambda^2}{E^2}\frac{1}{\sin^2\theta} + \frac{\lambda g_3}{4E^2}\right) \approx -\left(g_4 - \frac{\lambda^2}{E^2}\frac{1}{\sin^2\theta}\right).$$

  This is reminiscent of the Rutherford scattering and will frequently occur with massless mediators.

- The **total cross-section** has a threshold at $2m_H$. At energies near threshold we have

$$T \approx -\left(g_4 - \frac{2\lambda^2}{m_H^2} + \frac{\lambda g_3}{3m_H^2}\right)$$

  and the whole cross-section goes like $\sigma \sim \sqrt{E^2 - m_H^2}$. At high energies the behaviour is

$$\sigma = \frac{1}{128\pi}\frac{g_4^2}{E^2}.$$

If we also had loops, our T-matrix would have one integral for each loop. If more particles were produced, we would need to mend our phase space.

There are some other useful constructs we didn't cover – mainly the *Ward identity* which is the application of Ward-Takahashi identity to the S-matrix (when particles are on-shell). See the last chapter for further discussion. The other is the *optical theorem* which relates the imaginary part of the forward scattering amplitude to the total cross-section (see exercises).

Chapter 8

Renormalization

Consider the standard scalar Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{\lambda}{4!}\phi^4.$$

The tree level diagram at first order is just $iT_1 = -i\lambda$. The 1-loop correction consists of three different diagrams (s, t and u). Firstly, we will just focus on the t channel. The incoming and outgoing particles have momenta $p_1 + p_2 \to k_1 + k_2$. The propagators have momenta k and $k - p_1 - p_2 = k - p$. The T-matrix contribution for this diagram is

$$iT_2 = \frac{1}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{(p-k)^2}.$$

The factor 1/2 at the front is the symmetry factor. There are a few observations we can make:

- iT_2 is Lorentz invariant, it only depends on $p^2 = s$.
- It is dimensionless
- For large k we have
$$\int \mathrm{d}^4k\ \frac{1}{k^2}\frac{1}{(p-k)^2} \approx \int_C^\Lambda \mathrm{d}k\, k^3 \frac{1}{k^4} \approx \ln\Lambda.$$

Combining all three observations, the only possible form of the T-matrix is

$$iT_2 \sim \ln\left(\frac{s}{\Lambda^2}\right).$$

We can quickly check this by taking the derivative of T:

$$\frac{\partial}{\partial s}iT_2(s) = \frac{p^\mu}{2s}\frac{\partial}{\partial p^\mu}iT_2 = -\frac{\lambda^2}{2s}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}\frac{(p-k)p}{(p-k)^4} = -\frac{i\lambda}{32\pi^2}\frac{1}{s}.$$

Integrating the expression gives

$$T_2(s) = -\frac{\lambda^2}{32\pi^2}\ln s + C.$$

We choose C to be

$$C = \frac{\lambda^2}{32\pi^2} \ln(\Lambda^2).$$

The full T-matrix (up to 1-loop corrections) is

$$T(s) = -\lambda - \frac{\lambda^2}{32\pi^2} \ln\left(\frac{stu}{\Lambda^6}\right).$$

This is divergent! But it is not an observable yet, because we haven't defined how λ enters the expression. However, if we subtract T-matrices at two different energies, we get

$$T(s_1) - T(s_2) = \frac{\lambda^2}{32\pi^2} \ln\left(\frac{s_2}{s_1}\right).$$

The problem is that when we measure $\phi\phi \to \phi\phi$, we don't measure just λ, but also the higher order contributions. But what we want is that the measurement is finite. Thus, we define the *renormalized coupling* λ_{R} as the value of total matrix element at certain scale s_0:

$$\lambda_{\mathrm{R}} \equiv -T(s_0) = \lambda + \frac{\lambda^2}{32\pi^2} \ln\left(\frac{s_0}{\Lambda^2}\right) + \dots$$

Since this is an observable, it will clearly be finite. We will solve for λ using perturbation: $\lambda = \lambda_{\mathrm{R}} + a\lambda_{\mathrm{R}}^2 + \dots$. We get

$$\begin{aligned}\lambda_{\mathrm{R}} &= (\lambda_{\mathrm{R}} + a\lambda_{\mathrm{R}}^2 + \dots) + \frac{(\lambda_{\mathrm{R}} + a\lambda_{\mathrm{R}}^2 + \dots)^2}{32\pi^2} \ln\left(\frac{s_0}{\Lambda^2}\right) + \dots \\ &= \lambda_{\mathrm{R}} + a\lambda_{\mathrm{R}}^2 + \frac{\lambda_{\mathrm{R}}^2}{32\pi^2} \ln\left(\frac{s_0}{\Lambda^2}\right) + \mathcal{O}(\lambda_{\mathrm{R}}^3).\end{aligned}$$

Therefore,

$$a = -\frac{1}{32\pi^2} \ln\left(\frac{s_0}{\Lambda^2}\right)$$

and

$$\lambda = \lambda_{\mathrm{R}} - \frac{\lambda_{\mathrm{R}}^2}{32\pi^2} \ln\left(\frac{s_0}{\Lambda^2}\right) + \dots$$

We found that *bare parameter* λ is actually divergent. But with this we can calculate the cross-section at some different scale:

$$\begin{aligned}T(s) &= -\lambda - \frac{\lambda^2}{32\pi^2} \ln\left(\frac{s}{\Lambda^2}\right) \\ &= \left[\lambda_{\mathrm{R}} - \frac{\lambda_{\mathrm{R}}^2}{32\pi^2} \ln\left(\frac{s_0}{\Lambda^2}\right)\right] - \frac{\lambda_{\mathrm{R}}^2}{32\pi^2} \ln\left(\frac{s}{\Lambda^2}\right) + \dots \\ &= -\lambda_{\mathrm{R}} - \frac{\lambda_{\mathrm{R}}^2}{32\pi^2} \ln\left(\frac{s}{s_0}\right) + \dots\end{aligned}$$

The expansion in λ_{R} leads to finite predictions. It is also called the *on-shell perturbation theory*. Logarithms are characteristic of loop effects.

8.1 Counter-terms

Another way to get the same result is to add additional terms (called *counter-terms*) to our Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi - \frac{\lambda_{\mathrm{R}}}{4!}\phi^4 - \frac{\delta_\lambda}{4!}\phi^4.$$

The additional coupling constant is infinite, but formally we can think of it as being of order λ_{R}^2. The amplitude is

$$T(s) = -\lambda_{\mathrm{R}} - \delta_\lambda - \frac{\lambda_{\mathrm{R}}^2}{32\pi^2}\ln\left(\frac{s}{\Lambda^2}\right) + \mathcal{O}(\lambda_{\mathrm{R}}^4).$$

We find

$$\delta_\lambda = -\frac{\lambda_{\mathrm{R}}^2}{32\pi^2}\ln\left(\frac{s_0}{\Lambda^2}\right).$$

With this we get the same result as before

$$T(s) = -\lambda_{\mathrm{R}} - \frac{\lambda_{\mathrm{R}}^2}{32\pi^2}\ln\left(\frac{s}{s_0}\right) + \dots$$

This approach is easier in practice and is called *renormalized perturbation theory*. But where do divergences come from? The interactions are assumed to be local. We need to include infinite momenta in definition of our action:

$$\tilde{S} \propto \int \mathrm{d}^4k_1\mathrm{d}^4k_2\mathrm{d}^4k_3\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\delta^{(4)}(k_1+k_2+k_3).$$

Consider low-energy mode $k_1 \approx 0$. Then one needs an unsuppressed coupling to arbitrary high $k_2 \approx -k_3$ modes. The issue is to extrapolate theory to $E \to \infty$ to define local operators, but we are ignorant of physics at scales $E > \Lambda$. Thus, we often talk about *bare Lagrangian*:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0\partial^m u\phi_0 - m_0\phi_0^2) - \frac{\lambda_0}{4!}\phi_0^4$$

where every object with index 0 carries prefix "bare".

8.2 Self-energy calculation

In previous chapter, we calculated the two-point function:

$$\int \mathrm{d}^4(x-y)e^{ip(x-y)}\,\langle\Omega|\phi_0(x)\phi_0(y)|\Omega\rangle = \frac{i}{p^2 - m_0^2 - \Pi(p^2, m_0^2) + i\epsilon},$$

where ϕ_0 is the bare field for the full (bare) Lagrangian. As before, we have

$$-i\Pi(p^2, m_0^2) = \frac{1}{2}(-i\lambda_0)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - m_0^2 + i\epsilon} + \mathcal{O}(\lambda_0^2).$$

Next, we perform a *Wick rotation*. The new metric is $\eta_{\mu\nu}^{\mathrm{E}} = \delta_{\mu\nu}$, so the substitution is $k_{\mathrm{E}}^0 = -ik^0$. The 4-integral transforms as

$$\int \mathrm{d}^4k\, f(k^2) = \int -\infty^{\infty}\mathrm{d}k^0\int \mathrm{d}^3k\, f(k^2) = \int_{-i\infty}^{i\infty}\mathrm{d}k^0\int \mathrm{d}^3k\, f(k^2) = i\int \mathrm{d}^4k_{\mathrm{E}} f(-k_{\mathrm{E}}^2).$$

We can drop $i\epsilon$ term since we are never close to any pole:

$$-i\Pi(p^2, m_0^2) = -i\frac{\lambda_0}{2} i \int \frac{\mathrm{d}^4 k_\mathrm{E}}{(2\pi)^4} \frac{i}{-k_\mathrm{E}^2 - m_0^2} = -\frac{i\lambda_0}{32\pi^2} \int_0^\infty \mathrm{d}k_\mathrm{E}^2 \frac{k_\mathrm{E}^2}{k_\mathrm{E}^2 + m_0^2},$$

which is divergent. Using the regularization, we get

$$i\Pi(p^2, m_0^2) = -\frac{i\lambda_0}{32\pi^2} \left[\Lambda^2 - m_0^2 \ln\left(\frac{\Lambda^2 + m_0^2}{m_0^2}\right)\right] = -\frac{i\lambda_0}{32\pi^2} \left[\Lambda^2 - m_0^2 \ln\left(\frac{\Lambda^2}{m_0^2} + \mathcal{O}\left(\frac{m_0^4}{\Lambda^2}\right)\right)\right].$$

The physical mass is given implicitly by

$$m^2 - m_0^2 - \Pi(p^2 = m^2, m_0^2) = 0.$$

Our Π is quite special, because it doesn't depend on p^2. We get

$$m^2 = m_0^2 + \frac{\lambda_0}{32\pi^2} \left(\Lambda^2 - m_0^2 \ln\left(\frac{\lambda^2}{m_0^2}\right) + \dots\right).$$

The field strength renormalization can be calculated As

$$Z = 1 - \frac{1}{1 - \frac{\partial \Pi}{\partial p^2}|_{p^2=m^2}} = 1 + \mathcal{O}(\lambda_0^2).$$

When we do regularization and renormalization it is crucial to avoid breaking symmetries of QFT as much as possible. In recap, regularization is deformation of QFT to make it finite (UV cut-off), which depends on the regulator. Renormalization is a process that relates regularized amplitudes to observables. This procedure will make physical parameters scale-dependent (anomalous dimension).

8.3 Regularization methods and Feynman parameters

We will be interested in general integrals of form

$$A(a, \Delta) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\epsilon)^a} = (-1)^a i \int \frac{\mathrm{d}^4 k_\mathrm{E}}{(2\pi)^4} \frac{1}{(k_\mathrm{E}^2 + \Delta)^a}.$$

After regularization, we got

$$A(a, \Delta) = i(-1)^a \frac{\pi^2}{(2\pi)^4} \int_0^{\Lambda^2} \mathrm{d}k_\mathrm{E}^2 \frac{k_\mathrm{E}^2}{(k_\mathrm{E}^2 + \Delta)^a}.$$

For $a = 2$ it holds

$$A(2, \Delta) = \frac{i}{16\pi^2} \left(\ln\left(\frac{\Lambda^2}{\Delta} + 1\right) - \frac{\Lambda^2}{\Delta + \Lambda^2}\right).$$

This kind of regularization is called *cut-off regularization*. Another approach is the *Pauli-Villars regularization* where we add a mass to otherwise massless particles. But the most common method is what will be discussed in the following chapter.

8.3.1 Dimensional regularization

With *dimensional regularization* we will assume that space-time and momentum space is d-dimensional. This means that k_{E} is also d-dimensional. The integration measure changes to

$$\int \frac{\mathrm{d}^d k_{\mathrm{E}}}{(2\pi)^d} = \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d} \int_0^\infty \mathrm{d}k_{\mathrm{E}}^2 \frac{1}{2} (k_{\mathrm{E}}^2)^{d/2-1} = \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(d/2)} \int_0^\infty \mathrm{d}k_{\mathrm{E}}^2 (k_{\mathrm{E}}^2)^{d/2-1}.$$

The integral $A(a, \Delta)$ can be now written as

$$\begin{aligned} A(a,\Delta) &= \frac{i}{(4\pi)^{d/2}} \frac{(-1)^a}{\Gamma(d/2)} \tilde{\mu}^{4-d} \int_0^\infty \mathrm{d}\ell \frac{\ell^{d/2-1}}{(\partial+\Delta)^a} \\ &= \frac{i}{(4\pi)^{d/2}} \frac{(-1)^a}{\Gamma(d/2)} \Delta^{d/2-a} \tilde{\mu}^{4-d} \int_0^1 \mathrm{d}x\, x^{a-1-d/2} \overline{x}^{d/2-1}. \end{aligned}$$

The parameter $\tilde{\mu}$ is introduced to keep the dimension the same as in the original integral. The parameter k_{E}^2 was substituted for ℓ. In the last step, we performed the substitution

$$x \equiv \frac{\Delta}{\ell + \Delta}, \quad \text{and} \quad \overline{x} = 1 - x.$$

The integral measure was changed to $\mathrm{d}\ell = -\Delta/x^2\, \mathrm{d}x$. Using the definition of *Euler B-function*

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

we get

$$\boxed{A(a,\Delta) = \frac{i}{(4\pi)^2} \frac{\Gamma(a-d/2)}{\Gamma(a)} \left(\frac{\Delta}{4\pi\tilde{\mu}^2}\right)^{d/2-2} (-\Delta)^{2-a}.}$$

We will be able to transform any loop integral into this form.

8.4 Gamma function

Let's take a quick digression to the properties of the gamma function. The *gamma function* is defined as

$$\Gamma(z) = \int_0^\infty \mathrm{d}t\, t^{z-1} e^{-t}.$$

It has the following properties:

1. $\Gamma(x+1) = x\Gamma(x)$,
2. $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$,
3. $\Gamma(x)$ is analytic except for simple poles at $x = 0, -1, -2, \ldots$,
4. $$\Gamma(1+\epsilon) = \exp\left(-\epsilon\gamma + \sum_{n=2}^\infty \frac{(-1)^n}{n} \zeta_n \epsilon^n\right),$$
   where γ is the *Euler-Mascheroni constant* and ζ_n are the *Riemann ζ* values.

Using these properties, we can estimate

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon).$$

To calculate the original integral, we choose the typical parametrization of the dimension $d = 4 - 2\epsilon$. The problematic values of a are 1 and 2. So

$$\Gamma(a - d/2) = \Gamma(a - 2 + \epsilon) = \begin{cases} a = 1: & -\Gamma(\epsilon) = -\frac{1}{\epsilon} + \gamma - 1, \\ a = 2: & \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + 1. \end{cases}$$

Divergences are captured as simple $1/\epsilon$ poles in $d = 4 - 2\epsilon$ complex plane. The original integral A is defined only for $a < d/2$, but using analytic continuation, we can extend it to all a except $a = 1, 2$. In these cases we have:

$$\begin{aligned} A(a = 1, \Delta) &= \frac{i}{(4\pi)^2}(-\Delta)\frac{\Gamma(-1+\epsilon)}{\Gamma(1)}\left(\frac{\Delta}{2\pi\tilde{\mu}^2}\right)^{-\epsilon} \\ &= \frac{i}{(4\pi)^2}\Delta(1/\epsilon - \gamma + 1)\left[1 - \epsilon\ln\left(\frac{\Delta}{4\pi\tilde{\mu}^2}\right) + \dots\right] \\ &= \frac{i}{(4\pi)^2}\Delta\left(\frac{1}{\varepsilon} - \ln\left(\frac{\Delta}{e^{-\gamma}4\pi\tilde{\mu}^2}\right) + 1 - \gamma + \mathcal{O}(\epsilon)\right) \\ &= \frac{i}{(4\pi)^2}\Delta\left(\frac{1}{\varepsilon} - \ln\left(\frac{\Delta}{\mu^2}\right) + 1 - \gamma + \mathcal{O}(\epsilon)\right). \end{aligned}$$

We redefined $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$. For $a = 2$ we get

$$A(2, \Delta) = \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} - \ln\left(\frac{\Delta}{\mu^2}\right) + O(\epsilon)\right).$$

If we compare this result to the cut-off regularization, we see that the in dimensional regularization we didn't get any quadratic divergences.

8.4.1 Feynman parametrization

For calculation of more than one loop integrals we use

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x\, \frac{1}{(xA + (1-x)B)^2}$$

and a bit more generally

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 x\, \frac{x^{a-1}\overline{x}^{b-1}}{(xA'\overline{x}B)^{a+b}}$$

and even more generally

$$\frac{1}{P_1^{a_1}\dots P_n^{a_n}} = \frac{\Gamma(a_1 + \dots a_n)}{\Gamma(a_1)\dots\Gamma(a_n)}\int_0^1 \mathrm{d}x_1 \dots \mathrm{d}x_n\, \delta(1 - x_1 - \dots - x_n)\frac{x^{a_1-1}\dots x^{a_n-1}}{(a_1P_1 + \dots a_nP_n)^{a_1+\dots+a_n}}.$$

These are called *Feynman parameters*. It is left as an exercise to show that

$$I = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}\frac{(p-k)p}{(p-k)^4} = \frac{i}{16\pi^2}.$$

8.5 Revisiting $\phi\phi \to \phi\phi$ scattering

We have

$$\langle k_1, k_2; \text{out}|p_1, p_2; \text{in}\rangle_{\text{connected}} = (2\pi)^4\delta^{(4)}(k_1 + k_2 - p_1 - p_2)\tilde{G}^{(4)}_{\text{amp}}(k_1, k_2, -p_1, -p_2)|_{\text{on-shell}}.$$

At one-loop order in ϕ^4 theory it holds $\sqrt{Z} = 1 + \mathcal{O}(\lambda^2)$. The amputated Green function consists of the tree-level diagram and three one-loop diagrams (s, t and u channel):

$$\tilde{G}^{(4)}_{\text{amp}} = -i\lambda_0 + (-i\lambda_0)^2\frac{1}{2}\left[\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - m_0^2 + i\epsilon}\frac{i}{(p_1 + p_2 - k)^2 - m_0^2 - i\epsilon} + (s \to t) + (s \to u)\right].$$

In dimensional regularization, the dimensions of the quantities are:

$$[\mathcal{L}] = d, \quad [\partial_\mu] = 1, \quad [m_0] = 1, \quad [\phi_0] = d/2 - 1, \quad [\lambda_0] = d - 4 = 2\epsilon.$$

Since the coupling λ_0 is no longer dimensionless, we substitute it for $\lambda_0\tilde{\mu}^{-2\epsilon}$, which has dimension 0. Let's now calculate the loop integral in dimensional regularization:

$$\begin{aligned}
&\tilde{\mu}^{4-d}\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k^2 - m_0^2 + i\varepsilon)((p+k)^2 - m_0^2 + i\varepsilon)} \\
=&\tilde{\mu}^{4-d}\int_0^1\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(xk^2 + \overline{x}(p+k)^2 - m_0^2)^2 + i\varepsilon} \\
=&\tilde{\mu}^{4-d}\int_0^1 \mathrm{d}x\int \frac{\mathrm{d}^dk'}{(2\pi)^d}\frac{1}{(k'^2 - [m_0^2 - x\overline{x}p^2 + i\varepsilon])^2} \\
=&\frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} - \int_0^\infty \mathrm{d}x\,\ln\left(\frac{m_0^2 - x\overline{x}p^2 + i\varepsilon}{\mu^2}\right) + \mathcal{O}(\epsilon)\right) \\
=&\frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} - \ln\left(\frac{m_0^2}{\mu^2}\right) - \int_0^1 \mathrm{d}x\,\ln\left(1 - x\overline{x}\frac{p^2}{m_0^2} - i\varepsilon\right) + \mathcal{O}(\epsilon)\right).
\end{aligned}$$

In the second step we used

$$k^2 + 2\overline{x}kp + \overline{x}p^2 = (k + \overline{x}p)^2 - x\overline{x}p^2$$

and changed integration variable to $k' = k + \overline{x}p$. We denote

$$A(p^2)\int_0^1 \mathrm{d}x\,\ln\left(1 - x\overline{x}\frac{p^2}{m_0^2} - i\varepsilon\right).$$

So

$$\tilde{G}^{\text{amp}} = -i\lambda_0\left\{1 - \frac{\lambda_0\tilde{\mu}^{-2\epsilon}}{32\pi^2}\left[\frac{3}{\epsilon} - 3\ln\left(\frac{m_0^2}{\mu^2}\right) - A(s) - A(t) - A(u)\right] + \mathcal{O}(\lambda_0^2)\right\}.$$

To find the renormalized coupling, choose $s = 4m^2$ and $t = u = 0$. We get

$$\sqrt{Z}^4\tilde{G}^{(4)}_{\text{amp}} = -i\lambda_{\text{R}}\tilde{\mu}^{2\epsilon}$$

and

$$\lambda_{\text{R}}\tilde{\mu}^{2\epsilon} = \lambda_0\left\{1 - \frac{\lambda_0}{32\pi^2}\tilde{\mu}^{2\varepsilon}\left(\frac{3}{\epsilon} - 3\ln\left(\frac{m_0}{\mu^2} - A(4m^2)\right)\right) + \mathcal{O}(\lambda_0^2)\right\}.$$

The scattering amplitude can be expressed as

$$\sqrt{Z}^4 \tilde{G}^{(4)}_{\text{amp}} = -i\lambda_{\text{R}}\tilde{\mu}^{2\epsilon}\left[1 - \frac{\lambda_{\text{R}}}{32\pi^2}\left(A(s) + A(t) + A(u) - A(4m^2)\right) + \mathcal{O}(\lambda_{\text{R}}^2)\right].$$

The choice of λ_{R} is not unique. We could also have chosen $s = t = u = -4v^2$:

$$\sqrt{Z}^4 \tilde{G}^{(4)}_{\text{amp}} = -i\lambda'_{\text{R}}\tilde{\mu}^{2\epsilon}.$$

Analogously as before, we can write

$$\sqrt{Z}^4 \tilde{G}^{(4)}_{\text{amp}} = -i\lambda'_{\text{R}}\tilde{\mu}^{2\epsilon}\left[1 - \frac{\lambda'_{\text{R}}}{32\pi^2}\left(A(s) + A(t) + A(u) - 3A(4v^2)\right) + \mathcal{O}(\lambda_{\text{R}}^2)\right].$$

Since

$$\lambda'_{\text{R}} = \lambda_{\text{R}}\left[1 + \frac{\lambda_{\text{R}}}{32\pi^2}\left(3A(-4v^2) - A(4m^2)\right) + \mathcal{O}(\lambda_{\text{R}}^2)\right],$$

the numerical prediction for G_{amp} is the same for both choices of s, t, u. This procedure is called *fixed term perturbation theory*. But it becomes very messy really fast. So we introduce the *renormalized perturbation theory*:

$$\phi_0 = \sqrt{Z}\phi_{\text{R}}, \quad m_0^2 = m_{\text{R}}^2 + \delta m^2, \quad \lambda_0 = Z_\lambda \lambda_{\text{R}}\tilde{\mu}^{2\epsilon}.$$

Lagrangian is

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left(\partial_\mu\phi_0\partial_0^\mu - m_0^2\phi_0^2\right) - \frac{\lambda_0}{4!}\phi_0^4 \\
&= \underbrace{\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) - \frac{\lambda}{4!}\phi^4}_{\mathcal{L}_{\text{R}}} + \\
&\quad + \underbrace{\frac{1}{2}(Z-1)\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\left((Z-1)m^2 + Z\delta m^2\right)\phi^2 - (Z_\lambda Z^2 - 1)\frac{\lambda\tilde{\mu}^{2\epsilon}}{4!}\phi^4}_{\mathcal{L}_{\text{CT}}} \\
&= \mathcal{L}_{\text{R}} + \mathcal{L}_{\text{CT}}.
\end{aligned}$$

For clarity, we dropped all subscripts R (so $\phi_{\text{R}} \to \phi$). Next we compute the Green functions, which are already renormalized:

$$\frac{i}{p^2 - m^2 + i\varepsilon}.$$

Note that the renormalized mass is included in the propagator. The counter-terms

$$\delta_Z = Z - 1, \quad \delta_m = (Z-1)m^2 + Z\delta m^2, \quad \delta_\lambda = Z_\lambda Z^2 - 1$$

have to be determined order by order in λ. The *renormalization scheme* corresponds to is the choice of the renormalization conditions. In this case we have the exact renormalized two-point function is

$$\frac{1}{p^2 - m^2 - \Pi(p^2, m^2) + i\varepsilon} \xrightarrow{p^2 \to m^2} \frac{i}{p^2 - m^2 + i\varepsilon}$$

in all orders of λ. That means that

$$\Pi(p^2 = m^2, m^2) = 0, \quad \left.\frac{\partial\Pi}{\partial p}\right|_{p^2=m^2} = 0.$$

The amputated four-point Green function is $-i\lambda\tilde{\mu}^{2\epsilon}$ in all orders of λ. The Feynman rule for the counter-term is

$$-i(\delta_Z p^2 - \delta_m)$$

so

$$-i\Pi(p^2, m^2) = -\frac{i\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) + \mathcal{O}\left(\frac{m^4}{\Lambda^2}\right)\right] - i(\delta_Z p^2 + \delta_m)$$

and setting $p^2 = m^2$ gives

$$\delta_Z p^2 + \delta_m = -\lambda\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) + \mathcal{O}\left(\frac{m^4}{\Lambda^2}\right)\right].$$

By comparing the coefficients we read (in order $\mathcal{O}(\lambda)$)

$$\delta_Z = 0, \quad \delta_m = \delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) + \mathcal{O}\left(\frac{m^4}{\Lambda^2}\right)\right].$$

8.6 Renormalization schemes

So far we have defined m and Z as the location and the residue of the pole of the exact two-point function. Let's call these m_{phys} and Z_{OS} (on-shell). But in other renormalization schemes we can choose different conditions. We write

$$\phi_0 = \sqrt{Z_{\text{OS}}}\phi_{\text{OS}} = \sqrt{Z}\phi,$$

where Z and ϕ are in some other (generic) renormalization scheme. For the mass we have

$$m_0^2 = m_{\text{phys}}^2 + \delta m_{\text{phys}}^2 = m^2 + \delta m^2.$$

The masses and field renormalization can be related by

$$m_{\text{phys}} = m_{\text{phys}}(m, \lambda), \quad Z_{\text{OS}} = Z_{\text{OS}}(m, \lambda).$$

One of the most popular schemes is the $\overline{\text{MS}}$-*scheme* (minimal subtraction), where we choose δ_m and δ_λ so that they only include the divergences $1/\epsilon$ and higher powers. We calculate it order by order:

$$-i\Pi(p^2, m^2) = -\frac{i\lambda}{32\pi^2}(-m^2)\left[\frac{1}{\epsilon} - \ln\left(\frac{m^2}{\mu^2}\right) + 1\right] - i(\delta_Z p^2 + \delta_m).$$

In the on-shell scheme we had

$$\delta_m^{\text{OS}} = \frac{\lambda}{32\pi^2} m^2 \left[\frac{1}{\varepsilon - \ln\left(\frac{m^2}{\mu^2}\right)} + 1\right]$$

but for the $\overline{\text{MS}}$-scheme we have

$$\delta_m^{\overline{\text{MS}}} = \frac{\lambda}{32\pi^2} m^2 \frac{1}{\epsilon}.$$

We can check that the bare mass is the same in both schemes:

$$\begin{aligned} m_0^2 &= m_{\text{phys}}^2 - \delta m_{\text{phys}}^2 \\ &= m_{\text{phys}}^2\left[1 + \frac{\lambda}{32\pi^2}\left(\frac{1}{\varepsilon} - \ln\left(\frac{m_{\text{phys}}^2}{\mu^2}\right) + 1\right) + \mathcal{O}(\lambda^2)\right] \\ &= m_{\overline{\text{MS}}}^2(\mu) + \delta m_{\overline{\text{MS}}}^2(\mu) \\ &= m_{\overline{\text{MS}}}^2(mu)\left[1 + \frac{\lambda}{32\pi^2}\frac{1}{\epsilon} + \mathcal{O}(\lambda^2)\right] \end{aligned}$$

The relation between the two masses m_{phys} and $m_{\overline{\text{MS}}}$ is

$$m^2_{\text{phys}} = m^2_{\overline{\text{MS}}}(\mu)\left\{1 + \frac{\lambda}{32\pi^2}\left[\ln\left(\frac{m^2_{\overline{\text{MS}}}(\mu)}{\mu^2}\right) + 1\right] + \mathcal{O}(\lambda^2)\right\}.$$

The coupling constant is also renormalized in the $\overline{\text{MS}}$-scheme:

$$-i\lambda\mu^{2\epsilon}\left\{1 - \frac{\lambda}{32\pi^2}\left[\frac{3}{\epsilon} - 3\ln\left(\frac{m^2}{\mu^2}\right) - A(s) - A(t) - A(u)\right] + \delta_\lambda\right\}.$$

In the OS scheme we had

$$\delta^{\text{OS}}_\lambda = \frac{\lambda}{32\pi^2}\left[\frac{3}{\lambda} - 3\ln\left(\frac{m^2}{\mu^2}\right) - A(4m^2)\right],$$

but in the $\overline{\text{MS}}$-scheme we have

$$\delta^{\overline{\text{MS}}}_\lambda = \frac{\lambda}{32\pi^2}\frac{3}{\varepsilon}.$$

We can relate

$$\lambda = \left(\frac{\sqrt{Z_{\overline{\text{MS}}}}}{\sqrt{Z}}\right)^4 \left(\frac{Z_{\lambda,\overline{\text{MS}}}}{Z_\lambda}\right)\lambda_{\overline{\text{MS}}}(\mu) = (1 + \delta_{\lambda\overline{\text{MS}}} - \delta_{\lambda,\overline{\text{MS}}} + \mathcal{O}(\lambda^2))\lambda_{\overline{\text{MS}}}(\mu).$$

Hence, in the $\overline{\text{MS}}$-scheme, all couplings and masses depend on the scale μ!

8.7 The systematics of renormalization

The main question is: can we make all Green functions at any loop order finite by adding only counter-terms of the form of the original Lagrangian. Answering this question is not an easy task. To help us on the way to the answer, we introduce the (superficial) *degree of divergence* for a 1PI diagram γ:

$$D(\gamma) = \max_{\text{all terms}}\left(\begin{smallmatrix}\text{powers of loop}\\ \text{momenta in the numerator}\\ \text{including integration measure}\end{smallmatrix} - \begin{smallmatrix}\text{powers of loop}\\ \text{momenta in}\\ \text{the denominator}\end{smallmatrix}\right).$$

Example 11.

$$\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{1}{k_1^2}\frac{1}{(k_1-k_2)^2}\left(-2\eta_{\mu\nu} + \frac{k_{1\mu}k_{2\nu}}{m^2}\right)\frac{1}{((k_1+p)^2 - M^2)^2}.$$

Taking the first term we have $D_1(\gamma) = 4 + 4 - 2 - 2 + 0 - 4 = 0$ and if we take the second term in the parenthesis we get $D_2(\gamma) = 4 + 4 - 2 - 2 + 2 - 4 = 2$. The degree of divergence is then $D(\gamma) = \max\{0, 2\} = 2$.

If the superficial degree of divergence is less than 0, we call the diagram *convergent*. If $D(\gamma) = 0$, we call it *logarithmically divergent*. However, the superficial degree is just the upper bound for divergence, so even if it is positive the diagram may converge. Our propagators are of the form

$$\frac{1}{(k^2)^{1-s_f}}, \qquad s_f \geq 0,$$

where f is the field type and $k \gg m^2$. For scalar fields $s_f = 0$. For 1PI diagrams we have

$$D(\gamma) = \sum_f I_f(2s_f - 2) + \sum_i v_i a_i + dL,$$

where

- I_f is the number of internal lines,
- v_i is the number of vertices of type i,
- a_i is the number of derivatives on fields in vertex of type i,
- d is the number of space-time dimensions,
- L is the number of loops,
- E_f will be the number of external lines,
- n_{if} will be the number of fields of type f in vertex of type i.

Previously, we calculated

$$L = \sum_f I_f - \sum_i v_i.$$

If we include also the external lines E_f, we get

$$E_f + 2I_f = \sum_i v_i n_{if}.$$

Combining all these equations, we get

$$D(\gamma) = d - \sum_f E_f(d/2 - 1 + s_f) + \sum_i v_i \Delta_i$$

with

$$\Delta_i = d - a_i - \sum_f n_{if}(d/2 - 1 + s_f).$$

The value of Δ_i is characteristic for each vertex type i.

Case $\Delta_i \geq 0$: the estimate of the degree of divergence is

$$D(\gamma) \leq 4 - \sum_f E_f(1 + s_f)$$

so the degree of divergence decreases by increasing the number of external lines. This means that only a finite number of n-point functions will be divergent.

Example 12. Consider

$$\mathcal{L} = -\frac{1}{2}\phi_0(\Box + m^2)\phi_0 - \frac{\lambda_3}{3!}\phi_0^3 - \frac{\lambda_4}{4!}\phi_0^4.$$

It is easy to check that $[\lambda_3] = 1$ and $[\lambda_4] = 0$ and in general that Δ_i is the mass dimension of the coupling constant. Calculating δm^2 and Z would require the Green function with 2 external legs with the maximal $D(\gamma)$ of 2. For Z_{λ_3} the Green function would have 3 external legs and

$D(\gamma) = 1$. For Z_{λ_4} the Green function would have 4 external legs and $D(\gamma) = 0$. This suggests that the divergences can be removed by counter-terms corresponding to the field, mass and the couplings.

Note that we do not always need all counter-terms. If, for example, the Lagrangian is symmetric under $\phi \to -\phi$, the odd-point functions will be zero. It might also happen, that we need more terms that the ones that appear in the Lagrangian. For example

$$\mathcal{L} = -\frac{1}{2}\sum_{i=1}^{2}\phi_i(\Box + m^2)\phi_i - \frac{\lambda}{4}\phi_1^2\phi_2^2.$$

To cancel the divergence of the loop diagram $\phi_1\phi_1 \to \phi_2\phi_2 \to \phi_1\phi_1$ we would need the counter-terms of the form ϕ_1^4 and ϕ_2^4. But that means that we needed those terms in the Lagrangian from the beginning (they are not forbidden by symmetry).

QFT theory can be rendered finite by a reparametrization of a finite number of parameters in the Lagrangian if:
1. $\Delta_i \geq 0$ for all vertices and
2. $\mathcal{L}$ contains all vertices compatible with the symmetries of the (regularized) theory.

Such theory is called *renormalizable*. If there exists an interaction with $\Delta_i < 0$, the theory is *non-renormalizable*. If for all i $\Delta_i > 0$, the theory is called *super-renormalizable*. In 4 dimensions, the only super-renormalizable theory is the ϕ^3-theory.

8.8 Effective field theories

Consider a very general action

$$S = \int \mathrm{d}^4x \left(\frac{1}{2}\partial^\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda_3\phi^3 - \lambda_4\phi^4 - \lambda_5\phi^5 - \dots\right).$$

Consider QFT at low energies (long distances, $l \to \infty$):

$$x^\mu \to lx^\mu, \quad \mathrm{d}^4x \to \mathrm{d}^4x\, l^4.$$

The kinetic term becomes

$$\frac{1}{2}\int \mathrm{d}^4x\ \partial_\mu\phi\partial^\mu\phi \to \frac{1}{2}\int \mathrm{d}^4x\ l^2\partial_\mu\phi\partial^\mu\phi.$$

We will keep canonical kinetic terms to keep free-field as a reference point. So $\phi'(x) = l^{-1}\phi(lx)$. The action is

$$\int \mathrm{d}^4x \left(\frac{1}{2}\partial^\mu\phi\partial^\mu\phi - \underbrace{\frac{1}{2}m^2l^2\phi^2 - \lambda_3 l\phi^3}_{\text{relevant terms}} - \underbrace{\lambda_4\phi^4}_{\text{marginal interaction}} - \underbrace{\lambda_5 l^{-1}\phi^5 - \dots}_{\text{irrelevant operator}}\right).$$

The *relevant terms* will grow in importance at low energies, the *marginal terms* will stay the same and the *irrelevant terms* will become negligible. Since $[\lambda_5] = -1$, define $\tilde{\lambda}_5 = M\lambda_5$, where $[M] = 1$. Scaling the theory by l, the ϕ^5 term becomes

$$-\tilde{\lambda}\frac{l^{-1}}{M}\phi^5.$$

If $M \gg 1/l$ then the *effective coupling* $\lambda_5^{\text{eff}} = \lambda^5/(lM) \ll 1$. Historically, the renormalizability was imposed in order to preserve predictability. Modern viewpoint is that there is a hierarchy of scales $E_{\text{exp}} \sim 1/l \ll M$. Irrelevant terms are suppressed by powers of factor $lM \ll 1$.

Chapter 9

The renormalization group

The goal is to calculate the dependence of the effective coupling on the energy scale and a more general discussion of the beta function and scale behaviour of n-point function.

Let's consider the ϕ^4 theory and $\phi\phi \to \phi\phi$ scattering. The one-loop matrix element is

$$T(s) = -\lambda_{\mathrm{R}} - 3\frac{\lambda_{\mathrm{R}}^2}{32\pi^2}\ln\left(\frac{s}{s_0}\right) + \dots$$

In the high energy limit ($s \to \infty$) the perturbative expansion breaks down.

9.1 The coupling of scale-dependence

Consider a bare Lagrange with interacting part

$$\mathcal{L}^{\mathrm{int}} - \frac{\lambda}{4!}\phi_0^4 = -Z_\lambda^{\overline{\mathrm{MS}}}\tilde{\mu}^{2\epsilon}\lambda_{\mathrm{R}}(\mu)\frac{\phi_{\mathrm{R}}^4}{4!}$$

where $Z_\lambda^{\overline{\mathrm{MS}}} \equiv Z_\lambda Z^2$, so

$$\lambda_{\mathrm{R}}(\mu) = \tilde{\mu}^{-2\epsilon}(Z_\lambda^{\overline{\mathrm{MS}}})^{-1}\lambda_0.$$

Since λ_0 does not depend on the energy scale, we have

$$\frac{\mathrm{d}\lambda_0}{\mathrm{d}\mu^2}$$

and hence

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\lambda(\mu) = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\left(\tilde{\mu}^{-2}(Z_\lambda^{\overline{\mathrm{MS}}})^{-1}\right)\lambda_0 = -\epsilon\tilde{\mu}^{-2\epsilon}(Z_\lambda^{\overline{\mathrm{MS}}})^{-1} - \tilde{\mu}^{-2\epsilon}\frac{1}{(Z_\lambda^{\overline{\mathrm{MS}}})}^2\left(\mu^2\frac{\mathrm{d}Z_\lambda^{\overline{\mathrm{MS}}}}{\mathrm{d}\mu^2}\right)\lambda_0.$$

Inserting the expression for $\lambda_{\mathrm{R}}(\mu)$ we get

$$\boxed{\mu^2\frac{\mathrm{d}\lambda(\mu)}{\mathrm{d}\mu^2} = -\epsilon\lambda(\mu) - \frac{1}{(Z_\lambda^{\overline{\mathrm{MS}}})^2}\left(\mu^2\frac{\mathrm{d}Z_\lambda^{\overline{\mathrm{MS}}}}{\mathrm{d}\mu^2}\right)\lambda(\mu) \equiv \beta(\lambda(\mu)).}$$

This is called the *renormalization group equation* (RGE). In $\overline{\text{MS}}$-scheme, Z_i do not explicitly depend on μ, but in OS-scheme it does. This is why we use $\overline{\text{MS}}$-scheme. We also defined the *beta function* in the above expression. Separating variables, we can explicitly find the relation between $\lambda(\mu)$ and μ:

$$\ln\left(\frac{\mu_2^2}{\mu_1^2}\right) = \int_{\lambda(\mu_1)}^{\lambda(\mu_2)} \frac{\mathrm{d}\lambda'}{\beta(\lambda')}.$$

Let's calculate this dependence for the ϕ^4 theory. For convenience, we will write $Z_\lambda^{\overline{\text{MS}}}$ as Z_λ. We have

$$Z_\lambda = 1 + \frac{3\lambda}{32\pi^2}\frac{1}{\epsilon} + \dots,$$

so

$$\begin{aligned}
\beta(\lambda) = \frac{\mathrm{d}\lambda}{\mathrm{d}\ln\mu^2} &= -\epsilon\lambda(\mu) - Z_\lambda^{-1}\frac{\mathrm{d}Z_\lambda}{\mathrm{d}\ln\mu^2}\lambda(\mu) \\
&= -\epsilon\lambda(\mu) - \left(1 - \frac{1}{\epsilon}\frac{3\lambda}{32\pi^2} + \dots\right)\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\mu^2}\frac{3}{32\pi^2}\frac{1}{\epsilon}\lambda(\mu) \\
&= -\epsilon\lambda(\mu) - (-\epsilon\lambda(\mu))\frac{3}{32\pi^2}\frac{1}{\epsilon}\lambda(\mu) + \mathcal{O}(\lambda^3) \\
&= -\epsilon\lambda(\mu) + \frac{3}{32\pi^2}\lambda^2(\mu) + \mathcal{O}(\lambda^3).
\end{aligned}$$

As $\epsilon \to 0$ the expression is

$$\beta(\lambda) = \frac{3}{32\pi^2}\lambda^2 = \beta_0\lambda^2 + \mathcal{O}(\lambda)^3,$$

where we defined $\beta_0 = 3/(32\pi^2)$. Solving the renormalization group equation, we get

$$\ln\left(\frac{\mu_2^2}{\mu_1^2}\right) = \int_{\lambda(\mu_1)}^{\lambda(\mu_2)} \mathrm{d}\lambda' \left(\frac{1}{\beta_0\lambda^2} + \dots\right) = \frac{1}{\beta_0}\left(\lambda(\mu_1)^{-1} - \lambda(\mu_2)^{-1}\right) + \mathcal{O}(\ln(\mu))$$

and the solution is

$$\lambda(\mu_2) = \frac{\lambda(\mu_1)}{1 - \beta_0\lambda(\mu_1)\ln(\mu_2^2/\mu_1^2)}.$$

The same procedure can be used to calculate the scale dependence of the mass $m_0 = Z_m m(\mu)$. We get

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}m(\mu) = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}(Z_m^{-1})m_0 \equiv \gamma_m(\lambda)m(\mu),$$

where

$$\gamma_m(\lambda) \equiv \left(-\frac{1}{Z_m}\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}Z_m\right)$$

is the *gamma function*. Solving the differential equation gives

$$m(\mu_2) = m(\mu_1)\exp\left(\int_{\lambda(\mu_1)}^{\lambda(\mu_2)} \mathrm{d}\lambda\,\frac{\gamma_m(\lambda)}{\beta(\lambda)}\right).$$

This is only possible in the $\overline{\text{MS}}$-scheme, where Z_m does not depend on μ. The expression for Z_m can only include terms without dimension, since UV behaviour would just set them to zero. That is why the expression for Z_m doesn't include any ms.

9.2 General scale dependence and fixed points

A) Trivial IR fixed point (as in ϕ^4-theory):

$$\lambda(\mu_2) = \frac{\lambda(\mu_1)}{1 - \beta_0 \lambda(\mu_1) \ln(\mu_2^2/\mu_1^2)}.$$

This works for $\beta(\lambda) > 0$ and for small λ. Assuming $\beta_0 > 0$, this function has a pole (called *Landau pole*) at

$$\Lambda_{\mathrm{LP}} \approx \mu \exp\left(\frac{1}{2\beta_0 \lambda(\mu)}\right).$$

In perturbation, we require that for small changes in energy scale μ the value of $\lambda(\mu)$ does not change much. But close to Λ_{LP} we lose perturbative control. It is called *trivial IR fixed point* since the coupling and β-function go to zero as $\mu \to 0$, at which point it becomes a free theory. More generally, we define the *fixed point* as a point λ^* such that $\beta(\lambda^*) = 0$. It is called *trivial* if $\lambda^* = 0$. Using RGE we get certain terms for free – we get the automatic resummation of certain large terms. For $q^2 = s = -t = -u$ for the $\phi\phi \to \phi\phi$ scattering we get

$$iT_{\phi\phi\to\phi\phi} = -i\lambda(q)\,(1 + \dots) = -i\lambda_{\mathrm{R}} + i\frac{\lambda_{\mathrm{R}}^2}{32\pi^2} \ln(s/s_0)$$

after inserting the energy dependence.

B) *Trivial UV fixed point*: for $\beta_0 \leq 0$ we still have $\beta(\lambda) = \beta_0\lambda^2 + \dots$. The pole stands at point

$$\Lambda_{\mathrm{IR}} = \mu_{\mathrm{high}} \exp\left(-\frac{1}{2|\beta_0|\mu_{\mathrm{high}}}\right).$$

Such theory is called *asymptotically free* The natural emergence of a typical energy scale, here Λ_{IR}, from a theory which was formerly scale-invariant is called *dimensional transmutation*. One theory with this property is QCD: the measurement says that

$$\alpha_{\mathrm{s}}^{\overline{\mathrm{MS}}}(M_Z) \equiv \frac{g_{\mathrm{s}}^{\overline{\mathrm{MS}}}}{4\pi} \approx 0{,}118,$$

where M_Z is the mass of Z-bozon. For $\mu \to \infty$ the coupling is asymptotically free $\alpha_{\mathrm{s}}^{\overline{\mathrm{MS}}} \to 0$, but in the IR limit it explodes $\alpha_{\mathrm{s}}^{\overline{\mathrm{MS}}} \to \infty$ at $\mu \approx 200\,\mathrm{MeV}$. This is the reason why we cannot use perturbation theory for low energy QCD and why we have many bound states at low energies.

C) Non-trivial fixed points: for $\beta(\lambda_*) = 0$ at $\lambda_* \neq 0$. We have three options:

1) *UV-stable fixed point*: for $\lambda < \lambda_*$, $\beta(\lambda) > 0$ and λ increases with μ. For $\lambda > \lambda_*$, $\beta(\lambda) < 0$ and λ decreases with μ. The domain of attraction for $\mu \to \infty$ is independent on where we start. For $\mu \to 0$ the behaviour depends on whether we start at $\lambda < \lambda_*$ or $\lambda > \lambda_*$.

2) *IR-stable fixed point*: for $\lambda < \lambda_*$, $\beta(\lambda) < 0$ and λ decreases with μ. For $\lambda > \lambda_*$, $\beta(\lambda) > 0$ and λ increases with μ. The domain of attraction for $\mu \to 0$ is independent on where we start. For $\mu \to \infty$ the behaviour depends on whether we start at $\lambda < \lambda_*$ or $\lambda > \lambda_*$.

9.3 Renormalization group flow

Consider dimensionless couplings λ_i, $[\lambda_i] = \Delta_i = 0$. In the exercise we showed

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\lambda_i = \beta_i(\lambda_j) = \beta_i^{jk}\lambda_j\lambda'_k \ldots,$$

which describes a dynamical system with fixed points $\beta_i(\lambda_j^*) = 0$. We can expand the beta function around the fixed point:

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\lambda_i = \underbrace{\left.\frac{\partial\beta_i}{\partial\lambda_k}\right|_{\lambda_k=\lambda_k^*}}_{\equiv A_{ik}}(\lambda_k - \lambda_k^*).$$

When we diagonalize the matrix A_{ik}, we can separate the fixed points into attractive and repulsive. If all eigenvalues are positive, the fixed point is attractive, otherwise it is repulsive.

Let's study the RGE flow for relevant, marginal and irrelevant couplings. Consider

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2(\mu))\phi - \frac{\lambda_4(\mu)}{4!}\phi^4 - \frac{\lambda_6(\mu)}{6!}\phi^6 + \mathcal{L}_{\mathrm{CT}}.$$

Rewrite the dimensionless couplings:

$$\hat{\lambda}_2 = \frac{m^2}{\mu^2}, \quad \hat{\lambda}_4 = \lambda_4, \quad \hat{\lambda}_6 = \mu^2\lambda_6.$$

We have

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\hat{\lambda}_2 = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\left(\frac{m^2}{\mu^2}\right) = -\frac{m^2(\mu)}{\mu^2} + \frac{\mathrm{d}}{\mathrm{d}\mu^2}(m^2(\mu)) = -\hat{\lambda}_2 + \beta_{\hat{\lambda}_2}^{jk}\hat{\lambda}_j\hat{\lambda}_k.$$

For small $\hat{\lambda}_i \ll 1$, they are determined by $\lambda_i \approx (\mu^2)^{-\Delta_i/2}$. Assume we define our theory on some really large scale, that is well above any experimental results $\Lambda_i \gg E_{\mathrm{exp}}$. Then all couplings are of order $\hat{\lambda}_i \sim \mathcal{O}(1)$, which allows for perturbative expansion. In leading order we have

$$\frac{\mathrm{d}\hat{\lambda}_i}{\lambda_i} = \frac{\Delta_i}{2}\frac{\mathrm{d}\mu^2}{\mu^2},$$

which gives

$$\hat{\lambda}_i(\mu) = \hat{\lambda}_i(\Lambda)\left(\frac{\mu}{\Lambda}\right)^{-\Delta_i} + \text{corrections}.$$

Let's look at different types of couplings:

- Irrelevant (non-renormalizable, $\Delta_i < 0$) couplings become unimportant for $\mu \sim E \ll \Lambda$.
- Marginal (renormalizable, $\Delta_i = 0$) couplings are determined by perturbative calculations.
- Relevant (super-renormalizable, $\Delta_i > 0$) couplings grow for $\mu \ll \Lambda$. For example

  $$m^2(\mu) = \mu^2\hat{\lambda}_2(\mu) = \Lambda^2\hat{\lambda}_2(\Lambda) = \mathcal{O}(\Lambda^2).$$

  A mass term $m^2(\mu) \ll \Lambda^2$ requires fine-tuning $\hat{\lambda} \ll 1$.

9.4 RGE flow of Green functions

Collecting our results, we have

$$\begin{aligned}\beta_i(\lambda_j) &= \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\lambda_i \\ \gamma_m(\lambda_j) &= \frac{\mu^2}{m^2}\frac{\partial m}{\partial \mu^2} \quad \text{mass anomalous dimension} \\ \gamma_\phi(\lambda_j) &= \frac{\mu^2}{Z_\phi^2}\frac{\partial Z_\phi}{\partial \mu^2} \quad \text{filed anomalous dimension.}\end{aligned}$$

We will now derive the scale dependence for n-point function $\tilde{G}^{(n)}(p_i; m, \lambda)$ with mass dimension d_G expressed in terms of the renormalized couplings. We rescale the momenta $p_i \to a p_i$. Since $\tilde{G}^{(n)}$ depends on the scale μ, we should write

$$\tilde{G}^{(n)}(a p_i; m, \lambda, \mu) = a^{d_G}\tilde{G}^{(n)}(p; m/a, \lambda, \mu/a).$$

If a is too large, the Green function will include large quantum corrections. The bare Green function is independent of μ:

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\left(\sqrt{Z_\phi}^n \tilde{G}^{(n)}(p_i; m(\mu), \lambda(\mu), \mu)\right) = 0$$

so by the chain rule and after identifying some known functions we get *Callan-Symanzik-Gellman-Low equation*:

$$\left(\mu^2\frac{\partial}{\partial \mu^2} + \beta(\lambda)\frac{\partial}{\partial \lambda} + \gamma_m(\lambda), \frac{\partial}{\partial m} + \frac{n}{2}\gamma_\phi(\lambda)\right)\tilde{G}^{(n)} = 0.$$

We can simplify this equation by inserting solution to RGE for $\lambda(\mu)$ and $m(\mu)$ with initial conditions $\lambda(\mu_0) = \lambda$ and $m(\mu_0) = m$:

$$\left(\mu^2\frac{\partial}{\partial \mu^2} + \frac{n}{2}\gamma_\phi(\lambda(\mu))\right)F(\mu) = 0$$

and hence

$$F(\mu) = F(\mu_0)\exp\left(-\frac{n}{2}\int_{\mu_0^2}^{\mu^2}\frac{\mathrm{d}\mu'^2}{\mu'^2}\gamma_\phi(\lambda(\mu'))\right)$$

and

$$\tilde{G}^{(n)}(p_i; m(\mu), \lambda(\mu, \mu)) = \tilde{G}^{(n)}(p_i; m(\mu_0), \lambda(\mu_0), \mu_0)\exp\left(-\frac{n}{2}\int_{\mu_0^2}^{\mu^2}\frac{\mathrm{d}\mu'^2}{\mu'^2}\gamma_\phi(\lambda(\mu'))\right).$$

Next, we do the rescaling $p_i \to a p_i$:

$$a^{d_G}\tilde{G}^{(n)}(p_i; m(\mu)/a, \lambda(\mu), \mu/a) = \tilde{G}^{(n)}(p_i; m(\mu_0), \lambda(\mu_0), \mu_0)\exp\left(-\frac{n}{2}\int_{\mu_0^2}^{\mu^2}\frac{\mathrm{d}\mu'^2}{\mu'^2}\gamma_\phi(\lambda(\mu'))\right).$$

Choose $\mu = a\mu_0$ and renaming $\mu_0 \to \mu$ we get:

$$\tilde{G}^{(n)}(ap_i; m(\mu), \lambda(\mu), \mu) = a^{d_G} \exp\left(\int_1^a \frac{\mathrm{d}a'}{a'} n\gamma_\phi(\lambda(a'\mu))\right) \tilde{G}^{(n)}(p_i; m(a\mu)/a, \lambda(a\mu), \mu).$$

The exponential modifies the naive scaling a^{d_G} that we did before. Note that on both sides of the equation above, we have the same energy scale μ. If we have an amputated Green function

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}(\sqrt{Z_\phi}^{-n} \tilde{G}^{(n)}_{\text{amp}}) = 0$$

s o we have to substitute $n\gamma_\phi \to -n\gamma_\phi$.

Example 13. Let's study a two-point function $\tilde{G}^{(2)}(p^2; m(\mu), \lambda(\mu), \mu)$ with a mass dimension $d_G = -2$. We rescale the momenta $p = ap_0$ and we get

$$\tilde{G}^{(2)}(a^2p^2; m(\mu), \lambda(\mu), \mu) = \frac{p_0^2}{p^2} \exp\left(\int_1^a \frac{\mathrm{d}a'}{a'} 2\gamma_\phi(\lambda(a'\mu))\right) \tilde{G}^{(2)}(p^2; m(a\mu)/a, \lambda(a\mu), \mu).$$

Next, we distinguish two cases:

- *Small couplings*:

  $$\gamma_\phi(\lambda) = \gamma_0\lambda + \dots, \quad \beta(\lambda) = \beta_0\lambda^2 + \dots$$

  and so

  $$\begin{aligned}
  \exp\left(\int_1^a \frac{\mathrm{d}a'}{a'} 2\gamma_\phi(\lambda(a'\mu))\right) &= \exp\left(\int_{\mu^2}^{a\mu^2} \frac{\mathrm{d}\mu'^2}{\mu'^2} \gamma_\phi(\lambda(\mu'))\right) \\
  &= \exp\left(\int_{\lambda(\mu)}^{\lambda(a\mu)} \mathrm{d}\lambda' \frac{\gamma_\phi(\lambda')}{\beta(\lambda')}\right) \\
  &\approx \exp\left(\int_{\lambda(\mu)}^{\lambda(a\mu)} \mathrm{d}\lambda' \frac{\gamma_0}{\beta(\lambda')}\right) \\
  &= \exp\left[\frac{\gamma_0}{\beta_0} \ln\left(\frac{\lambda(a\mu)}{\lambda(\mu)}\right)\right] \\
  &= \left(\frac{\lambda(a\mu)}{\lambda(\mu)}\right)^{\gamma_0/\beta_0}.
  \end{aligned}$$

  Thus,

  $$\tilde{G}^{(2)}(p^2; m(\mu), \lambda(\mu)) \sim \frac{1}{p^2} \lambda(p)^{\gamma_0/\beta_0} (1 + \mathcal{O}(\lambda, m^2/p^2)).$$

- *Non-trivial fixed points*: we have $a \to \infty$ and $\lambda \to \lambda_*$. We can approximate

  $$\exp\left(\int_1^a \frac{\mathrm{d}a'}{a'} 2\gamma_\phi(\lambda(a'\mu))\right) = \exp\left(2\gamma_\phi(\lambda_*) \int_1^a \frac{\mathrm{d}a'}{a'}\right) = \left(\frac{p^2}{p_0^2}\right)^{\gamma_\phi(\lambda_*)}$$

  and

  $$\tilde{G}^{(2)}(p^2; m(\mu), \lambda(\mu)) \sim \frac{1}{(p^2)^{1-\gamma_\phi(\lambda_*)}} (1 + \mathcal{O}(\lambda, m^2/p^2)).$$

We can see that near fixed point the field dimension is changed. This means that some naively non-renormalizable theories can in fact be renormalizable. This is called *asymptotic safety*.

Chapter 10

Poincaré group, spin and relativistic particles

10.1 Symmetries and their representations

In this section we will denote the group with G, its element with g and the *group representation* with the mapping $g \mapsto U(g)$. The map U is called *unitary* if it preserves scalar product:

$$\langle\psi|\phi\rangle \mapsto \langle\psi|U^\dagger(g)U(g)|\phi\rangle = \langle\psi|\phi\rangle\,.$$

If G is a *Lie group*, then we can write g as

$$g = \exp(-i\epsilon^A t^A) \approx \mathbb{1} + i\epsilon^A t^A$$

where ϵ^A are the coordinates of group manifold and t^A are *group generators*. This induces representation

$$U(g) = \exp(-i\epsilon^A T^A)$$

where T^A are representations of generators t^A. The group generators satisfy the commutation relations

$$[T^A, T^B] = if^{ABC}T^C + iZ^{AB}\mathbb{1}.$$

Coefficients f^{ABC} are called *structure constants* and Z^{AB} is the *central charge*.

Example 14. Let's consider the group $SU(2)$ with the generators t^i that obey

$$[t^i, t^j] = i\epsilon^{ijk}t^k,$$

where ϵ^{ijk} is the Levi-Civita symbol. The group representation can be written as

$$U = \exp(-i\theta^i T^i).$$

The condition $U^\dagger U = 1$ gives that T^i are Hermitian, and the condition $\det(U) = 1$ gives that T^i are traceless. The matrices that obey that are exactly spanned by the Pauli matrices: $T^i = \sigma^i/2$.

10.2 Unitary representations of the Poincaré group

The Poincaré group is the group of translations and Lorentz transformations ISO(1, 3), i.e. the group of isometries of Minkowski space. States transform as $|\psi\rangle \mapsto U|\psi\rangle$ where U is the representation of the Poincaré group. In the basis $|\psi_i\rangle$ the transformation is $|\psi_i\rangle \mapsto U_{ij}|\psi_j\rangle$. If there is no subset of states $\{\psi_i\}$ which transform only among themselves under U_{ij}, we call such representation *irreducible*.

Particles transform under irreducible unitary representations of the Poincaré group.

If we want to represent 4-vectors, we can write them as

$$|\psi\rangle = c_0|V_0\rangle + c_1|V_1\rangle + c_2|V_2\rangle + c_3|V_3\rangle$$

with norm

$$\langle\psi|\psi\rangle = |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2.$$

If we start with $|\psi\rangle = |V_0\rangle$ with $\langle\psi|\psi\rangle = 1$. When we boost the state, we get

$$|\psi'\rangle = \cosh\beta|V_0\rangle + \sinh\beta|V_1\rangle.$$

The norm of the new state is $\langle\psi'|\psi'\rangle = \cosh^2\beta + \sinh^2\beta \neq 1$. So there is a conflict between unitarity and Lorentz invariance. We could modify our norm to be Lorentz invariant, but it would no longer be positive definite.

10.3 Poincaré group

The transformation of a vector x under the Poincaré group is

$$x' = \Lambda x + a.$$

If we transform vector twice we get

$$(x'')^\mu = (\Lambda_2)^\mu{}_\nu (x')^\nu + a_2^\mu = (\Lambda_2\Lambda_1)^\mu{}_\nu x^\nu + (\Lambda_2 a_1 + a_2)^\mu.$$

For $\Lambda = \mathbb{1}$ we have space-time translations and for $a = 0$ we have the Lorentz group. Lorentz transformations leave the scalar product unchanged:

$$x^\mu y_\mu = \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta$$

or

$$\eta_{\alpha\beta}\Lambda^\mu{}_\alpha \eta_{\mu\nu}\Lambda^\nu{}_\beta.$$

It follows that $\det(\lambda) = \pm 1$. We can separate the Lorentz group into those with positive determinant and those with negative determinant. We can further separate into those with the element $\Lambda^0{}_0 \geq 0$ and those with $\Lambda^0{}_0 < 0$. These form four connected components of the Lorentz group. The component of the Lorentz group that is connected to the identity is called *proper orthochronous Lorentz group* ($\mathcal{L}_+^\uparrow$). There are two operations that change the sign of the determinant and the sign of the time component of the Lorentz transformation: the *parity transformation* $\mathcal{P}$ and the *time reversal* $\mathcal{T}$:

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can write the whole Lorentz group as

$$\mathcal{L} = \mathcal{L}_+^\uparrow \otimes \mathcal{P} \otimes \mathcal{T}.$$

The multiplication in Poincaré group is

$$U(\Lambda_2, a_2)U(\lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2).$$

We have shown in the first homework that the Lorentz group can be infinitesimally written as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

where $\omega^{\mu\nu}$ is an antisymmetric tensor. For a given representation $U(\Lambda, a)$ we can write

$$U(\Lambda, a) \approx (\mathbb{1} + \omega, \epsilon) \approx \mathbb{1} - \frac{i}{2}\omega_{\mu_\nu} J^{\mu\nu} + i\epsilon_\mu P^\mu,$$

where $J^{\mu\nu} = -J^{\nu\mu}$ are six generators of the Lorentz group and P^μ are four generators of translations. We can find the commutation relations

$$\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= i(\eta^{\mu\sigma}J^{\nu\sigma} + \eta^{\nu^\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho}) \\
[P^\mu, J^{\nu\rho}] &= i(\eta^{\mu\nu}P^\rho - \eta^{\mu\rho}P^\nu) \\
[P^\mu, P^\nu] &= 0.
\end{aligned}$$

The generators of spatial rotations are

$$\mathbf{J} = (J^1, J^2, J^2) = (J^{23}, J^{31}, J^{12})$$

satisfying

$$[J^i, J^j] = i\epsilon^{ijk}J^k.$$

Generators of boosts are

$$\mathbf{K} = (K^1, K^2, K^3) = (J^{10}, J^{20}, J^{30}),$$

satisfying

$$[K^i, K^j] = -i\epsilon^{ijk}J^k, \quad \text{and} \quad [J^i, K^j] = i\epsilon^{ijk}K^k.$$

Momentum commutes in the following way:

$$[P^0, J^i] = [P^0, P^\mu] = 0, \quad [P^0, K^i] = -iP^i, \quad [P^i, J^j] = i\epsilon^{ijk}P^k, \quad [P^i, K^j] = -i\delta^{ij}P^0.$$

If one-particle state is transformed under the Poincaré group, we should again get a one-particle state. Therefore, a relativistic particle is defined by an irreducible representation of the Poincaré group. We characterize the irreducible unitary representations by invariant operators. Invariant operators are also Lorentz scalars (but not also the other way around). We construct them from

$$p^\mu, \quad J^{\mu\nu}, \quad W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma,$$

where W_μ is the *Pauli-Lubanski vector*. The two Poincaré invariants are

$$P^2 = P_\mu P^\mu, \quad W^2 = -W_\mu W^\mu,$$

which commute with all generators of the Poincaré group. Other possible combinations are

$$P^\mu W_\mu = 0, \quad P^\mu P^\nu J_{\mu\nu} = 0, \quad W^\mu W^\nu J_{\mu\nu} = 0.$$

There is also $J^{\mu\nu}J_{\mu\nu}$, which is not Poincaré invariant.

10.4 One-particle basis states

A relativistic particle is characterized by two basic invariant quantities: the eigenvalues of P^2 (mass) of W^2 (spin). To describe the one-particle states, we use the last three components of P^μ and the last component of W^μ: $\langle \mathbf{p}, s\rangle$, where $\mathbf{p}$ are eigenvalues of $\mathbf{P}$ and s is the eigenvalue of W^3. We don't need P^0 to characterize the particle due to the on-shell condition

$$(P^0)^2 = m^2 + \mathbf{p}^2.$$

The $\mathbf{p}$ spectrum is continuous, but we still don't know the spectrum of s and the eigenvalues of W^2. The eigenvalues of P^2

$$P^2 \left|\mathbf{p}, s\right\rangle = m^2 \left|\mathbf{p}, s\right\rangle$$

c can be $m^2 > 0$, $m^2 = 0$, $m^2 < 0$ (tachyons) or $m^2 = p = 0$ (vacuum). For massive particles we have $m^2 > 0$. We can always write $p^\mu = L^\mu{}_\nu k^\nu$ with $k^\mu = m(1,0,0,0)^T$, so we define

$$\left|\mathbf{p}, s\right\rangle = U(L(p)) \left|\mathbf{k}, s\right\rangle .$$

The interpretation of $\left|\mathbf{k}, s\right\rangle$ is the rest frame state:

$$W_0 = 0, \quad W_i = m^2 J^i \implies W^2 = m^2 \mathbf{J}^2.$$

We see that W^2 is related to angular momentum at $\mathbf{p} = 0$, which is the intrinsic spin. The label s is hence the eigenvalue of $W_3 \sim J^i$, which can take values

$$-j, -j+1, \ldots, j-1, j,$$

w where j is an integer or half-integer. The group SO(3) leaves $k^\mu = m(1,0,0,0)^T$ invariant. It is called the *little group* and its representations induce the representations of the full Poincaré group. The translations of the state $\left|\mathbf{p}, s\right\rangle$ are

$$U(\mathbb{1}, a) \left|\mathbf{p}, s\right\rangle = e^{ia^\mu P_\mu} \left|\mathbf{p}, s\right\rangle .$$

We can show that

$$U(\Lambda, 0) \left|\mathbf{p}, s\right\rangle = U(L(\Lambda p) L^{-1}(\Lambda p) \Lambda L(p), 0) \left|\mathbf{k}, s\right\rangle = U(L(\Lambda p) w(\Lambda, p), 0) \left|\mathbf{k}, 0\right\rangle ,$$

where w is pure rotation if Λ is pure rotation (the rotations are preserved in the relativistic case, which means we can use spherical harmonics and Clebsch-Gordan coefficients in the relativistic case). More generally $w = L^{-1}(\Lambda p)\Lambda L(p)$ can be thought of as firs boots to rest frame, then transformation by Λ and then again boost in the initial frame. The general Lorentz transformation of the state $\left|\mathbf{p}, s\right\rangle$ is

$$U(\Lambda, 0) \left|\mathbf{p}, s\right\rangle = \exp(-i\omega_{\mu\nu} J^{\mu\nu}/2) \left|\mathbf{p}, s\right\rangle = \sum_{s'=-j}^{j} D^{(j)}_{ss'}(w) \left|\Lambda\mathbf{p}, s'\right\rangle ,$$

where $D^{(j)}_{ss'}(w)$ is the *Wigner D-function* (see maths lecture). We still have to check, that we have a unitary representation. General transformation is

$$U(\Lambda, a) \left|\mathbf{p}, s\right\rangle = e^{ia^\mu (\Lambda p)_\mu} \sum_{s'=-j}^{j} D^{(j)}_{ss'}(w) \left|\Lambda\mathbf{p}, s'\right\rangle .$$

We have

$$\langle \mathbf{p}, s|U^\dagger(\Lambda,a)U(\Lambda,a)|\mathbf{p}', s'\rangle = \langle \Lambda\mathbf{p}, s|\Lambda\mathbf{p}', s'\rangle = \langle \mathbf{p}, s|\mathbf{p}', s'\rangle = (2\pi)^3 2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{ss'}.$$

This holds because

$$e^{ia^\mu(\Lambda p)_\mu}e^{-ia^\mu(\Lambda p)_\mu} = 1$$

and because Wigner functions are orthonormal

$$\sum_{s''} D^{(j)}_{s''s}(w)D^{(j)}_{s''s'}(w) = \delta_{ss'}.$$

Now, we will return to the massless case. There is no rest frame for such particles, but we can choose the direction of momentum to be $k^\mu = n(1,0,0,1)^T$, where n is some real number. We again use $p^\mu = L(p)^\mu{}_\nu k^\nu$ and with this it is possible to show that

$$U(\Lambda,0)\,|\mathbf{p},s\rangle = U(L(p),0)U(w,0)\,|\mathbf{k},s\rangle\,.$$

In this case, $w(p)$ leaves the reference frame momentum k^μ invariant. With $w^\mu{}_\nu k^\nu = k^\mu$ we replace w with Λ and expand Λ infinitesimally $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ and get $\omega^\mu{}_\nu k^\nu = 0$. We start with general antisymmetric matrix

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & -\alpha & -\beta & C \\ \alpha & 0 & \theta & D \\ \beta & -\theta & 0 & E \\ -C & -D & -E & 0 \end{pmatrix}.$$

The condition gives us $C = 0$, $D = -\alpha$ and $E = -\beta$, so

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & -\alpha & -\beta & 0 \\ \alpha & 0 & \theta & -\alpha \\ \beta & -\theta & 0 & -\beta \\ 0 & \alpha & \beta & 0 \end{pmatrix}.$$

Therefore, the representation of the Wigner rotation is

$$U(\mathbb{1}+\omega,0) = \mathbb{1} - i\left(\theta J^3 + \alpha(K^1+J^2) + \beta(K^2+J^1)\right).$$

If we denote $A = K^1 + J^2$ and $B = K^2 + J^1$. These parameters form the $m = 0$ *little group algebra*:

$$[A,B] = 0, \quad [J^3,A] = iB, \quad [J^3,B] = -iA.$$

We can interpret this algebra as the isometries of 2-dimensional Euclidean plane ISO(2), consisting of rotations (J^3) and translations (A and B). We found three eigenvalues for s, namely the eigenvalues (α,β,σ) of (A,B,J_3). Eigenvalues α and beta are continuous, but since we haven't observed a continuous spin yet, we set them to $\alpha = \beta = 0$. With $s = \sigma$, we get

$$\begin{aligned} J^3\,|\mathbf{p},\sigma\rangle &= \sigma\,|\mathbf{k},\sigma\rangle \\ U(w(\theta),0)\,|\mathbf{p},\sigma\rangle &= e^{-i\theta\sigma}\,|\mathbf{p},\sigma\rangle \\ U(\Lambda,0)\,|\mathbf{p},\sigma\rangle &= e^{-i\theta(\Lambda,p)\sigma}\,|\mathbf{p},\sigma\rangle \end{aligned}$$

For a 2π rotation, $U(\Lambda, 0) = \pm 1$, and hence σ can only be integer or half-integer. In this case, representation space is only one-dimensional. The eigenvalue σ is called the *helicity* of the particle. We can show that

$$W_\mu |\mathbf{k}, \sigma\rangle = \sigma k_\mu |\mathbf{k}, \sigma\rangle .$$

Unlike spin s, the helicity σ is Lorentz invariant.

Now that we found the unitary one-particle-state representations, we can construct the creation and annihilation operators:

$$U(\Lambda) a^\dagger(\mathbf{p}, s) U^{-1}(\Lambda) = \sum_{s'=-j}^{j} D^{(j)}_{ss'}(w(\Lambda, \mathbf{p})) a^\dagger(\Lambda \mathbf{p}, s').$$

Next, we will look at the *inversion* and *time reversal*. For $m^2 > 0$, the inversion is

$$U_\mathcal{P} |\mathbf{p}, s\rangle = \eta |\mathbf{p}, s\rangle ,$$

where $|\eta|^2 = 1$. It holds

$$\begin{aligned} \text{scalar}: &\quad \phi(x) \xrightarrow{\mathcal{P}} \phi(\mathcal{P}x) \\ \text{pseudo-scalar}: &\quad a(x) \xrightarrow{\mathcal{P}} -a(\mathcal{P}x) \end{aligned}$$

The time reversal is

$$U_\mathcal{T} |\mathbf{p}, s\rangle = (-1)^{j-s} |-\mathbf{p}, -s\rangle .$$

For $m^2 = 0$, the inversion is

$$U_\mathcal{P} |\mathbf{p}, \sigma\rangle = \eta e^{\mp i\pi\sigma} |\mathcal{P}\mathbf{p}, -\sigma\rangle$$

and the time inversion is

$$U_\mathcal{T} |\mathbf{p}, \sigma\rangle = \xi e^{\pm i\pi\sigma} |\mathcal{T}\mathbf{p}, \sigma\rangle ,$$

where $|\eta| = |\xi| = 1$. Note that operator $\mathcal{T}$ is anti-unitary:

$$\hat{U}_\mathcal{T} i \hat{U}_\mathcal{T}^{-1} = -i.$$

There is an important theorem, called the *CPT theorem*:

> Any local relativistic QFT is invariant under simultaneous $\mathcal{CPT}$ transformations.

In quantum mechanics, if the object $|\langle\alpha|\beta\rangle|$ is invariant under some transformation, then the transformation is either unitary (U) or anti-unitary ($\hat{U}$).

10.5 Internal symmetries

Let S be the full symmetry group of our QFT. Group S must contain Poincaré group $\mathrm{ISO}(1,3)$. Hypothetically we could have a tensor charge $Q_{\mu\nu}$ corresponding to the $U(1)$ internal symmetry. This would yield a conserved current $J^\mu_{\alpha\beta} = 0$, which would over-constrain the S-matrix (reducing the number of free parameters, for $2 \to 2$ scattering it would be less than two). There is an important theorem called *Coleman-Mandula theorem*, which says that conserved charges cannot

have Lorentz indices attached. So the only possibility for the full internal symmetry group is the direct product of the Poincaré group and the internal symmetry group G:

$$S = \mathrm{ISO}(1,3) \otimes G.$$

It also means that the corresponding algebra between the generators $A_{\mathrm{ISO}\,1,3}$ and B_G is trivial

$$[A_{\mathrm{ISO}(1,3)}, B_G] = 0.$$

The only exception is the case of supersymmetry. The particle state can be represented as $|\mathbf{p}, s; n\rangle$, where n is the set of eigenvalues of the maximal set of commuting operators. So far, there were only finite symmetry groups observed in the Nature so the action of the internal symmetry group G can be written with a matrix $D_{nn'}$:

$$U(g)\,|\mathbf{p}, s; n\rangle = \sum_{n'} D_{nn'}\,|\mathbf{p}, s; n'\rangle\,.$$

Example 15. Two complex scalar fields $\phi = (\phi_1, \phi_2)$ transform under SU(2) as

$$\phi_n \mapsto U(g)_{nn'}\phi_{n'} = \exp(i\theta^a\sigma^a)_{nn'}\phi_{n'}.$$

10.5.1 Antiparticles

Antiparticles are represented by the state $|\mathbf{p}, s; \overline{n}\rangle$, which means:

1. they have the same mas m as $|\mathbf{p}, s; n\rangle$,
2. they have the same spin j if $m^2 > 0$ and opposite helicity σ if $m^2 = 0$,
3. they transform under all internal symmetries in the same way as particles, but with the complex conjugate representation:

$$U(g)\,|\mathbf{p}, s; \overline{n}\rangle = \sum_{n'} D^*_{nn'}\,|\mathbf{p}, s; \overline{n}'\rangle\,.$$

With this we can define the *charge conjugation* operator $\mathcal{C}$ (if the antiparticle exists):

$$U_{\mathcal{C}}\,|\mathbf{p}, s; n\rangle = \xi_{\mathcal{C}}\,|\mathbf{p}, s; \overline{n}\rangle\,, \quad |\xi_{\mathcal{C}}| = 1,$$

$$U_{\mathcal{C}}\,|\text{particle}\rangle = |\text{anitparticle}\rangle\,.$$

10.6 Relativistic fields

We can write the field operator as $\phi_{\alpha,n}(x)$, where α is the Lorentz index, and n the internal symmetry indices. The Poincaré transformation of the field is

$$U(\Lambda, a)\phi_{\alpha,n}(x)U^{-1}(\Lambda, a) = \sum_{\alpha'} D_{\alpha\alpha'}(\Lambda^{-1})\phi_{\alpha',n}(\Lambda x + a)$$

and the internal symmetry transformation is

$$U(g)\phi_{\alpha,n}(x)U^{-1}(g) = \sum_{n'} D_{nn'}(g)\phi_{\alpha,n'}(x).$$

Local fields with definite Lorentz transformation properties allow us to construct Lorentz invariant theories that satisfy relativistic causality. We know that $D(\Lambda)$ is a representation of the Lorentz group, so it holds $D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2)$. We will require our representations to be finite-dimensional and irreducible. However, on one-particle states $D(\Lambda, a)$, the $D(\Lambda)$ need not be unitary. The Lorentz group is generated by

$$J^{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & -J^3 & J^2 \\ -K^2 & J^3 & 0 & -J^1 \\ -K^3 & -J^2 & J^1 & 0 \end{pmatrix}$$

and

$$\mathbf{J} = (J^{23}, J^{31}, J^{12}), \quad \mathbf{K} = (J^{10}, J^{20}, J^{30}).$$

Next we *complexify* the generators:

$$\mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}), \quad \mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}).$$

This is good, because $\mathbf{A}$ and $\mathbf{B}$ don't talk to each other. In fact,

$$[A^i, A^j] = i\varepsilon^{ijk}A^k, \quad [B^i, B^j] = i\varepsilon^{ijk}B^k, \quad [A^i, B^j] = 0.$$

Both $\mathbf{A}$ and $\mathbf{V}$ form the SU(2) algebra:

$$\mathrm{SO}(1,3) \xrightarrow{\text{complexification}} \mathrm{SU}(2)_A \times \mathrm{SU}(2)_B.$$

In other words

$$\text{Complexified Lie algebra of } \mathrm{SO}(1,3) \cong \text{Lie algebra of } \mathrm{SU}(2)_A \otimes \mathrm{SU}(2)_B.$$

Irreducible representations of Lorentz group can be relabelled as "spins" of $\mathbf{A}$ and $\mathbf{B}$ (A, B), where A and B are integers or half-integers. The dimension of the chosen representation is $(2A+1)(2B+1)$. It is easy to check that

$$\mathcal{P}\mathbf{A}\mathcal{P} = \mathbf{B}, \quad \mathcal{P}\mathbf{B}\mathcal{P} = \mathbf{A}.$$

Clearly $(A, B) \xrightarrow{\mathcal{P}} (B, A)$ and the only representation invariant under $\mathcal{P}$ is when $A = B$. The *trivial representation* describes a scalar field: $(A, B) = (0, 0)$, $D(\Lambda) = \mathbb{1}$ and

$$U(\Lambda, a)\phi(x)U^{-1}(\Lambda, a) = \phi(\Lambda x + a).$$

We start with the free field operators for a general Lorentz representation. The creation and annihilation operators are

$$\phi^{(-)}_{\alpha,n}(x) = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} v_{\alpha,n}(x, \mathbf{p}, s)a^\dagger(\mathbf{p}, s; n),$$

$$\phi^{(+)}_{\alpha,n}(x) = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} v_{\alpha,n}(x, \mathbf{p}, s)a(\mathbf{p}, s; n),$$

where $v_{\alpha,n}(x,\mathbf{p},s)$ are coefficients we still need to determine. The operator $\phi^{(\pm)}_{\alpha,n}$ transforms in the matrix representation of $\mathcal{L}^{\uparrow}_{+}$. Operators a and $a^\dagger$ inherit the transformation properties from the unitary representations of Poincaré group in one-particle states. Hence,

$$|\mathbf{p},s\rangle = a^\dagger(\mathbf{p},s)\,|\Omega\rangle\,.$$

In the case $m^2>0$, we have

$$U(\Lambda,b)a^\dagger(\mathbf{p},s)U^{-1}(\Lambda,b) = e^{ib^\mu(\Lambda p)_\mu}\sum_{s'} D^{(j)}_{ss'}(w(\Lambda,b))a^\dagger(\Lambda,p,s'),$$

$$U(\Lambda,b)a(\mathbf{p},s)U^{-1}(\Lambda,b) = e^{-ib^\mu(\Lambda p)_\mu}\sum_{s'} D^{(j)}_{ss'}(w(\Lambda,b))a(\Lambda,p,s').$$

In massless case, we have

$$U(\Lambda,b)a^\dagger(\mathbf{p},s)U^{-1}(\Lambda,b) = e^{ib^\mu(\Lambda p)_\mu}e^{-i\theta(\Lambda p)\sigma}a^\dagger(\Lambda,p,\sigma),$$

$$U(\Lambda,b)a(\mathbf{p},s)U^{-1}(\Lambda,b) = e^{-ib^\mu(\Lambda p)_\mu}e^{i\theta(\Lambda p)\sigma}a(\Lambda,p,\sigma).$$

Next, we want to find the x-dependence of $u_{\alpha,n}$ and $v_{\alpha,n}$. We have (dropping index n)

$$\begin{aligned}
\phi^{(+)}_\alpha(x+b) &= U(\mathbb{1},b)\phi^{(+)}_\alpha U^{-1}(\mathbb{1},b)\\
&= \sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}u_\alpha(x,\mathbf{p},s)U(\mathbb{1},b)a(\mathbf{p},s)U(\mathbb{1},b)\\
&= \sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}u_\alpha(x,\mathbf{p},s)e^{-ib^\mu p_\mu}a(\mathbf{p},s)\\
&= \sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}u_\alpha(x+b,\mathbf{p},s)a(\mathbf{p},s).
\end{aligned}$$

It must then hold

$$u_\alpha(x+p,\mathbf{p},s) = e^{-ibp}u_\alpha(x,\mathbf{p},s)$$

and

$$u_\alpha(x,\mathbf{p},s) = e^{-ipx}u_\alpha(0,\mathbf{p},s) \equiv e^{-ipx}u_\alpha(\mathbf{p},s).$$

Similarly,

$$v_\alpha(x,\mathbf{p},s) = e^{ipx}v_\alpha(\mathbf{p},s).$$

The creation and annihilation fields can be rewritten as

$$\phi^{(+)}_\alpha(x) \equiv \sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ipx}u_\alpha(\mathbf{p},s)a(\mathbf{p},s,n),$$

$$\phi^{(-)}_\alpha(x) \equiv \sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{ipx}u_\alpha(\mathbf{p},s)a^\dagger(\mathbf{p},s,n).$$

Next, we will find the Lorentz transformation properties of u and v. For massive case, we have

$$U(\Lambda,0)\phi^{(\pm)}_\alpha(x)U^{-1}(\Lambda,0) = \sum_{\alpha'} D_{\alpha\alpha'}(\Lambda^{-1})\phi^{(\,pm)}_{\alpha'}(\Lambda x).$$

On one side we have

$$U(\Lambda,0)\phi^{(\pm)}_\alpha(x)U^{-1}(\Lambda,0)=\sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ipx}u_\alpha(\mathbf{p},s)\sum_{s'}D^{(j)*}_{ss'}(w)a(\Lambda x,s').$$

On the other side we have

$$\begin{aligned}\sum_{\alpha'}D_{\alpha\alpha'}(\Lambda^{-1})\phi^{(\,pm)}_{\alpha'}(\Lambda x)&=\sum_{\alpha'}D_{\alpha\alpha'}(\Lambda^{-1})\sum_{s'}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\Lambda x}u_{\alpha'}(\mathbf{p},s')a(\mathbf{p},s')\\&=\sum_{\alpha'}D_{\alpha\alpha'}(\Lambda^{-1})\sum_{s'}\int\frac{\mathrm{d}^3\Lambda p}{(2\pi)^3}\frac{1}{E_{\Lambda p}}e^{-ip\Lambda x}u_{\alpha'}(\Lambda p,s')a(\Lambda p,s').\end{aligned}$$

Comparing the two sides, we get

$$\sum_s u_\alpha(\mathbf{p},s)D^{(j)*}_{ss'}(w)=\sum_{\alpha'}D_{\alpha\alpha'}(\Lambda^{-1})u_{\alpha'}(\Lambda p,s').$$

Here, $D_{ss'}$ is unitary little group transformation and $D_{\alpha\alpha}$ is non-unitary Lorentz group transformation. We can multiply from left by $D_{\beta\alpha}(\Lambda)$ and from right by $D^{(j)}_{s't}(w)$ to get

$$D_{\beta\alpha}(\Lambda)u_\alpha(\mathbf{p},s)\underbrace{D^{(j)*}_{ss'}(w)D^{(j)}_{s't}(w)}_{\delta_{st}}=\underbrace{D_{\alpha\beta}(\Lambda)D_{\alpha\alpha'}(\Lambda^{-1})}_{\delta_{\beta\alpha'}}u'(\Lambda p,s')D^{(j)}_{s't}(w).$$

W e can do the same procedure for v, and we get

$$\boxed{D_{\beta\alpha}(\Lambda)u_\alpha(\mathbf{p},s)=u_\beta(\Lambda p,s')D^{(j)}_{s's}(w(\Lambda,p))}$$

and

$$\boxed{D_{\beta\alpha}(\Lambda)v_\alpha(\mathbf{p},s)=v_\beta(\Lambda p,s')D^{(j)*}_{s's}(w(\Lambda,p)).}$$

These two equations relate the transformation of Lorentz transformation of index α to the Little group transformation of index s. The equations are implicitly summed over α and s'. For massless particles the result is much simpler – we just substitute $D^{(j)}_{s's}\to e^{-i\theta\sigma}$.

We will simplify the above result using the reference momentum $k^\mu=(m,0,0,0)^T$ for $m^2>0$ and $k=n(1,0,0,1)^T$ for $m^2=0$. Remember that $L(p)k=p$ and $L(k)=\mathbb{1}$. The Wigner rotation becomes (using $\Lambda=L(p)$)

$$w(\Lambda,k)=L^{-1}(\Lambda k)\Lambda L(k)=L^{-1}(L(p)k)L(p)\mathbb{1}=L^{-1}(p)L(p)\mathbb{1}=\mathbb{1}.$$

The Wigner function also simplifies to

$$D^{(j)}_{s's}(w(\Lambda,k))=D^{(j)}_{s's}(\mathbb{1})=\delta_{s's}.$$

For massless case, we have $e^{-i\theta(\Lambda,k)}=1$ and $\theta(\Lambda,k)=0$. The transformations of u and v now read

$$\boxed{\begin{aligned}u_\beta(\mathbf{p},s)&=\sum_\alpha D_{\beta\alpha}(L(p))u_\alpha(\mathbf{k},s),\\v_\beta(\mathbf{p},s)&=\sum_\alpha D_{\beta\alpha}(L(p))v_\alpha(\mathbf{k},s).\end{aligned}}$$

This means that we only need to find u and v for reference momentum, and then we can use the equations above to find u and v for any momentum.

Next, we will consider the transformations by the Little group (SO(3) or ISO(2)), which we will call R. We replace Λ with R and p with k and using $Rk = k$ we get

$$D_{\beta\alpha}(R)u_\alpha(\mathbf{k}, s) = u_\alpha(\mathbf{k}, s')D^{(j)}_{s's}(R),$$
$$D_{\beta\alpha}(R)v_\alpha(\mathbf{k}, s) = v_\alpha(\mathbf{k}, s')D^{(j)*}_{s's}(R).$$

Example 16. For scalar field, we have $D(\Lambda) = \mathbb{1}$, so

$$u_\alpha(\mathbf{k}, s) = \sum_{s'} u_\alpha(\mathbf{k}, s')D^{(j)}_{s's}(R)$$

for all R. We find that $j = 0$ or $\sigma = 0$, so this field represents particles with spin 0 (or helicity 0). The solution is simple $u_\alpha(\mathbf{k}, s) = v_\alpha(\mathbf{k}, s) = 1$ and so

$$\phi^{(+)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}e^{-ipx}a_\mathbf{p},$$
$$\phi^{(-)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}e^{ipx}a^\dagger_\mathbf{p}.$$

Relativistic causality requires creation and annihilation operators, so $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$.

For a complex scalar field we require an antiparticle. If the internal symmetry group is G, then

$$U(g)a(\mathbf{p}, s, n)U^{-1}(g) = \sum_{n'} D^*_{nn'}(g)a(p, s, n'),$$
$$U(g)a^\dagger(\mathbf{p}, s, n)U^{-1}(g) = \sum_{n'} D_{nn'}(g)a^\dagger(p, s, n').$$

This means that we cannot combine a and $a^\dagger$ into the same field as it would transform as $D^*(g)a + D(g)a^\dagger$. So the field must contain an antiparticle that transforms under complex conjugated representation:

$$U(g)b^\dagger(\mathbf{p}, s, n)U^{-1}(g) = \sum_{n'} D^*_{nn'}(g)b^\dagger(\mathbf{p}, s, n').$$

A field $\phi \sim a + b^\dagger$ transforms covariantly as $\phi' = D^*(g)a + D^*(g)b^\dagger = D^*(g)\phi$. There is one caveat, however. Sometimes the conjugate representation can be related to the original representation by a similarity transformation S: $D^*(g) = SD(g)S^{-1}$. If $S = S^T$ the representation is called *real* and if $S^T = -S$, the representation is called *pseudo-real*. In this case the introduction of antiparticles might not be necessary.

10.7 Classification of particles

We have seen that the group $\mathrm{SO}(1,3)$ can be written locally as $\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$, with the eigenvalues of

$$\mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}) \in \mathfrak{su}(2)_A \text{ and } \mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}) \in \mathfrak{su}(2)_B$$

being integers or half-integers:

$$A, B = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots .$$

The dimension of this representation is dim $= (2A + 1)(2B + 1)$. For any combination of eigenvalues of A and B we get a representation of the Lorentz group. For $m^2 > 0$ we have the following examples.

- $(A, B) = (0, 0)$ is the fundamental representation (element **1** of SO(3)) and represents a scalar field (dimension is 1).
- $(A, B) = (1/2, 0), (0, 1/2)$ are the Weyl spinor representation for left- and right-handed spinors (dimension 2, elements **2** of SO(3)).
- $(A, B) = (1/2, 1/2)$ is the representation of a vector field (dimension 4, element $\mathbf{1} \oplus \mathbf{3}$ of SO(3)).
- $(A, B) = (1/2, 0) \oplus (0, 1/2)$ is the Dirac spinor representation (dimension 4, elements $\mathbf{2} \oplus \mathbf{2}$ of SO(3)).

The maximal spin of the particle in the representation (A, B) is $J_{\max} = |\mathbf{A} + \mathbf{B}|$.

Chapter 11

Spin 1/2 particles and spinor fields

11.1 Spinor representations (1/2,0) and (0,1/2)

Let's start with representation $(1/2, 0)$. The generators are

$$\mathbf{A} = \frac{\boldsymbol{\sigma}}{2}, \quad \mathbf{B} = 0$$

where $\boldsymbol{\sigma}$ are *Pauli matrices*:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From the previous chapter we know the representation of $J^{\mu\nu}$:

$$J^{i0} = K^i - i(A^i - B^i), \quad J^{ij} = \varepsilon^{ijk}(A^k + B^k) = \frac{1}{2}\varepsilon^{ijk}\sigma^k.$$

Next, we extend the vector of Pauli matrices to a 4-vector and a tensor:

$$\sigma^\mu = (\mathbb{1}, \sigma^i)^T, \qquad \sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu),$$

$$\overline{\sigma}^\mu = (\mathbb{1}, -\sigma^i)^T, \qquad \overline{\sigma}^{\mu\nu} = \frac{i}{4}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu).$$

The representation of $J^{\mu\nu}$ is thus

$$\boxed{[J^{\mu\nu}]_{(1/2,0)} = \overline{\sigma}^{\mu\nu}.}$$

The operator

$$D_{(1/2,0)}(\Lambda) \equiv D_R(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}\right)$$

acts on a two-component object ψ_R called *right-handed Weyl spinor*.

The story is completely equivalent for the $(0, 1/2)$ representation. The generators are

$$\mathbf{A} = 0, \quad \mathbf{B} = \frac{\boldsymbol{\sigma}}{2}$$

and $J^{\mu\nu}$ is

$$[J^{\mu\nu}]_{(0,1/2)} \equiv \sigma^{\mu\nu}.$$

The finite transformations are

$$D_L(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)$$

which act on *left-handed Weyl spinors* ψ_L.

There are certain issues with these two representations. Namely, $D_L(\Lambda)$ and $D_R(\Lambda)$ are not unitary matrices! Well, this is somehow expected as Λ is not unitary. Even if $\mathbf{A}$ and $\mathbf{B}$ were hermitian, boosts are not. However, $D_{L/R}(\Lambda)$ are elements of $\mathrm{SL}(2,\mathbb{C})$. To check this, look at

$$\det\left[\exp\left(\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\right] = \exp\left(\frac{i}{2}\omega_{\mu\nu}\,\mathrm{tr}(\sigma^{\mu\nu})\right) = 1$$

as $\mathrm{tr}\,\sigma^{\mu\nu} = 0$. The group $\mathrm{SL}(2,\mathbb{C})$ is 6-dimensional. So we have constructed $\mathrm{SL}(2,\mathbb{C})$ spinors, which differ from $\mathrm{SU}(2)$ spinors from quantum mechanics. For the full spatial rotations we have $D_{L/R}(2\pi\text{-rotation}) = -1$. Hence, $\mathrm{SL}(2,\mathbb{C})$ is the universal covering group of $\mathrm{SO}(1,3)$ just as $\mathrm{SU}(2)$ is the universal covering group of $\mathrm{O}(3)$. Also, note that ψ_L and ψ_R are not equivalent, i.e. no matrix S exists, such that

$$D_{(0,1/2)}(\Lambda) = S D_{(1/2,0)}(\Lambda) S^{-1}, \qquad \forall\Lambda.$$

However, the following holds:

$$D_L^\dagger(\Lambda) = D_R^{-1}(\Lambda)$$

and

$$D_L(\Lambda) = \varepsilon D_R^*(\Lambda)\varepsilon^{-1}, \qquad \text{where } \varepsilon = -i\sigma^2.$$

This holds because

$$\begin{aligned}(\sigma^{\mu\nu})^\dagger &= \frac{i}{4}\left(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu\right)^\dagger = -\frac{i}{4}\left((\overline{\sigma}^\nu)^\dagger(\sigma^\mu)^\dagger - (\overline{\sigma}^\mu)^\dagger(\sigma^\nu)^\dagger\right)\\ &= -\frac{i}{4}\left(\overline{\sigma}^\nu\sigma\mu - \overline{\sigma}^\mu\sigma^\nu\right) = -\overline{\sigma}^{\nu\mu} = \overline{\sigma}^{\mu\nu}\end{aligned}$$

and hence

$$D_L^\dagger(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})^\dagger\right) = \exp(-\frac{i}{2}\omega_{\mu\nu}\overline{\sigma}^{\mu\nu})^{-1} = D_R^{-1}(\Lambda).$$

For the proof of the second identity, see exercises. An important consequence of this is that charge conjugated left-handed spinor $(\psi_L)^{\mathcal{C}} = \varepsilon\psi_L^*$ transforms as a right-handed spinor:

$$\varepsilon(D_L(\Lambda)\psi_L)^* = \varepsilon D_L^*(\Lambda)\varepsilon^{-1}\varepsilon\psi_L^* = D_R(\Lambda)(\varepsilon\psi_L^*).$$

Another important relation is that σ^μ transform as 4-vectors:

$$D_L^{-1}\sigma^\mu D_R(\Lambda) = \Lambda^\mu{}_\nu\sigma^\nu,$$

$$D_R^{-1}\sigma^\mu D_L(\Lambda) = \Lambda^\mu{}_\nu\overline{\sigma}^\nu.$$

11.2 Dirac spinors

Since QED is parity-invariant theory and ψ_L and ψ_R are not parity-invariant. Since $(1/2,0) \xrightarrow{\mathcal{P}} (0,1/2)$ we need both ψ_R and ψ_L to get irreducible representation of $\mathcal{L}_+^\uparrow \oplus \mathcal{P}$. Define

$$\begin{pmatrix} D_L(\Lambda) & 0 \\ 0 & D_R(\Lambda) \end{pmatrix} \equiv D_{(1/2,0)\oplus(0,1/2)}(\Lambda).$$

This is reducible representation for $\mathcal{L}_*^\uparrow$, but not for $\mathcal{L}_+^\uparrow \oplus \mathcal{P}$. Define

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \qquad \Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix},$$

such that

$$D_{(1/2,0)\oplus(0,1/2)}(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)$$

which acts on a four-dimensional object $\Psi = (\psi_L, \psi_R)^T$ called *Dirac spinor*. Matrices γ^μ are called *Dirac matrices* and the definition above is called *Weyl representation* of Dirac matrices or *chiral basis*. The Dirac matrices satisfy the *Clifford algebra*:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}.$$

Replacing the notation $D_{(1/2,0)\oplus(0,1/2)}$ with D_{Dirac} we can write

$$D_{\text{Dirac}}^{-1}(\Lambda)\gamma^\mu D_{\text{Dirac}}(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu.$$

Changing the basis for Dirac matrices can be done using unitary transformations U: $U^\dagger\gamma^\mu U$. One handy basis is called *Dirac basis*:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

11.3 Field operator for spinors

Let's first return to Weyl spinors. We have already shown, that it is enough to know the action of the little group transformation R on u_α and v_α in the reference momentum $\mathbf{k}$ (for u and v coefficient functions for creation and annihilation operators):

$$u_\alpha(\mathbf{k}, s')D^{(j)}_{s's}(R) = D_L(R)_{\alpha\alpha'}u_{\alpha'}(\mathbf{k}, s)$$

and similar for v_α. For any given j we sum over $2j+1$ values of s'. Since there are two indices for α, u is $2\cdot(2j+1)$ dimensional matrix. A general result from the representation theory (*Schur's lemma*) says that two irreducible representations D_1 and D_2 for which we can write

$$uD_1 = D_2 u$$

can only exist if D_1 and D_2 have the same dimensions. Hence, in our case we have $D_1 = D^{(j)}$ and $D_2 = D_L$, so $j = 1/2$. For pure rotation $R \in \mathrm{SO}(3)$ we can write

$$D^{(1/2)}_{ss'}(R) = \exp\left(-i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)_{s's}, \quad D_L(R)_{\alpha\alpha'} = \exp\left(-i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)_\alpha.$$

Infinitesimally, it holds

$$u\sigma^i = \sigma^i u, \quad v(-(\sigma^i)^*) = \sigma^i v$$

and hence

$$[u, \sigma^i] = 0, \quad [v\varepsilon^{-1}, \sigma^i] = 0.$$

We found that u and $v\varepsilon^{-1}$ commute with all σ^i. This means that u and $v\varepsilon^{-1}$ are proportional to the unit matrix (actually to the Casimir operator σ^2 which is proportional to the identity). Therefore,

$$u_\alpha(\mathbf{k}, s) = C_u \delta_{\alpha s}, \quad v_\alpha(\mathbf{k}, s) = -C_v \varepsilon_{\alpha s}.$$

In 2-spinor notation, we have

$$u(\mathbf{k}, 1/2) = \begin{pmatrix} C_u \\ 0 \end{pmatrix}, \quad u(\mathbf{k}, -1/2) = \begin{pmatrix} 0 \\ C_u \end{pmatrix}$$

and

$$v(\mathbf{k}, 1/2) = \begin{pmatrix} 0 \\ C_v \end{pmatrix}, \quad v(\mathbf{k}, -1/2) = \begin{pmatrix} -C_v \\ 0 \end{pmatrix}.$$

For the general $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ we boost the standard momentum (for $m^2 > 0$):

$$D_{L/R}(L(p)) = \frac{\sqrt{m^2 + \mathbf{p}^2} + m^2 \pm \boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{2m(\sqrt{m^2 + \mathbf{p}^2} + m)}}.$$

To derive this, start with particle at rest $k^\mu = (m, \mathbf{0})^T$ and move it to a general momentum $p^\mu = (p^0, \mathbf{p})^T$. For the boost in the x direction, we have

$$L(p) = \Lambda = \begin{pmatrix} \cosh(\omega) & -\sinh(\omega) & 0 & 0 \\ -\sinh(\omega) & \cosh(\omega) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We let it act on k^μ:

$$\Lambda \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh(\omega) \\ m\sinh(\omega) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p^0 \\ p^1 \\ 0 \\ 0 \end{pmatrix} = e^{i\omega_{01} K^1} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Denote

$$\cosh(\omega) = \frac{p^0}{m} = \gamma, \qquad \sinh(\omega) = \frac{|\mathbf{p}|}{m} = \sqrt{\gamma^2 - 1}.$$

Boosting in any direction $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$ we get

$$D_{R/L}(L(p)) \exp(i\omega \mathbf{n} \cdot \mathbf{K}).$$

Using the representation for $\mathbf{K} = \mp i\boldsymbol{\sigma}/2$ and the half angle identities for hyperbolic functions, we get

$$D_{R/L}(L(p)) = \cosh(\omega/2)\mathbb{1} \pm \mathbf{n}\cdot\boldsymbol{\sigma} \sinh(\omega/2) = \sqrt{\frac{p^0/m + 1}{2}}\mathbb{1} \pm \mathbf{n}\cdot\boldsymbol{\sigma} \frac{|\mathbf{p}|/m}{\sqrt{2p^0/m + 1}} = \frac{p^0 + m \pm \boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{2m(p^0 + m)}}.$$

11.4 Neutral spinor fields

We will define a more general commutation operator:

$$[\psi_\alpha(x), \psi_\beta(y)]_\mp = \psi_\alpha(x)\psi_\beta(y) \pm \psi_\beta(y)\psi_\alpha(x).$$

Writing a single field with creation and annihilation operators

$$\psi_\alpha(x) = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left(e^{-ipx}u_\alpha(\mathbf{p},s)a(\mathbf{p},s) + \lambda e^{ipx}v_\alpha(\mathbf{p},s)a^\dagger(\mathbf{p},s)\right)$$

the commutators are

$$[\psi_\alpha(x), \psi_\beta(y)]_\mp = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left(e^{-ip(x-y)}\lambda u_\alpha(\mathbf{p},s)v_\beta(\mathbf{p},s) \mp e^{ip(x-y)}\lambda v_\alpha(\mathbf{p},s)u_\beta(\mathbf{p},s)\right),$$

$$[\psi_\alpha(x), \psi^\dagger_\beta(y)]_\mp = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left(e^{-ip(x-y)}u_\alpha(\mathbf{p},s)u^*_\beta(\mathbf{p},s) \mp e^{ip(x-y)}|\lambda|^2 v_\alpha(\mathbf{p},s)v^*_\beta(\mathbf{p},s)\right),$$

where we wrongly assumed standard commutation relations for creation and annihilation operators. We can continue with the help of the following identities (in order to derive them one uses $(\boldsymbol{\sigma}\cdot\mathbf{p})^2 = \mathbf{p}^2$ and $D_L(L(p))D_L^\dagger(L(p)) = \sigma^\mu p_\mu/m$):

$$\begin{aligned}
\sum_s u_\alpha(\mathbf{p},s)u^*_\beta(\mathbf{p},s) &= \frac{|C_u|^2}{m}\sigma^\mu_{\alpha\beta}p_\mu,\\
\sum_s v_\alpha(\mathbf{p},s)v^*_\beta(\mathbf{p},s) &= \frac{|C_v|^2}{m}\sigma^\mu_{\alpha\beta}p_\mu,\\
\sum_s u_\alpha(\mathbf{p},s)v_\beta(\mathbf{p},s) &= -C_uC_v\varepsilon_{\alpha\beta},\\
\sum_s v_\alpha(\mathbf{p},s)u_\beta(\mathbf{p},s) &= C_uC_v\varepsilon_{\alpha\beta}.
\end{aligned}$$

Using these, we find

$$[\psi_\alpha(x), \psi_\beta(y)]_\mp = -C_uC_v\lambda\varepsilon_{\alpha\beta}\left(\Delta_+(x-y) \pm \Delta_+(y-x)\right),$$

$$[\psi_\alpha(x), \psi^\dagger_\beta(y)]_\mp = -\sigma^\mu_{\alpha\beta}\frac{|C_u|^2}{m}\left(\partial^{(x)}_\mu\Delta_+(x-y) \mp |\lambda|^2\left|\frac{C_v}{C_u}\right|^2\partial^{(y)}_\mu\Delta_+(y-x)\right),$$

with

$$\Delta_+(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}e^{-ipx}.$$

Recall from the discussion of scalar fields that for $x^2 < 0$ the right-hand side vanishes.

> The commutators $[\bullet,\bullet]_\mp$ vanish for space-like separation only if we choose the anti-commutation relations and $|\lambda|^2|C_v/C_u|^2 = 1$. Relativistic causality requires that spin 1/2 fields are fermionic.

This is a special case of *spin-statistics theorem*:

$$\begin{aligned} \text{integer spin} &= \text{bosons}, \\ \text{half-integer spin} &= \text{fermions}. \end{aligned}$$

Without loss of generality we can choose

$$\lambda = 1, \quad C_u = \sqrt{m}, \quad C_v = \sqrt{m}e^{i\delta}.$$

Such field satisfies the *Majorana equation*

$$i\overline{\sigma}^\mu \partial_\mu \psi - me^{i\delta}(\psi^T \varepsilon)^\dagger = 0.$$

In components this reads

$$i\overline{\sigma}^\mu_{\alpha\beta} \partial_\mu \psi_\beta - me^{i\delta}\varepsilon_{\alpha\beta}\psi^*_\beta = 0.$$

- For single *Majorana field* we can redefine $\psi \to e^{i\delta/2}\psi$ which sets $\delta = 0$.
- For a left-handed spinor field we must replace $\overline{\sigma}^\mu$ with σ^μ.
- We can show that Majorana equation implies the Klein-Gordon equation. First we remember two relations:

  $$\sigma^\mu \overline{\sigma}^\mu + \sigma^\mu \overline{\sigma}^\nu = 2\eta^{\mu\nu}, \quad \text{and} \quad (\overline{\sigma}^\mu)^* = -\varepsilon \sigma^\mu \varepsilon.$$

  We multiply the Majorana equation with $i\sigma^\mu \partial_\mu$ from the left:

  $$-\sigma^\nu_{\alpha\beta} \partial_\nu \overline{\sigma}^\mu \partial_\mu \psi + mi\sigma^\mu \partial_\mu (\varepsilon \psi^*) = 0.$$

  The first part can be rewritten as

  $$-\sigma^\nu \partial_\nu \overline{\sigma}^\mu \partial_\mu \psi = -\frac{1}{2}(\sigma^\nu \overline{\sigma}^\mu + \sigma^\mu \overline{\sigma}^\nu)\partial_\mu \partial_\nu \psi = -\Box \psi.$$

  Complex-conjugated Majorana can be simplified to

  $$i\varepsilon \sigma^\mu \varepsilon \partial_\mu \psi^* + m\varepsilon \psi = 0.$$

  Multiplying this with ε from the left and using $\varepsilon^2 = -\mathbb{1}_{2\times 2}$ we get

  $$i\sigma^\mu \partial_\mu (\varepsilon \psi^*) = -m\psi.$$

  Joining the two results we obtain

  $$(\Box + m^2)\psi = 0.$$

11.5 Field Lagrangian and equations of motion

The goal is to find a bilinear form of the field $\psi_\alpha(x)$ which is a Lorentz scalar in order to get the Lagrangian density. The term $\psi_L^\dagger \overline{\sigma}^\mu \psi_L$ transforms as Lorentz vector. Hence,

$$\psi_L^\dagger i\overline{\sigma}^\mu \partial_\mu \psi_L, \qquad \psi_R^\dagger i\sigma^\mu \partial_\mu \psi_R$$

are Lorentz scalars. Also, $\psi_L^T \varepsilon \psi_L$ and $\psi_R^T \varepsilon \psi_R$ are scalars (we can see this from the fact that $D_L^T(\Lambda)\varepsilon D_L(\Lambda) = \varepsilon$). The *Majorana Lagrangian* is

$$\mathcal{L} = \psi_L^\dagger i\overline{\sigma}^\mu \partial_\mu \psi_L - \frac{m}{2}\left(\psi_L^T \varepsilon \psi_L - \psi_L^\dagger \varepsilon \psi_L^*\right).$$

The dimension of the field is $[\psi_L] = 3/2$. It can easily be shown that the Lagrangian is real. The fields ψ_α must be anti-commuting, since $\psi_L^T \varepsilon \psi_L$ must be non-vanishing. The equations of motion are

$$\frac{\partial \mathcal{L}}{\partial \psi_{L,\alpha}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_{L,\alpha})} = 0.$$

It is easier to use $\psi_L^\dagger$ since

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_{L,\alpha}^\dagger)} = 0.$$

We have

$$\frac{\partial \mathcal{L}}{\partial \psi_{L,\alpha}^\dagger} = -i\overline{\sigma}^\mu_{\alpha\beta}\psi_{L,\beta} + \frac{m}{2}\left(-\varepsilon_{\alpha\beta}\psi_{L,\beta}^\dagger + \psi_{L,\gamma}^\dagger \varepsilon_{\gamma\alpha}\right).$$

We have to be very careful with the derivatives of $\psi_L^\dagger$ – it acts from the right, and we have to consider anti-commutation. This gives the desired result

$$i\overline{\sigma}^\mu_{\alpha\beta}\partial_\mu \psi_{L,\beta} + m\varepsilon_{\alpha\beta}\psi_\beta^\dagger = 0.$$

The conjugate momentum is simply

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_{L,\alpha})} = \psi_{L,\beta}^\dagger i\overline{\sigma}^0_{\beta\alpha} = i\psi_\alpha^\dagger.$$

We can proceed by requiring canonical anti-commutation relations to get

$$H_0 = \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} E_{\mathbf{p}} a^\dagger(\mathbf{p}, s) a(\mathbf{p}, s).$$

Imposing equal-time anti-commutation relations we get

$$\begin{aligned}
\{\pi_\alpha(t,\mathbf{x}), \psi_\beta(t,\mathbf{y})\} &= i\delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
\{\psi_\alpha(t,\mathbf{x}), \psi_\beta(t,\mathbf{y})\} &= 0,\\
\{\pi_\alpha(t,\mathbf{x}), \pi_\beta(t,\mathbf{y})\} &= 0.
\end{aligned}$$

Can we add interactions to this theory? The term

$$\frac{\tilde{g}}{\Lambda^2}(\psi_L^T \varepsilon \psi_L)^2$$

has dimension 6, so it is irrelevant at low energies. If there is another scalar field ϕ, we can write only one interaction term, called *Yukawa coupling*:

$$\mathcal{L}^{\text{int}} = -g\phi\psi_L^T \varepsilon \psi_L + \text{h.c.}$$

If ϕ is complex, then we could perform a $U(1)$ transformation $\phi \to e^{i\alpha}\phi$ and $\psi_L \to e^{i\alpha}\psi_L$. However, the mass term $m\psi_L^T\varepsilon\psi_L$ is not invariant under this transformation. This means that (massive) Majorana spinor cannot carry any $U(1)$ charge or couple to complex scalar fields. There is also a possible coupling with a vector field A_μ:

$$\mathcal{L}^{\text{int}} = g\psi_L^\dagger i\overline{\sigma}^\mu A_\mu \psi_L.$$

11.6 Dirac fermions

Let's begin with a wrong attempt to express ψ_α (left or right-handed):

$$\psi_\alpha \sim u_\alpha(\mathbf{p}, s)a(\mathbf{p}, s) + v_\alpha(\mathbf{p}, s)b^\dagger(\mathbf{p}, s).$$

This is fine for internal symmetries (as we have already seen before), but it is not Lorentz invariant! Charge conjugation gives

$$\psi_\alpha^{\mathcal{C}} \sim u_\alpha(\mathbf{p}, s)b(\mathbf{p}, s) + v_\alpha(\mathbf{p}, s)a^\dagger(\mathbf{p}, s).$$

We were able to do this for scalar fields, because by making a field complex, we increased the degrees of freedom. However, Majorana field is already complex, and its complex conjugate is not independent (it is proportional to the conjugate momentum)! We can mend this by introducing *Dirac spinor* as a construction from two Majorana fields:

$$\Psi_\alpha(x) = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ipx}u_\alpha(\mathbf{p}, s)a(\mathbf{p}, s) + e^{-ipx}v_\alpha(\mathbf{p}, s)b^\dagger(\mathbf{p}, s)\right)$$

with $\alpha = 1, 2, 3, 4$. It transforms under the Lorentz group as

$$\Psi \xrightarrow{\Lambda} \begin{pmatrix} D_L(\Lambda) & 0 \\ 0 & D_R(\Lambda) \end{pmatrix}\Psi.$$

We will write $\Psi = (\psi_L, \psi_R)^T$ and use γ^μ to derive the following expressions. The anti-commutation relations are

$$\{a(\mathbf{p}, s), a^\dagger(\mathbf{p}', s')\} = 2E_{\mathbf{p}}\delta_{ss'}\delta^{(3)}(\mathbf{p} - \mathbf{p}'),$$
$$\{b(\mathbf{p}, s), b^\dagger(\mathbf{p}', s')\} = 2E_{\mathbf{p}}\delta_{ss'}\delta^{(3)}(\mathbf{p} - \mathbf{p}'),$$

with all others being zero. Spin polarization coefficients are (remember $k^\mu = (m, 0, 0, 0)^T$)

$$u(\mathbf{p}, s) = \frac{\not{p} + m}{\sqrt{2m(E_{\mathbf{p}} + m)}}u(\mathbf{k}, s),$$

$$v(\mathbf{p}, s) = \frac{\not{p} - m}{\sqrt{2m(E_{\mathbf{p}} + m)}}v(\mathbf{k}, s).$$

We used the *Feynman slash notation*: $\not{a} = a_\mu\gamma^\mu$. Moreover, we have

$$u(\mathbf{k}, 1/2) = \sqrt{m}(1, 0, 1, 0)^T, \quad u(\mathbf{k}, -1/2) = \sqrt{m}(0, 1, 0, 1)^T,$$
$$v(\mathbf{k}, 1/2) = \sqrt{m}(0, 1, 0, -1)^T, \quad v(\mathbf{k}, -1/2) = \sqrt{m}(-1, 0, 1, 0)^T.$$

The *Dirac adjoint* is $\overline{\Psi}_\alpha \equiv (\Psi^\dagger\gamma^0)_\alpha = \Psi_\beta^\dagger\gamma^0_{\beta\alpha}$. We use this to set

$$\overline{u}(\mathbf{p}, s) = u^\dagger(\mathbf{p}, s)\gamma^0,$$

$$\overline{v}(\mathbf{p},s) = v^\dagger(\mathbf{p},s)\gamma^0.$$

The spin sums are

$$\sum_{s=\pm 1/2} u_\alpha(\mathbf{p},s)\overline{u}_\beta(\mathbf{p},s) = (\not{p}+m)_{\alpha\beta},$$

$$\sum_{s=\pm 1/2} u_\alpha(\mathbf{p},s)\overline{u}_\beta(\mathbf{p},s) = (\not{p}-m)_{\alpha\beta}.$$

Skipping a lot of steps, we can arrive to the *Dirac equation*

$$(i\not{\partial}-m)\Psi(x) = 0.$$

Component-wise, we have

$$i\overline{\sigma}^\mu\partial_\mu\psi_L - m\psi_R = 0,$$
$$i\overline{\sigma}^\mu\partial_\mu\psi_R - m\psi_L = 0.$$

The Lagrangian is

$$\mathcal{L} = \overline{\Psi}(i\not{\partial}-m)\Psi.$$

It is straightforward to check that it is indeed Lorentz-invariant. Two possible (marginal) interaction with scalar ϕ and vector A_μ are

$$-g\phi\overline{\Psi}\Psi + \text{h.c.},$$
$$-g\phi\overline{\Psi}\gamma^\mu A_\mu\Psi + \text{h.c.}$$

11.7 Massless spin 1/2 particles

If we take the limit $m \to 0$ in the Majorana field (we can do it since it is continuous, contrary to vector fields):

$$\psi_\alpha(x) = \sum_{s=\pm 1/2} u(\mathbf{p},s)a(\mathbf{p},s) + v(\mathbf{p},s)a^\dagger(\mathbf{p},s).$$

The spin polarization coefficients become

$$u(\mathbf{p},s) = \frac{\sigma^\mu p_\mu}{\sqrt{2E_\mathbf{p}}}\xi_s, \quad v(\mathbf{p},s) = \frac{\sigma^\mu p_\mu}{\sqrt{2E_\mathbf{p}}}\varepsilon\xi_s e^{i\delta},$$

where $\xi_{1/2} = (1,0)^T$ and $\xi_{-1/2} = (0,1)^T$. These are correct equations for the field, but it is wrong to interpret them as a creation and annihilation operators. In order to find the correct interpretation, we need to go to the roots. The consistency conditions are

$$u(\mathbf{k},\sigma)e^{-i\theta(w)\sigma} = D(w)u(\mathbf{k},\sigma),$$
$$v(\mathbf{k},\sigma)e^{i\theta(w)\sigma} = D(w)v(\mathbf{k},\sigma),$$

where $\theta(w)$ is the angle of rotation around the z-axis generated by J^3. We have

$$D(w) = \mathbb{1} - i[\theta J^3 + \alpha(J^2+K^1) + \beta(K^2-J^1)] + \ldots$$

and

$$J^3 = \frac{\sigma^3}{2}, \quad K^2 + K^1 = \frac{\pm i\sigma^1 + \sigma^2}{2}, \quad K^2 - J^1 = \frac{\pm i\sigma^2 - \sigma^1}{2}.$$

Bringing all terms together, we obtain

$$D(w) = \mathbb{1} - i\theta\frac{\sigma^2}{2} - i(\pm i\alpha - \beta)\frac{\sigma^1 \mp i\theta^2}{2} + \dots$$

Since $\mathbf{A} = \mathbf{B} = 0$, we know

$$(\sigma^1 \mp i\sigma^2)u(\mathbf{k}, \sigma) = 0,$$
$$(\sigma^1 \mp i\sigma^2)v(\mathbf{k}, \sigma) = 0,$$

where

$$\sigma^1 - i\sigma^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \qquad \sigma^1 + i\sigma^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

The J^3 eigenvalue gives

$$\frac{\sigma^3}{2}u(\mathbf{k}, \sigma) = \sigma u(\mathbf{k}, \sigma),$$
$$\frac{\sigma^3}{2}v(\mathbf{k}, \sigma) = -\sigma v(\mathbf{k}, \sigma).$$

The solution is that for left-handed massless spinor we take the lower sign everywhere:

$$(\sigma^1 - i\sigma^2)u(\mathbf{k}, \sigma) = 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} u(\mathbf{k}, \sigma) = 0$$

and hence

$$u(\mathbf{k}, s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \xi_{-1/2}.$$

This means it has the helicity $-1/2$. In the same way we get

$$v(\mathbf{k}, s) = \begin{pmatrix} 0 \\ * \end{pmatrix}, \quad \text{helicity } +1/2.$$

We choose

$$v(\mathbf{k}, \sigma = 1/2) = \varepsilon\xi_{1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We found that massless left-handed spinor annihilates particles with helicity $\sigma = -1/2$ and creates anti-particles with helicity $\sigma = +1/2$. Similarly, massless right-handed spinor annihilates particles with helicity $\sigma = +1/2$ and creates anti-particles with helicity $\sigma = -1/2$. If ψ_L carries charge (it can since it is massless), then the particle and anti-particle are *distinguishable*. These massless two-component spinors are called *Weyl spinors*.

Chapter 12

Path integral for fermion fields

The coordinates in the functional integration are now anti-commuting:

$$\{\Psi_\alpha(t,\mathbf{x}), i\Psi^\dagger_\beta(t,\mathbf{y})\} = i\delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}).$$

Ψ_α and $\Psi^\dagger_\alpha$ play similar role as did Q_n and P_n in the normal path integral. We can then equivalently write

$$\{Q_n, P_m\} = i\delta_{mn}, \quad \{Q_m, Q_n\} = \{P_m, P_n\} = 0.$$

Let $|q\rangle$ be a position eigenstate, such that

$$Q_n |q\rangle = q_n |q\rangle .$$

q_n are not operators, but also not ordinary numbers. We have

$$0 = \{Q_m, Q_n\} |q\rangle = (q_m q_n + q_m q_n) |q\rangle = 0.$$

So our numbers satisfy

$$q_n q_m = -q_m q_n, \qquad \forall m, n.$$

12.1 Grassmann variables

Grassmann variables are anti-commuting numbers. In particular, it means that for single Grassmann number ξ it holds $\xi^2 = 0$. Expanding functions of Grassmann numbers in Taylor series, we get

$$f(\xi) = f_0 + f_1\xi$$

for $f_0, f_1 \in \mathbb{C}$. Pretty boring, right? The situation gets a bit more interesting when we introduce the differentiation. We can define it in two ways:

$$f(\xi + \mathrm{d}\xi) = \begin{cases} f(\xi) + \mathrm{d}\xi\, f'(\xi), & \text{left-derivative,} \\ f(\xi) + f'(\xi)\,\mathrm{d}\xi, & \text{right-derivative.} \end{cases}$$

We will always choose the right-derivative. The real fun begins, when we introduce multiple variables:

$$g(\xi,\eta) = a + b\xi + c\eta + d\xi\eta$$

for $a, b, c, d \in \mathbb{C}$. The right-derivative is

$$g(\xi + \mathrm{d}\xi, \eta) = g(\xi, \eta) + g'(\xi, \eta)\,\mathrm{d}\xi.$$

We need to be really careful, when commuting Grassmann variables. The partial derivative of the general two-variable function is

$$\frac{\partial}{\partial \xi} g(\xi, \eta) = b - d\eta.$$

The right-derivative is consistent with the definition of conjugate momentum

$$\pi_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_a)} = i\psi_a^\dagger.$$

The product rule for differentiation is

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(f(\xi)g(\xi)) = f(\xi)g'(\xi) + (-1)^{|g|} f'(x)g(x),$$

where

$$|g| = \begin{cases} 0, & \text{if } g(\xi)\text{is completely complex-valued;} \\ 1, & \text{if } g(\xi)\text{is completely Grassmann-valued.} \end{cases}$$

In order to integrate Grassmann functions, we will use the linearity of Grassmann variables and the fact that integration over total derivative vanishes. We write

$$0 = \int \mathrm{d}\xi \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi) = f_1 \int \int \mathrm{d}\xi \cdot 1$$

which gives $\int \mathrm{d}\xi \cdot 1 = 0$. The linearity implies

$$\int \mathrm{d}\xi\, f(\xi) = f_1 \int \mathrm{d}\xi\, \xi = f_1.$$

We have defined $\int \mathrm{d}\xi\, \xi = 1$. Let's see how the variable transformation works. Let $\xi' = a\xi$, $a \in \mathbb{C}$. We get

$$1 = \int \mathrm{d}\xi\, \xi = \frac{1}{a} \int \mathrm{d}\xi\, \xi' = \frac{1}{a} \int \mathrm{d}\xi \left\| \frac{\mathrm{d}\xi}{\mathrm{d}\xi'} \right\| \xi'.$$

From this it follows that

$$\|\mathrm{d}\xi/\mathrm{d}\xi'\| = a$$

which is exactly the inverse of what we get with ordinary functions. The delta function in Grassmann space is defined as $\delta(\xi) = \xi$. To check this, write

$$\begin{aligned} \int \mathrm{d}\xi\, \delta(\xi - \xi_0) f(\xi) &= \int \mathrm{d}\xi\, (\xi - \xi_0)(f_0 + f_1\xi) \\ &= \int \mathrm{d}\xi\, (f_0\xi - \xi_0 f_1 \xi) \\ &= f_0 + f_1 \int \mathrm{d}\xi\, \xi\xi_0 \\ &= f_0 + f_1\xi_0 = f(\xi_0). \end{aligned}$$

The integration of exponential function works as follows:

$$\int \mathrm{d}x\, e^{\xi\xi_0} = \int \mathrm{d}\xi\, (1 + \xi\xi_0) = \xi_0 = \delta(\xi_0).$$

To generalize integration to multiple variables we use the following anti-commutation relations:

$$0 = \{\xi_i, \xi,\} = \{\mathrm{d}\xi_i, \xi_j\} = \{\mathrm{d}\xi_i, \mathrm{d}\xi_j\}.$$

A general function can be written as

$$f(\xi) = f_0 + f_1^i \xi_i + f_2^{ij} \xi_i \xi_j + f_N \xi_N \xi_{N-1} \dots \xi_1.$$

The product of ξs is always written in descending order of indices. The multivariable integral is defined in *ascending* order:

$$\int \mathrm{d}^N\xi\, f(\xi) = \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \dots \mathrm{d}\xi_N\, f(\xi) = f_N.$$

For each coordinate it still holds

$$\int \mathrm{d}\xi_i \cdot 1 = 0, \qquad \int \mathrm{d}\xi_i\, \xi_i = 1.$$

Performing the coordinate transformation $\xi_i' = A_{ij}\xi_j$ gives

$$\int \mathrm{d}^N\xi\, f(\xi) = \int \mathrm{d}^N x i' \left\| \frac{\mathrm{d}\xi}{\mathrm{d}\xi'} \right\| f(\xi') = f_n \left\| \frac{\mathrm{d}\xi}{\mathrm{d}\xi'} \right\|.$$

In particular

$$\int \mathrm{d}^N\xi\, f_N (A\xi)_N \dots (A\xi)_1 = \int \mathrm{d}^N\xi\, f_N A_{Ni_N} \dots A_{1i_1} \xi_N \dots \xi_1$$

$$f_n \int \mathrm{d}^N\xi, \left(\sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) A_{1\sigma(1)} \dots A_{N\sigma(N)} \right) \xi_N \dots \xi_1$$

$$= f_N \det(A).$$

Hence,

$$\boxed{\left\| \frac{\mathrm{d}\xi}{\mathrm{d}xi'} \right\| = \det(A).}$$

The multidimensional delta function is defined in descending order: $\delta^{(N)}(\xi) = \xi_N \dots \xi_1$. Grassmann integrals are also translationally invariant:

$$\int \mathrm{d}^N\xi\, f(\xi + \eta) = f_N \int \mathrm{d}^N\xi\, (\xi_N + \eta_N) \dots (\xi_1 + \eta_1) = \int \mathrm{d}^N\xi\, f(\xi).$$

We can also introduce complex Grassmann variables such that

$$\int \mathrm{d}\xi\, \xi = \int \mathrm{d}\xi^*\, \xi^* = 1 = \int \mathrm{d}\xi \mathrm{d}\xi^*\, \xi^* \xi = 1$$

where we treat ξ and ξ^* as two independent real variables. We define

$$\mathrm{d}^N\xi\mathrm{d}^N\xi^* = \mathrm{d}\xi_1\mathrm{d}\xi_1^*\mathrm{d}\ldots\xi^N\mathrm{d}\xi_N^*.$$

This definition gives the following:

$$\begin{aligned}
\int \mathrm{d}^N\xi\mathrm{d}^N\xi^*\, e^{\xi^\dagger A\xi} \overset{\xi'=A\xi}{=} & \int \mathrm{d}^N\xi'\mathrm{d}^N\xi^* \left\|\frac{\mathrm{d}\xi}{\mathrm{d}\xi'}\right\| e^{\xi'^\dagger\xi'} \\
&= \det(A)\int \mathrm{d}^N\xi'\mathrm{d}^N\xi^*\frac{1}{N!}(\xi^\dagger\xi')^N \\
&= \det(A)\int \mathrm{d}^N\xi'\mathrm{d}^N\xi^*\frac{1}{N!}(\xi_1^*\xi_1' + \cdots + \xi_N^*\xi_N')^n \\
&= \det(A)\int \mathrm{d}^N\xi'\mathrm{d}^N\xi^*\xi_N^*\xi_N'\ldots\xi_1^*\xi_1' \\
&= \det(A).
\end{aligned}$$

12.2 Path integral formula

The notation will be much simpler if we use the discrete notation for Ψ_n and $\Psi_n^\dagger$. The anti-commutation relation is

$$\{\Psi_n, \Psi_m^\dagger\} = \delta_{nm}.$$

In the ground sate, we have $\Psi_n|0\rangle = 0$ and $\langle 0|\Psi_n^\dagger$ for all n. We can always define the ground state as

$$|0\rangle \equiv \prod_n \Psi_n|\Psi\rangle.$$

A *coherent fermion state* is defined as

$$|\xi\rangle \equiv \exp\left(-\sum_n \xi_n\Psi_n^\dagger\right)|0\rangle$$

which is an eigenstate for all Ψ_n:

$$\Psi_n|\xi\rangle = \xi_n|\xi\rangle.$$

We can show this by writing the difference:

$$\begin{aligned}
(\Psi_n - \xi_n)|\xi\rangle &= (\Psi_n - \xi_n)\exp\left(-\sum_i \xi_i\Psi_i^\dagger\right)|0\rangle \\
&= (\Psi_n - \xi_n)e^{-\xi_n\Psi_n^\dagger}\exp\left(-\sum_{i\neq n} \xi_i\Psi_i^\dagger\right)|0\rangle \\
&= (\Psi_n - \xi_n)(1 - \xi_n\Psi_n^\dagger)\exp\left(-\sum_{i\neq n} \xi_i\Psi_i^\dagger\right)|0\rangle \\
&= (\Psi_n - \xi_n\Psi_n^\dagger\Psi_n)\exp\left(-\sum_{i\neq n} \xi_i\Psi_i^\dagger\right)|0\rangle \\
&= 0.
\end{aligned}$$

We used the anti-commutation relation $\{\Psi_n, \Psi_n^\dagger\} = 1$. Similarly,

$$\langle\xi| = \langle 0| \exp\left(-\sum_n \Psi_n \xi_n^*\right)$$

with

$$\langle\xi|\,\Psi_n^\dagger = \langle\xi|\,\xi_n^*.$$

The complex conjugation of the product of Grassmannians is order-reversing: $(\xi\eta)^* = \eta^*\xi^*$. We have the scalar product

$$\langle\xi|\eta\rangle = \exp\left(\sum_n \xi_n^*\eta_n\right)$$

and the completeness relation

$$\mathbb{1} = \int \prod_n \mathrm{d}\xi_n^* \mathrm{d}\xi_n \; |\xi\rangle \exp\left(-\sum_n \xi_n \xi_n^*\right) \langle\xi|\,.$$

Using this we can derive the path integral formula. The evolution of states is

$$\langle\xi_f; t_f|\xi_i; t_i\rangle = \langle\xi_f|e^{-iH(t_f-t_i)}|\xi_i\rangle$$

where in the Hamiltonian $H[\Psi_n^\dagger, \Psi_m]$ all $\psi_n^\dagger$ are positioned left of the Ψ_n. The transition between two states is

$$\langle\xi^k|e^{iH\Delta t}|\xi^{k-1}\rangle \approx e^{-iH(\xi^{k*},\xi^{k-1})\Delta t} \exp\left(\sum_n \xi_n^{k-1^*}\xi_n^{k-1}\right).$$

The transition matrix element is

$$\begin{aligned}&\langle\xi_f; t_f|\xi_i; t_i\rangle = \\ &= \exp\left(\sum_n \xi^*(t_f)\xi_n(t_f)\right) \int \mathcal{D}[\xi_n^*(t), \xi_n(t)] \exp\left[i\int_i^{t_f} \mathrm{d}t \left(i\sum_n \xi_n^*(t)\dot{\xi}_n(t) - \mathcal{H}(\xi_n^*(t), \xi_n(t))\right)\right]\end{aligned}$$

with the boundary conditions $\xi(t_i) = \xi_i$ and $\xi^*(t_f) = \xi_f^*$.

12.3 Green functions of fermionic fields

Now we transition to the continuous case

$$\Psi_n \to \Psi_\alpha(x), \qquad \sum_n \to \sum_\alpha \int \mathrm{d}^3x.$$

As in the scalar case we start with

$$\langle\Psi_f; t_f|T\{O_a(x_a)\cdot\ldots\}|\Psi_i; t_i\rangle\,.$$

The T^*-ordering is defined as

$$T^*\{\mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\} = \theta(x_a^0 - x_b^0)\mathcal{O}_a(x_a)\mathcal{O}_b(x_b) \pm \theta(x_b^0 - x_a^0)\mathcal{O}_b(x_b)\mathcal{O}_a(x_a)$$

where we use the "$-$" sign if we are dealing with two Grassmann-valued operators. As before, we will denote T^*-ordering just with T. Multiply with $\langle\Omega|\Psi_f;t_f\rangle$ from the left and with $\langle\Psi_i;t_i|\Omega\rangle$ from the right and use the $i\varepsilon$ prescription:

$$\begin{aligned}\langle\Omega|T\{\mathcal{O}_a(x_a)\cdot\ldots\}|\Omega\rangle &= \\ &= |N|^2\int\mathcal{D}[\Psi_\alpha(x),\Psi^*_\alpha(x)]\exp\left(i\int \mathrm{d}^4x\,(\Psi^*_\alpha i\partial_0\Psi_\alpha-\mathcal{H}(\Psi^*,\Psi)+i\varepsilon\Psi^*_\alpha\Psi_\alpha)\right)\mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\ldots.\end{aligned}$$

The Lagrangian is $\mathcal{L}=\Psi^\dagger i\partial_0\Psi-\mathcal{H}$, so the path integral in terms of Lagrangian is

$$\langle\Omega|T\{\mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\ldots\}|\Omega\rangle=|N|^2\int\mathcal{D}[\Psi,\Psi^*]\,\exp\left(i\int\mathrm{d}^4x\,\mathcal{L}(\Psi,\Psi^*)\right)\mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\ldots.$$

12.4 Feynman rules for spinor fields

The generating functional is

$$Z[\eta,\overline{\eta}]=|N|^2\in\mathcal{D}[\psi,\overline{\psi}]\,\exp\left(i\int\mathrm{d}^4x\,\left(\mathcal{L}+\overline{\eta}(x)\psi(x)+\overline{\psi}(x)\eta(x)\right)\right).$$

Note that η and $\overline{\eta}=\eta^\dagger\gamma^0$ are Grassmann-valued fields. The functional derivative acting on generating functional gives

$$\frac{1}{i}\frac{\delta}{\delta\eta(x)}\leftrightarrow\overline{\psi},\qquad -\frac{1}{i}\frac{\delta}{\delta\overline{\eta}}\leftrightarrow\psi.$$

Acting on the free Lagrangian

$$Z_0[\eta,\overline{\eta}]=|N|^2\in\mathcal{D}[\psi,\overline{\psi}]\,\exp\left(i\int\mathrm{d}^4x\,\left(\overline{\psi}(i\partial\!\!\!/-m+i\varepsilon)\psi+\overline{\eta}\psi+\overline{\psi}\eta\right)\right).$$

Performing the change of variables

$$\psi(x)=\psi'(x)-\frac{1}{i\partial\!\!\!/-m+i\varepsilon}\eta(x)$$

gives

$$\begin{aligned}Z_0[\eta,\overline{\eta}] &= |N_0|^2\int\mathcal{D}[\psi',\overline{\psi}']\,\exp\left(i\int\mathrm{d}^4x\,(\overline{\psi}'(i\partial\!\!\!/-m+i\varepsilon)\psi'-\overline{\eta}(i\partial\!\!\!/-m+i\varepsilon)^{-1}\eta)\right)\\ &= |N_0|^2\det(i(i\partial\!\!\!/-m+i\varepsilon))\exp\left(i\int\mathrm{d}^4x\,\overline{\eta}\frac{1}{i\partial\!\!\!/-m+i\varepsilon}\eta\right).\end{aligned}$$

The factor $|N_0|^2\det(i(i\partial\!\!\!/-m+i\varepsilon))$ is independent on η and $\overline{\eta}$ and must be equal to 1 in order for $Z_0[0,0]=1$ to hold. Hence,

$$Z_0[\eta,\overline{\eta}]=\exp\left(-\int\mathrm{d}^4x\,\mathrm{d}^4y\,\overline{\eta}_\alpha(x)\Delta_F^{\alpha\beta}(x-y)\eta_\beta(y)\right)$$

where

$$\left(\frac{i}{i\partial\!\!\!/-m+i\varepsilon}\right)^{\alpha\beta}\eta_\beta(x)=\int\mathrm{d}^4y\,\Delta_F^{\alpha\beta}(x-y)\eta_\beta(y).$$

Δ_F is the Feynman propagator for the Dirac spinor field. The Fourier transform is

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)} \tilde{\Delta}_F(p).$$

Inserting this into the path integral gives

$$-i(i\partial\!\!\!/_x - m + i\varepsilon)\Delta_F(x-y) = \delta^{(4)}(x-y)$$

yields

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4}(-i)(p\!\!\!/ - m + i\varepsilon)\tilde{\Delta}_F(p) = \delta^{(4)}(x-y).$$

Hence,

$$\tilde{\Delta}_F(p) = \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\varepsilon} = \frac{i}{p\!\!\!/ - m + i\varepsilon}.$$

With this we can define the contraction

$$\Psi(x)\overline{\Psi}(y) = \delta_F(x-y) = \quad x \longrightarrow y$$

or in the momentum space

$$\xrightarrow{\quad p \quad} = \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\varepsilon}.$$

The other contractions vanish:

$$\Psi(x)\Psi(y) = \overline{\psi}(x)\overline{\psi}(y) = \overline{\psi}(x)\Psi(y) = 0.$$

The n-point function in free theory is

$$\langle\Omega|\Psi(x_1)\dots\overline{\Psi}(x_l)\dots|\Omega\rangle = \dots\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x_l)}\dots\frac{-1}{i}\frac{\delta}{\delta\eta(x_1)}Z_0[\eta,\overline{\eta}]\Big|_{\eta=\overline{\eta}=0}.$$

Using the same argumentation as in the scalar theory, the general n-point function is given by the sum over all non-vanishing pairwise contractions. However, we need to keep track of the signs when exchanging the fields

$$\langle\Omega|T\{\Psi(x_1)\overline{\Psi}(x_2)\}|\Omega\rangle = -\langle\Omega|T\{\overline{\Psi}(x_2)\Psi(x_1)\}|\Omega\rangle\,.$$

Let's see for example the 4-point function:

$$\begin{aligned}
\langle\Omega|T\{\Psi_1\overline{\Psi}_2\Psi_3\overline{\Psi}_4\}|\Omega\rangle &= \frac{(-1)^2}{i^2}\frac{1}{i^2}\frac{\delta}{\delta\eta_4}\frac{\delta}{\delta\overline{\eta}_3}\frac{\delta}{\delta\eta_2}\frac{\delta}{\delta\overline{\eta}_1}e^{-\overline{\eta}_x\Delta^F_{xy}\eta_y}\Big|_{\eta=\overline{\eta}=0} \\
&= \frac{\delta}{\delta\eta_4}\frac{\delta}{\delta\overline{\eta}_3}\frac{\delta}{\delta\eta_2}\left(-\left(-\Delta^F_{1y}\eta_y\right)e^{-\overline{\eta}_x\Delta^F_{xy}\eta_y}\right)\Big|_{\eta=\overline{\eta}=0} \\
&= \frac{\delta}{\delta\eta_4}\frac{\delta}{\delta\overline{\eta}_3}\left(\left(\Delta^F_{12}-\Delta^F_{1y}\eta_y\overline{\eta}_x\Delta^F_{x2}\right)e^{\overline{\eta}_{z_1}\Delta^F_{z_1z_2}\eta_{z_2}}\right)\Big|_{\eta=\overline{\eta}=0} \\
&= \frac{\delta}{\delta\eta_4}\left(\Delta^F_{12}\Delta^F_{3x}\eta_x-\Delta^F_{1y}\eta_y\Delta^F_{32}+\text{terms vanishing for }\eta=0\right)\Big|_{\eta=\overline{\eta}=0} \\
&= \Delta^F_{12}\Delta^F_{34}-\Delta^F_{14}\Delta^F_{32} \\
&= \overbrace{\Psi_1\overline{\Psi}_2}\overbrace{\Psi_3\overline{\Psi}_4}+\overbrace{\Psi_1\overline{\Psi}_2\Psi_3\overline{\Psi}_4}.
\end{aligned}$$

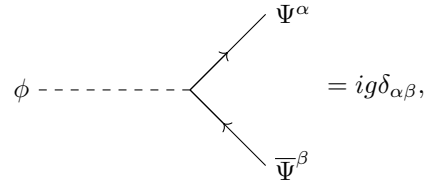
The rule is the same as for scalar fields, we just have to use a minus sign every time we commute two fields. For interactions, we use the expansion

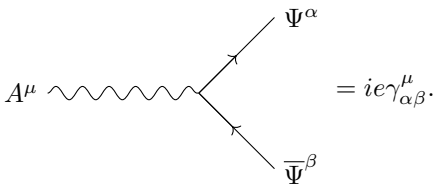
$$\exp\left(i\int \mathrm{d}^4x\,\mathcal{L}^{\text{int}}\right)=\sum_{m=0}^{\infty}\frac{1}{m!}\left(i\int \mathrm{d}^4x\,\mathcal{L}^{\text{int}}\right)^m.$$

Example 17. Assume the following Lagrangian:

$$\mathcal{L}=\mathcal{L}_0+g\overline{\Psi}\Psi\phi+e\overline{\Psi}\gamma^\mu\Psi A_\mu.$$

This is the coupling of spinor field with a vector A^μ and a scalar ϕ. The vertex factor for $\phi\to\Psi_\alpha\overline{\Psi}_\beta$ vertex is $ig\delta_{\alpha\beta}$ and the vertex factor for $A^\mu\to\Psi_\alpha\overline{\Psi}_\beta$ is $ie\gamma^\mu_{\alpha\beta}$.

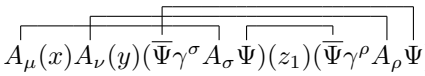




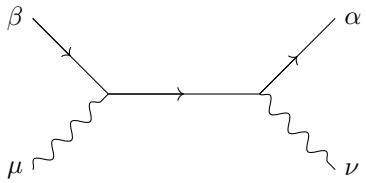
Consider the two-point function for the vector field:

$$\begin{aligned}
\int \mathrm{d}^4x\,\mathrm{d}^4y e^{ip_1x+ip_2y}\,\langle\Omega|T\{A_\mu(x)A_\nu(y)\}|\Omega\rangle &= (2\pi)^4\delta^{(4)}(p_1+p_2)\left(\tilde{\Delta}^{\mu\nu}_F(p_1)+\right. \\
&\left.+\tilde{\Delta}^{\mu\sigma}_F(p_1)\tilde{\Delta}^{\rho\nu}_F(p_2)(ie)^2(-1)\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\gamma^\rho_{\alpha\beta}i(\not{k}+m)_{\beta\gamma}\gamma^\sigma_{\beta\gamma}i(\not{p}_1+\not{k}+m)_{\delta\alpha}}{(k^2-m^2+i\varepsilon)((p_1+k)^2-m^2+i\varepsilon)}\right).
\end{aligned}$$

The minus sign in front of the integral comes from the interchanges of spinor fields in the contraction



It turns out that every spinor loop contributes a minus sign. Note that spinors are contracted starting at the end of the fermion line and in the opposite direction of the arrow.



$$= \gamma^\mu_{\beta\delta} i(\not{p} + m)_{\delta\gamma}\gamma^\nu_{\gamma\alpha}.$$

12.5 LSZ reduction formula for spinors

We want to find the expression for

$$\langle k_1, s_1, \ldots, k_m, s_m; \text{out}|p_1, s_1', \ldots, p_n, s_n'; \text{in}\rangle$$

in terms of Green functions. The asymptotic in/out fields are

$$\Psi_\alpha(x) \to \sqrt{Z}\Psi_\alpha^{\text{in/out}}(x)$$

where

$$\Psi_\alpha^{\text{in/out}}(x) = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ipx}u_\alpha(\mathbf{p},s)a^{\text{in/out}}(\mathbf{p},s) + e^{ipx}v_\alpha(\mathbf{p},s)b^{\dagger\text{in/out}}(\mathbf{p},s)\right).$$

If the field is real, then $a^\dagger = b^\dagger$.

The main result is that the S-marix is given by the on-shell, amputated Green function multiplied by the polarization functions $u(\mathbf{p},s)$ and $v(\mathbf{p},s)$. For $S = \mathbb{1} + iT$ we have

$$iT_{k_1,s_1,\ldots;p_1,s_1',\ldots} = (\sqrt{Z})^{n+m}\prod_{i=1}^m \overline{u}_{\alpha_i}(\mathbf{k}_i,s_i)\prod_{j=1}^n u_{\beta_j}(\mathbf{p}_j,s_j)\tilde{G}^{\text{amp}}(k_1,\ldots,k_m;-p_1,\ldots,-p_n)_{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_n}.$$

For antiparticles, we exchange $u \to \overline{v}$ and $\overline{u} \to v$. The general rule is in the following table.

incoming particle	$u(\mathbf{p},s)$
incoming antiparticle	$\overline{v}(\mathbf{p},s)$
outgoing particle	$\overline{u}(\mathbf{p},s)$
outgoing antiparticle	$v(\mathbf{p},s)$

If we have Weyl spinors, then the formula works exactly the same, we just need to change $\overline{u}, \overline{v} \to u^*, v^*$. If the particle is its own antiparticle, then $u = v^*$ or there is some other relation between u and v^*.

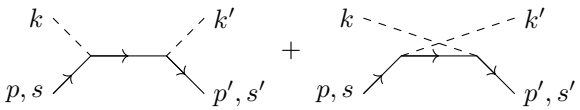
Example 18. Consider the *Compton scattering* of an electron on a scalar particle ϕ. The interaction term is Yukawa $\mathcal{L}^{\text{int}} = -g\overline{\Psi}\Psi\phi$. The incoming electrons are unpolarized, so we need to average the spin (average over initial spins and sum over final spins):

$$\overline{|T|^2} = \frac{1}{2}\sum_{s,s'} |T|^2.$$

We will use the spin sum that we derived in the previous chapter

$$\sum_{s=\pm 1/2} u(\mathbf{p},s)\overline{u}(\mathbf{p},s) = \not{p} + m.$$

There are two relevant (firs order) Feynman diagrams giving the following expression

$iT =$ 

$$\begin{aligned} &= \overline{u}(\mathbf{p}',s)\left(ig\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i\varepsilon}ig - ig\frac{i(\not{p}+\not{k}'+m)}{(p-k')^2-m^2+i\varepsilon}\right)u(\mathbf{p},s) \\ &= (ig)^2 i\overline{u}_\alpha(\mathbf{p}',s')M_{\alpha\beta}u_\beta(\mathbf{p},s). \end{aligned}$$

We will need $|T|^2 = T^*T$, so we use this handy formula:

$$(\overline{u}(\mathbf{p}',s')\gamma^{\alpha_1}\dots\gamma^{\alpha_n}u(\mathbf{p},s))^* = \overline{u}(\mathbf{p},s)\gamma^{\alpha_n}\dots\gamma^{\alpha_1}u(\mathbf{p}',s').$$

We get

$$|T|^2 = g^4[u(\mathbf{p},s)\overline{u}(\mathbf{p},s)]_{\alpha\beta}\overline{M}_{\beta\gamma}[u(\mathbf{p}',s')\overline{u}(\mathbf{p}',s')]_{\gamma\delta}M_{\delta\alpha}$$

and

$$\overline{|T|^2} = \frac{1}{2}g^4\,\mathrm{tr}((\not{p}+m)\overline{M}(\not{p}'+m)M)$$

where

$$M = \frac{\not{p}+\not{k}+m}{(p+k)^2-m^2} + \frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2}.$$

Since ϕ is massless, $k^2 = k'^2 = 0$ and also

$$(p+k)^2 - m^2 = 2pk, \quad (p-k')^2 - m^2 = -2pk'.$$

Since $(\not{p}+m)(\not{p}-m) = 0$ it holds

$$(\not{p}+m)(\not{p}+\not{k}+m) = (\not{p}+m)(\not{k}+2m).$$

So the T-matrix simplifies to

$$\overline{|T|^2} = \frac{1}{2}g^4\,\mathrm{tr}\left[(\not{p}+m)\left(\frac{\not{k}+2m}{2pk} + \frac{-\not{k}'+2m}{-2k'p}\right)(\not{p}'+m)\left(\frac{\not{k}+2m}{2pk} + \frac{-\not{k}'+2m}{-2k'p}\right)\right].$$

To simplify the last M matrix, we cyclically permuted $(\not{p}+m)$ to the end of the trace and simplified M from the right.

When evaluating traces all the previously stated trace identities come extremely handy. Most of all, the trace of odd number of γ matrices is zero. One of the terms, for example, is

$$\begin{aligned}\mathrm{tr}[(\not{p}+m)(\not{k}+2m)(\not{p}'+m)(-\not{k}'+2m)] &= \\ = 16m^4 + 4m^2(2pk + 4pp' - 2pk'2 + p'k - kk' - 2p'k') &\\ - 4((pk)(p'k') - (pp')(kk') + (pk')(p'k)).&\end{aligned}$$

We can express the final result with Mandelstam variables, or we can move to a specific frame and calculate everything for that. For scattering on an electron, one usually assumes that the electron is initially at rest and the scalar comes from the z direction with energy E. The scalar is then scattered in the xz plane under the angle θ. We have

$$p = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad k = \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix}, \quad k' = \begin{pmatrix} E' \\ E'\sin\theta \\ 0 \\ E'\cos\theta \end{pmatrix}, \quad p' = \begin{pmatrix} m+E-E' \\ -E'\sin\theta \\ 0 \\ E-E'\cos\theta \end{pmatrix}.$$

The scalar products between them are

$$pk = p'p' = mE, \quad pk' = p'k = mE'$$

and

$$kk' = pp' - m^2 = EE'(1-\cos\theta).$$

There is one other expression relating these quantities:

$$m(E-E') = EE'(1-\cos\theta).$$

The full T-matrix is

$$\overline{|T|^2} = g^4\left(\frac{E}{E'} + \frac{E'}{E} + 2\cos 2\theta\right).$$

The Lorentz invariant phase space is

$$\mathrm{d}\Pi_{\mathrm{LIPS}} = \frac{1}{8\pi}\frac{E'^2}{mE}\mathrm{d}\cos\theta.$$

Using $|v_a - v_b| = c = 1$ we get the final result for the scattering cross-section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{g^4}{32\pi m^2}\left(\frac{E'}{E}\right)^2\left(\frac{E}{E'} + \frac{E'}{E} + 2\cos 2\theta\right).$$

Chapter 13

Vector fields and gauge redundancy

We will discuss fields that transform according to the fundamental representation of the Lorentz group:

$$\Lambda \mapsto D(\Lambda)^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\nu}.$$

In the (A, B) representation, this is equivalent to the irreducible representation $(1/2, 1/2)$ which (as we will find out) corresponds to massive particles with spin one or massless particles with helicity one. Since we can write

$$(1/2, 1/2) = (1/2, 0) \otimes (0, 1/2)$$

we can represent a Lorentz 4-vector V^{μ} with a Hermitian 2×2 matrix $V \equiv \sigma^{\mu} V_{\mu}$ (one can check that both representations have 4 degrees of freedom). The matrix V transforms under the Lorentz group as

$$\Lambda^{\mu}{}_{\nu} V^{\nu} \longrightarrow D_{0,1/2}(\Lambda) V D^{-1}_{(1/2,0)}(\Lambda)$$

which *defines* the action of $(1/2, 1/2)$ representation on Hermitian matrices. Denote $D_{(1/2,0)} = D_L$ and $D_{(0,1/2)} = D_R$. Now we use some known identities:

$$D_R^{\dagger}(\Lambda) \sigma^{\mu} D_R(\Lambda) = \Lambda^{\mu}{}_{\nu} \sigma^{\nu}, \quad D_R^{\dagger}(\Lambda) = D_L^{-1}(\Lambda) = D_L(\Lambda^{-1}), \quad D_R(\Lambda) = D_R^{-1}(\Lambda^{-1})$$

to get

$$D_L(\Lambda) \sigma^{\mu} D_R^{-1}(\Lambda) V_{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu} \sigma^{\nu} V_{\mu}.$$

This gives us

$$V'_{\mu} = \Lambda_{\mu}{}^{\nu} V_{\nu}.$$

So V^{μ} transforms as a 4-vector if V transforms under the representation $(1/2, 1/2)$.

13.1 Massive vector fields

Notice that a general Lorentz tensor $A^{\mu_1 \cdots \mu_n}$ has 4^n components, but a massive particle has only $2j + 1$ spin degrees of freedom. Usually, the representation of Lorentz group is equivalent to an irreducible representation of the little group (SO(3) in our case). Schematically, we can write $4 = 1 \oplus 3$, where 4 is a four-vector V^{μ}, 1 is a scalar under spatial rotations V^0 and 3 is

a three-vector under SO(3) rotations V^i. We expect an extra condition on the vector field that eliminates one degree of freedom. For example, rank 2 symmetric tensor decomposes as

$$(4 \times 4)_{\text{sym}} = (1 \oplus 3) \otimes (1 \oplus 3)_{\text{sym}} = 1 \oplus 3 \oplus (3 \otimes 3)_{\text{sym}} = 1 \oplus 3 \oplus 5 \oplus 1.$$

Particularly, we can write $h = h_{00} \oplus h_{i0} \oplus \overline{h}_{ij} \oplus \text{tr}(h_{ij})$ where $\overline{h}_{ij}$ is traceless.

The most general Lorentz invariant quadratic terms are

$$A_\mu A^\mu, \quad \partial_\mu A^\nu \partial^\mu A_\nu, \quad \partial_\mu A^\nu \partial_\nu A^\mu, \quad \partial_\mu A^\mu \partial_\nu A^\nu.$$

For convenience, we define $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The most general Lagrangian we can write (up to second order) is

$$\mathcal{L} = -\frac{a}{4} F_{\mu\nu} F^{\mu\nu} + \frac{b}{2} (\partial_\mu A^\nu)(\partial_\mu A^\nu) + \frac{c}{2} (\partial_\mu A^\mu)(\partial_\mu A^\nu) + \frac{m^2}{2} A_\mu A^\mu.$$

The conjugate momentum is

$$\pi_\nu = a F_\nu{}^0 + b \partial_\nu A^0 + c \delta^0{}_\nu \partial_\mu A^\mu.$$

If for a given i the conjugate momentum π_i vanishes, then A_i is not a generalized coordinate. Since we want to get rid of one degree of freedom (4 components of A_μ may only represent three different spins 1, 0 and -1), we choose to get rid of the scalar component in the $1 \oplus 3$ sum:

$$\pi^0 = (b + c) \partial_0 A^0 - c \partial_i A^i = 0.$$

This gives $b = c = 0$ and after choosing $a = 1$ (in order to get the canonical form of the Hamiltonian) the Lagrangian reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu.$$

We could even generalize this procedure for a complex vector field in the same way we did it with scalars. The component A^0 still appears in the Lagrangian, but it is not independent as we can solve for it with equation $\pi^0(A^0, \mathbf{A}) = 0$. Now we will add sources to the Lagrangian:

$$\mathcal{L} = \mathcal{L}_0 - J_\mu A^\mu.$$

The conjugate momenta are $\pi^0 = 0$ and $\pi_i = -\pi^i = -F^{i0}$ which is exactly the electric field. The Euler-Lagrange equations give

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = m^2 A^\mu - J^\mu, \quad \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} = F^{\mu\nu}$$

and finally

$$\boxed{\partial_\nu F^{\nu\mu} - m^2 A^\mu = J^\mu.}$$

This is the *Maxwell-Proca equation*. In the free theory ($J^\mu = 0$) we have $\partial_\mu \partial_\nu F^{\mu\nu} + m^2 \partial_\mu A^\mu = 0$ and since $\partial_\mu \partial_\nu F^{\mu\nu} = 0$ it holds

$$\partial_\mu A^\mu = 0.$$

This Lorentz invariant condition projects out the scalar degree of freedom. Since $\partial_\nu F^{\nu\mu} = \Box A^\mu$, we can write

$$(\Box + m^2)A^\mu = 0$$

which is again *Klein-Gordon equation* (for a massive vector field). The field can be expanded as

$$A^\mu(x) = \sum_{\lambda=0,\pm1} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ipx} \epsilon^\mu(\mathbf{p},\lambda) a(\mathbf{p},\lambda) + e^{ipx} \epsilon^{*\mu}(\mathbf{p},\lambda) a^\dagger(\mathbf{p},\lambda) \right).$$

The *polarization vectors* ϵ^μ corresponds to the u_α in the general case. The polarization vectors are orthogonal to the momentum p ($\epsilon^\mu p_\mu = 0$) and satisfy

$$\sum_\lambda e^\mu(\mathbf{p},\lambda)\epsilon^{*\nu}(\mathbf{p},\lambda) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}.$$

In the reference frame where $p^\mu = k^\mu = (m,\mathbf{0})^T$, this becomes

$$\sum_\lambda e^i(\mathbf{k},\lambda)e^{*j}(\mathbf{k},\lambda) = \delta^{ij}, \qquad \epsilon^0(\mathbf{k},\lambda) = 0.$$

To check this, we use $\epsilon^\mu(\mathbf{p},\lambda) = L(p)^\mu{}_\nu \epsilon^\nu(\mathbf{k},\lambda)$ and choose

$$\epsilon^\mu(\mathbf{k},0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \epsilon^\mu(\mathbf{k},\pm1) = \mp\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}$$

and $p^\mu = (p^0,0,0,p)^T$ where $p^0 = \sqrt{p^2+m^2}$. Notice that it indeed holds $\epsilon^\mu k_\mu = 0$. Now we set $L(p)k = p$ which gives

$$L(p) = \begin{pmatrix} \cosh\beta & 0 & 0 & \sinh\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta & 0 & 0 & \cosh\beta \end{pmatrix}$$

where $\cosh\beta = p^0/m$ and $\sinh\beta = p/m$. We see that

$$\epsilon^\mu(\mathbf{p},\pm1) = \epsilon^\mu(\mathbf{k},\pm1).$$

Hence,

$$\sum_{\lambda=\pm1} \epsilon^\mu(\mathbf{k},\lambda)\epsilon^{*\nu}(\mathbf{k},\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $\lambda = 0$ we get

$$\epsilon^\mu(\mathbf{k},0) = \frac{1}{m}\begin{pmatrix} p \\ 0 \\ 0 \\ p^0 \end{pmatrix}, \quad \epsilon^\mu(\mathbf{k},0)\epsilon^{*\nu}(\mathbf{k},0) = \frac{1}{m^2}\begin{pmatrix} p^2 & 0 & 0 & pp^0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ pp^0 & 0 & 0 & (p^0)^2 \end{pmatrix}.$$

Combining all together we get

$$\sum_\lambda e^\mu(\mathbf{p},\lambda)\epsilon^{*\nu}(\mathbf{p},\lambda) = \frac{1}{m^2}\begin{pmatrix} p^2 & 0 & 0 & pp^0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ pp^0 & 0 & 0 & (p^0)^2 \end{pmatrix} = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}.$$

Let's check the compatibility between spin-1 particles and field transformations of Lorentz representations. We need to satisfy

$$\sum_{s'} u_\alpha(\mathbf{k},s')D^{(j)}_{s's} = \sum_{\alpha'} D_{\alpha\alpha'}(R)u_{\alpha'}(\mathbf{k},s),$$

and so we get

$$\sum_{\lambda'} \epsilon^i(\mathbf{k},\lambda')D^{(1)}_{\lambda'\lambda}(R) = \sum_j R^{ij}\epsilon^{ij}(\mathbf{k},\lambda).$$

The Wigner function is

$$D^{(1)}_{\lambda'\lambda}(R) = \exp(-i\theta n^k J^k)_{\lambda'\lambda}$$

and the rotation matrix is $R^{ij} = \delta^{ij} - \varepsilon^{ijk}n^k\theta + \dots$. We can write this infinitesimally as

$$\sum_{\lambda'} \epsilon^i(\mathbf{k},\lambda')J^k_{\lambda'\lambda} = -i\varepsilon^{ijk}\epsilon^k(\mathbf{k},\lambda)$$

and

$$J^k_{\lambda'\lambda} = \langle j=1,\lambda'|J^k|j=1,\lambda\rangle$$

are the matrix elements of the angular momentum operator J^k in the $|jm\rangle$ basis. We can use for example the representation

$$J^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus, we can conclude that A^μ creates and annihilates particles with spin 1. We want to check if the spin-1 particles are bosons or fermions. Let's write

$$A_\mu(x) = A^{(+)}_\mu(x) + \lambda A^{(-)}_\mu(x)$$

where $(+)$ denotes the annihilation and $(-)$ creation operator. For space-like separations $(x-y)^2 < 0$ we have

$$[A^\mu(x), A^\nu(x)]_\mp = \lambda(1\mp 1)\left(\eta^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta_+(x-y)$$

and

$$[A^\mu(x), (A^\dagger)^\nu(x)]_\mp = (1\mp|\lambda|^2)(-1)\left(\eta^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta_+(x-y).$$

If we take $|\lambda|$ and normal commutators, both expressions vanish. Hence, the spin-1 particles are bosons.

13.2 Path integrals for spin 1 fields

The Hamiltonian version of the path integral follows exactly the same steps in the scalar case. The n-point function is

$$\langle\Omega|\mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\dots|\Omega\rangle = |N|^2\int\mathcal{D}[\mathbf{A},\boldsymbol{\pi}]\exp\left(i\int \mathrm{d}^3x\ (\boldsymbol{\pi}\cdot\mathbf{A}-\mathcal{H}(\boldsymbol{\pi},\mathbf{A})+i\varepsilon)\right)\mathcal{O}_a(x_a)O_b(x_b)\dots$$

Note that A^0 is not a canonical coordinate, so the path integral only uses $\mathbf{A}$ and $\boldsymbol{\pi}$. We eliminate A^0 by writing

$$A^0 = -\frac{1}{m^2}\left(\nabla\cdot\mathbf{E}-J^0\right)$$

and

$$\partial_0\mathbf{A} = -\mathbf{E}-\nabla A^0 = -\mathbf{E}+\frac{1}{m^2}\left(\nabla^2\mathbf{E}-\nabla J^0\right).$$

Now we compute $\mathcal{H}$ in terms of $\mathbf{A}$ and $\boldsymbol{\pi}=\mathbf{E}$:

$$\mathcal{H} = \pi_\mu\dot{A}^\mu-\mathcal{L} = -\boldsymbol{\pi}\cdot\partial_0\mathbf{A}-\mathcal{L} = \frac{\mathbf{E}^2}{2}+\frac{(\nabla\times\mathbf{A})^2}{2}+\frac{(\nabla\cdot\mathbf{E})^2}{(2m)^2}+\frac{m^2\mathbf{A}^2}{2}-\mathbf{J}\cdot\mathbf{A}-\frac{J^0\nabla\cdot\mathbf{E}}{m^2}+\frac{(J^0)^2}{2m^2}.$$

To get a covariant form, notice that the change of variables

$$A^0 = A'^0+\frac{1}{m^2}\left(\nabla\cdot\boldsymbol{\pi}-J^0\right)$$

has a unit Lagrangian, so

$$\int\mathcal{D}[A^0]\exp\left(i\frac{m^2}{2}\int\mathrm{d}^3x\ \left(A^0-(\nabla\cdot\boldsymbol{\pi}-J^0)\right)^2\right) = \text{const.}$$

is independent of $\boldsymbol{\pi}$, so it doesn't change the value of the path integral. When we add this term to the Lagrangian, we get the Hamiltonian

$$\mathcal{H}' = \frac{1}{2}\boldsymbol{\pi}^2+(\nabla\times\mathbf{A})^2-\frac{1}{2}A_\mu A^\mu+J_\mu A^\mu+A^0\nabla\cdot\boldsymbol{\pi}.$$

After integrating over $\boldsymbol{\pi}$ (as in the scalar case) we get the Lagrangian path integral

$$\langle\Omega|T\{\mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\dots\}|\Omega\rangle = |N|^2\int\mathcal{D}[A^\mu]\ \exp\left(i\int\mathrm{d}^4x\ \mathcal{L}\right)\mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\dots$$

where $\mathcal{L}$ can be written in manifestly covariant way:

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^2A_\mu A^\mu-J_\mu A^\mu.}$$

In this case, A^0 is an independent field configuration, not determined by $\mathbf{A}$. The generating functional is

$$Z[\mathcal{J}] = \frac{\int\mathcal{D}[A^\mu]\exp(i\int\mathrm{d}^4x\ (\mathcal{L}+\mathcal{J}_\mu A^\mu))}{\int\mathcal{D}[A^\mu]\exp(\int\mathrm{d}^4x\ \mathcal{L})}.$$

In the free theory, the result is

$$Z_0[\mathcal{J}] = \exp\left(-\frac{1}{2}\int\mathrm{d}^4x\ \mathrm{d}^4y\ \mathcal{J}_\mu(x)\Delta_F^{\mu\nu}(x-y)\mathcal{J}_\nu(y)\right).$$

Since

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(m^2 - i\varepsilon)A_\mu A^\mu = -\frac{1}{2}A^\mu\left(-\eta_{\mu\nu}(\Box_x + m^2 + i\varepsilon) + \partial_\mu\partial_\nu\right)$$

it holds

$$\left(-\eta_{\mu\nu}(\Box_x + m^2 + i\varepsilon) + \partial_\mu\partial_\nu\right) i\Delta_F^{\mu\nu}(x-y) = \delta^\rho{}_\mu\delta^{(4)}(x-y).$$

When we Fourier transform this equation, we get

$$(\eta_{\mu\nu}(p^2 - m^2 + i\varepsilon) - p_\mu p_\nu)i\tilde{\Delta}_F^{\nu\rho}(p) = \delta^\rho{}_\mu.$$

There are only two Lorentz structures which can fulfil this equation, and thus we make the ansatz

$$\tilde{\Delta}_F^{\nu\rho}(p) = A(p^2)\eta^{\nu\rho} + B(p^2)p^\nu p^\rho$$

which leads to the form

$$\boxed{\tilde{\Delta}_F^{\mu\nu}(p) = \frac{i}{p^2 - m^2 + i\varepsilon}\left(-\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right).}$$

Schematically, we have

$$\overbrace{A^\mu(x)A^\nu}(y) = \Delta_F^{\mu\nu}(x-y)$$

$$\nu \sim\sim_{p} \mu = \tilde{\Delta}_F^{\mu\nu}(p).$$

The polarization vector $\epsilon^\mu(\mathbf{p},\lambda)$ is associated with the incoming particle, while the vector $\epsilon^{*\mu}(\mathbf{p},\lambda)$ is associated with the outgoing one.

The marginal interactions we can write down for massive vector fields are

$$A_\mu A^\mu \partial_\nu A^\mu, \quad A_\mu A_\nu \partial^\nu A^\mu, \quad (A_\mu A^\mu)^2$$

and with other fields

$$\psi_L^\dagger \overline{\sigma}^\mu \psi_L A_\mu, \quad \psi_R^\dagger \sigma^\mu \psi_R A_\mu, \quad \overline{\Psi}\gamma^\mu \Psi A_\mu, \quad \phi\partial^\mu\phi A_\mu, \quad \phi^2 A_\mu A^\mu, \quad \phi A_\mu A^\mu.$$

However, *a theory with massive scalar interactions is not renormalizable!* Remember the quantity (in four space-time dimensions)

$$\Delta_i = 4 - a_i - \sum_f n_{if}(1 + s_f),$$

where s_f is power of the propagator of the field f: $1/(k^2)^{1-s_f}$, a_i is the number of derivatives in the vertex v_i and n_{if} is the number of fields of type f in the vertex i. For scalar field, we had $s_f = 0$ and for spinor we had $s_f = 1/2$. However, for the massive vector it holds

$$\tilde{\Delta}_F^{\mu\nu}(p) = \frac{i}{p^2 - m^2 + i\varepsilon}\left(-\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right) \sim \frac{p^\mu p^\nu}{p^2} \sim \text{const.}$$

Hence, $s_f = 1$. We can check that for all marginal interactions listed above, it holds $\Delta_i < 0$, so *massive vector fields have no renormalizable interactions!*. We can check explicitly that for $A_\mu A^\mu \partial_\nu A^\mu$, $\Delta_i = -3$ and for $\phi^2 A_\mu A^\mu$, $\Delta_i = -2$. As an EFT, the theory is valid for $E \ll m$, so that the contribution $p^\mu p^\nu/m^2 \sim E^2/m^2 \ll 1$.

13.3 Massless vector fields

The key insight for the massless case is

$$U(\Lambda)A_\nu(x)U^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu A_\mu(\Lambda x)+\partial_\mu\lambda_\Lambda(x).$$

The blue term is not covariant, so even though A_μ looks like a 4-vector, it is not!. The Lorentz group for massless particle is $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$. In order to leave the reference momentum $k^\mu = (k,0,0,k)^T$ invariant, it takes the form

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & -\alpha & -\beta & 0 \\ \alpha & 0 & \theta & -\alpha \\ \beta & -\theta & 0 & -\beta \\ 0 & \alpha & \beta & 0 \end{pmatrix}.$$

We parametrize it as

$$U(\mathbb{1}+\omega,1) = \mathbb{1} - i(\theta J^3 + \alpha A + \beta B),$$

where $A = K^1 + J^2$ and $B = K^2 - J^1$. We choose the eigenvalues for A and B to be zero and $J_3 = \pm 1$. The polarization vectors are

$$\epsilon^\mu_{\pm1}(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}.$$

We can check this for a general matrix for a rotation around z axis (J^3):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & s & 0 \\ 0 & -s & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ c \pm is \\ -is \pm c \\ 0 \end{pmatrix} = e^{\pm i\sigma}\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}.$$

The polarization vector generalizes to

$$\epsilon^\mu_{\pm1}(p) = L(p)^\mu{}_\nu \epsilon^\nu_{\pm1}(k).$$

The problem is that $\epsilon^0_{\pm1}(p) = 0$ which is not Lorentz invariant. Thus, we impose the Lorentz invariance condition $k_\mu\epsilon^\mu_{\pm1}(k) = 0$ or more generally $p_\mu\epsilon^\mu_{\pm1}(p) = 0$. The free field can be expressed as

$$A^\mu(x) = \sum_{\sigma=\pm1}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ipx}\epsilon^\mu(\mathbf{p},\sigma)a(\mathbf{p},\sigma) + e^{ipx}\epsilon^{*\mu}(\mathbf{p},\sigma)a^\dagger(\mathbf{p},\sigma)\right).$$

From $\epsilon^0_{\pm1}(p) = 0$ follows $A^0 = 0$ and $\partial_\mu A^\mu = 0$ ($\nabla\cdot\mathbf{A} = 0$). The Lorentz transformation of massless spin-1 field can be written infinitesimally as

$$\Lambda(\alpha,\beta,\theta=0)^\mu{}_\nu\epsilon^\nu_{\pm1}(k) = \epsilon^\mu_{\pm1}(k) - (\alpha \pm i\beta)\frac{k^\mu}{\sqrt{2}k} + \ldots$$

Moreover

$$\begin{aligned}\Lambda^\mu{}_\nu\epsilon^\nu_{\pm1}(p) &= (L(\Lambda p)w(\Lambda,p))^\mu{}_\nu\epsilon^\nu_{\pm1}(k) \\ &= L(\Lambda p)e^{\pm i\theta(\Lambda,p)}\left(\epsilon^\mu_{\pm1}(k) - (\alpha\pm i\beta)\frac{k^\mu}{\sqrt{2}k}\right) \\ &= e^{\pm i\theta(\Lambda,p)}\left(\epsilon^\mu_{\pm1}(\Lambda p) - (\alpha\pm i\beta)\frac{(\Lambda p)^\mu}{\sqrt{2}k}\right).\end{aligned}$$

The solution for $\epsilon^\mu_{\pm1}(p)$ is

$$e^{\pm\theta(\Lambda,p)}\epsilon^\mu_{\mp1}(p) = (\Lambda^{-1})^\mu{}_\nu\epsilon^\nu_{\pm1}(\Lambda p) - (\alpha \pm i\beta)\frac{1}{\sqrt{2}k}p^\mu.$$

Define $c_\pm = (\alpha \pm i\beta)p^\mu/(\sqrt{2}k)$. With one-particle-state the creation and annihilation operators transform as

$$U(\Lambda)a(\mathbf{p},\sigma)U^{-1}(\Lambda) = e^{\mp i\sigma\theta(\Lambda,p)}a(\Lambda p,\sigma)$$

so

$$\begin{aligned}
U(\Lambda)A^\mu U^{-1}(\Lambda) &= \sum_{\sigma=\pm1}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left(e^{-ipx}e^{\mp i\sigma\theta(\Lambda,p)}\epsilon^\mu(\mathbf{p},\sigma)a(\Lambda p,\sigma) + \text{h.c.}\right)\\
&= \sum_{\sigma=\pm1}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left(e^{-ipx}((\Lambda^{-1})^\mu{}_\nu\epsilon^\nu(\Lambda p,\sigma) + c_\sigma p^\mu)a(\Lambda p,\sigma) + \text{h.c.}\right)\\
&= (\Lambda^{-1})^\mu{}_\nu\sum_{\sigma=\pm1}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left(e^{-ip\Lambda x}\epsilon^\nu(\mathbf{p},\sigma)a(\mathbf{p},\sigma) + \text{h.c.}\right) + \partial\mu\lambda_\Lambda(x)\\
&= (\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x) + \partial^\mu\lambda_\Lambda(x).
\end{aligned}$$

The derivative term comes from $c_\sigma p^\mu$ in the integral and this term is not covariant. So we cannot just require that $\mathcal{L}$ is a scalar since, for example $m^2A_\mu A^\mu$ wouldn't transform as scalar. What we need is

1) Lorentz invariance and
2) Lagrangian unchanged by replacement $A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x)$.

The kinetic term already satisfies these requirements:

$$F_{\mu\nu} \to \partial_\mu(A_\nu + \partial_\nu\lambda) - \partial_\nu(A_\mu + \partial_\mu\lambda) = F_{\mu\nu} + (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\lambda = F_{\mu\nu}.$$

If we transform the field operator, we get

$$U(\Lambda)F_{\mu\nu}U^{-1}(x) \longrightarrow (\Lambda^{-1})^\sigma{}_\mu(\Lambda^{-1})^\rho{}_\nu F_{\sigma\rho}.$$

This is covariant and $F^{\mu\nu}$ is indeed a Lorentz tensor. We can also couple A^μ to conserved currents $\partial_\mu\mathcal{J}^\mu(x) = 0$:

$$\mathcal{L}'^{\text{int}} = \mathcal{L}^{\text{int}} + (\partial_\mu\lambda)\mathcal{J}^\mu \overset{\text{P.I.}}{=} \mathcal{L}^{\text{int}} - \lambda\partial_\mu\mathcal{J}^\mu = \mathcal{L}^{\text{int}}.$$

13.4 Einstein-Hilbert action from gauge redundancy

Consider a massless spin-2 particle $h_{\mu\nu}(x) = h_{\nu\mu}(x)$. Polarization terms describe helicity **2** eigenstates:

$$\epsilon^{\mu\nu}_{\pm2}(k) = \epsilon^\mu_{\pm1}(k)\epsilon^\nu_{\pm1}(k).$$

We can check that

$$R_z(\theta)^\mu{}_\rho R_z(\theta)^\nu{}_\sigma\epsilon^{\mu\nu}_{\pm2}(k) = e^{\pm i2\theta}\epsilon^{\mu\nu}_{\pm2}(k).$$

It holds

$$\epsilon^{\mu\nu}_{\pm2}(k) = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & \pm i & 0\\ 0 & \pm i & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

and $\epsilon^{\mu\nu}_{\pm 2}(p) = L(p)^{\mu}{}_{\rho} L(p)^{\nu}{}_{\sigma} \epsilon^{\rho\sigma}_{\pm 2}(k)$. The field expansion is

$$h^{\mu\nu}(x) = \sum_{\sigma=\pm 2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ipx} \epsilon^{\mu\nu}(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + \text{h.c.} \right).$$

Similarly, as before we can derive

$$e^{\mp 2i\theta(\Lambda,p)} \epsilon^{\mu\nu}_{\pm 2}(p) = (\Lambda^{-1})^{\mu}{}_{\rho} (\Lambda^{-1})^{\nu}{}_{\rho} \epsilon^{\sigma\rho}_{\pm 2}(\Lambda p) + p^{\mu} v^{\nu}_{\pm}(p) + p^{\nu} v^{\mu}_{\pm}(p)$$

where $v^{\mu}_{\pm}(p) = (\Lambda^{-1})^{\mu}{}_{\rho}(\alpha \pm i\beta)$. Finally, we get

$$U(\Lambda) h_{\rho\sigma} U^{-1}(\Lambda) = (\Lambda^{-1})^{\mu}{}_{\rho} (\Lambda^{-1})^{\nu}{}_{\rho} h_{\mu\nu}(\Lambda x) + \partial^{\mu} \xi^{\nu}_{\Lambda}(x) + \partial^{\nu} \xi^{\mu}_{\Lambda}(x).$$

So gravity also requires gauge redundancy (diffeomorphism invariance)

$$h'^{\mu\nu}(x) = h^{\mu\nu}(x) + \partial^{\mu} \xi^{\nu}(x) + \partial^{\nu} \xi^{\mu}(x).$$

The simplest way we can write it is using the linearized *Riemann tensor*

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} (\partial_{\nu}\partial_{\rho} h_{\mu\sigma} - \partial_{\mu}\partial_{\rho} h_{\nu\sigma} - \partial_{\nu}\partial_{\sigma} h_{\mu\rho} + \partial_{\mu}\partial_{\sigma} h_{\nu\rho}).$$

The *Einstein-Hilbert action* is defined as

$$S_{\text{EH}} = \frac{1}{16\pi M_{\text{p}}^2} \int \mathrm{d}^4 x \, \sqrt{g} R$$

where $R = g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma}$. The gravity can couple to conserved 2-tensors. It turns out that there is only one conserved 2-tensor in QFT – the energy momentum tensor $\partial_{\mu} T^{\mu\nu} = 0$ so the only possible interaction term is thus

$$\mathcal{L}^{\text{int}} \propto h_{\mu\nu} T^{\mu\nu}.$$

From this we can also conclude that there are no massless spin-3 particles. We would need a conserved 3-tensors to couple to the matter. They don't exist, so there doesn't exist such long-range force.

Now we know that gauge-invariant Lagrangian looks like

$$\mathcal{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \mathcal{J}_{\mu} A^{\mu}.$$

Assume that our theory has U(1) symmetry. Let's see how the QED coupling transforms:

$$\overline{\Psi} \gamma^{\mu} A_{\mu} \Psi \xrightarrow{\Lambda} \overline{\Psi} \gamma^{\mu} A_{\mu} \Psi + \overline{\Psi} \gamma^{\mu} \Psi \partial_{\mu} \alpha(x).$$

So we substitute $\Psi(x)$ with $e^{i\alpha(x)}(x)$ (which doesn't change Lagrangian, since the Lagrangian has this symmetry) and ∂_{μ} with

$$\boxed{\mathrm{D}_{\mu} = \partial_{\mu} - i A_{\mu}(x).}$$

This is called the *covariant derivative*. Then

$$\begin{aligned} \mathrm{D}'_{\mu} \Psi'(x) &= (\partial_{\mu} - i(A_{\mu} + \partial_{\mu}\alpha(x))) e^{i\alpha(x)} \Psi(x) \\ &= e^{i\alpha(x)} (\partial_{\mu} - iA_{\mu} - i(\partial_{\mu}\alpha(x)) + i(\partial_{\mu}\alpha(x))) \Psi(x) \\ &= e^{i\alpha(x)} \mathrm{D}_{\mu} \Psi(x). \end{aligned}$$

The electron kinetic term is now invariant:

$$i\overline{\Psi}'\not{\mathrm{D}}'\Psi' = i\overline{\Psi}e^{-i\alpha(x)}e^{i\alpha(x)}\not{\mathrm{D}}\Psi = i\overline{\Psi}\not{\mathrm{D}}\Psi.$$

In other words, we introduced a local U(1) redundancy

$$\begin{aligned}\Psi(x) &\to e^{i\alpha(x)}\Psi(x),\\ A_\mu(x) &\to A_\mu(x) + \partial_\mu\alpha(x).\end{aligned}$$

We found that the mass term $m\overline{\Psi}\Psi$ is invariant under global U(1) transformation, but the kinetic term $i\overline{\Psi}\not{\partial}\Psi$ is not. The problematic part is the derivative. Hence, we relaxed the global U(1) invariance to local invariance, and thus we replaced the normal derivative with the covariant derivative

$$\partial_\mu\Psi \to \mathrm{D}_\mu\Psi = (\partial_\mu - iA_\mu)\Psi.$$

This is exactly the same as in the case of connection (Christoffel symbol) in GR:

$$\partial_\mu V^\nu \to \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda}V^\lambda.$$

The covariant derivative transforms in a much more convenient way:

$$\not{\mathrm{D}}\Psi(x) \xrightarrow{\mathrm{U}(1)_{\mathrm{local}}} e^{i\alpha(x)}\not{\mathrm{D}}\Psi.$$

Since the photon has no charge it holds $\partial_\mu A_\nu = \mathrm{D}_\mu A_\nu$.

Let's consider a gauge invariant Lagrangian with a charged scalar ϕ and spinor Ψ and a massless vector A_μ field:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}(i\not{\mathrm{D}} - m)\Psi + (\mathrm{D}_\mu\phi)^\dagger(\mathrm{D}^\mu\phi) - \frac{m_\phi^2}{2}\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2.$$

We introduce the gauge coupling g, such that

$$\begin{aligned}A_\mu &\to gA_\mu\\ \mathrm{D}_\mu &= \partial_\mu - igA_\mu\\ A_\mu &\to A_\mu + \frac{1}{g}\partial_\mu\alpha(x)\\ \mathcal{L}^{\mathrm{int}} &= g\overline{\Psi}\gamma^\mu\Psi A_\mu = \mathcal{J}^\mu A_\mu.\end{aligned}$$

The issue we encounter is that the equation of motion $\partial_\nu F^{\mu\nu} = \mathcal{J}^\mu$ is not invertible. In the scalar case we knew we can invert the equation $(\Box + m^2)\phi = \mathcal{J}$ as $\phi = 1/(\Box + m^2)\mathcal{J}$. However, in the vector case we have

$$(\eta_{\mu\nu}(\partial^\alpha\partial_\alpha) - \partial_\mu\partial_\nu)A^\nu = J^\mu.$$

We know that if $A_\mu(x)$ is a solution, then $A_\mu(x) + \partial_\mu\lambda(x)$ will also be a solution. The operator on A_μ has some eigenvalues equal to zero, so we cannot invert it. We can solve the equation if we *fix the gauge* (essentially removing one degree of freedom). There are two common choices:

- *Lorentz gauge* $\partial_\mu A^\mu = 0$ and $\partial_\mu\partial^\mu\alpha = 0$,
- *Coulomb gauge* $\nabla\cdot\mathbf{A} = 0$ and $A^0 \equiv 0$.

Choosing the Lorentz gauge, we get $\Box A_\mu = \mathcal{J}_\mu$ for the equation of motion. In QFT2, we will meet the Faddeev-Popov method, which introduces new particles (called ghosts) to get rid of the gauge redundancy and this leads to BRST symmetry.

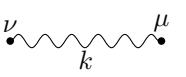
13.5 Quantum electrodynamics (QED)

After the gauge fixing we get the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \overline{\Psi}(i\not{\partial} - m)\Psi$$

with $\mathrm{D}_\mu = \partial_\mu - ieA_\mu$. The Feynman rules are

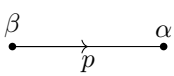
- Photon propagator:



$$= \tilde{\Delta}^F_{\mu\nu}(k) = \frac{i}{k^2 - i\varepsilon}\left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right).$$

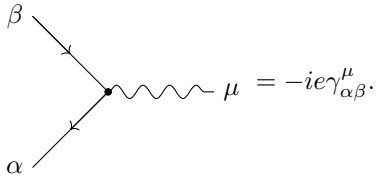
  We typically choose $\xi = 1$ (Feynman gauge).

- Electron propagator:



$$= \tilde{\Delta}^F_{\alpha\beta}(p) = \frac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon}.$$

- Electron-photon vertex:



$$= -ie\gamma^\mu_{\alpha\beta}.$$

We also have the following:

incoming electron	$u(\mathbf{p}, s)$
incoming positron	$\overline{v}(\mathbf{p}, s)$
incoming photon	$\epsilon^\mu(\mathbf{k}, \lambda)$
outgoing electron	$\overline{u}(\mathbf{p}, s)$
outgoing positron	$v(\mathbf{p}, s)$
outgoing photon	$\epsilon^{*\mu}(\mathbf{k}, \lambda)$

For QED, every vertex (the only vertex) has $\Delta_i = 0$ so by power-counting QED is renormalizable.

13.5.1 Renormalization of QED

Introduce the following bare quantities:

$$\Psi^0 = \sqrt{Z}\Psi, \quad A^0_\mu = \sqrt{Z_3}A_\mu, \quad m_0 = Z_m m, \quad e_0 = Z_e e, \quad \xi_0 = Z_\xi \xi.$$

The Lagrangian with counter-terms is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \overline{\Psi}(i\not{\partial} - m)\Psi + \\ &- \frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} - \left(\frac{Z_3}{Z_\xi} - 1\right)\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \\ &+ (Z_2 - 1)\overline{\Psi}i\not{\partial}\Psi - (Z_2 Z_m - 1)m\overline{\Psi}\Psi + (Z_e Z_2\sqrt{Z_3} - 1)e\overline{\Psi}\not{A}\Psi.\end{aligned}$$

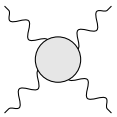
We will show that quantum corrections respect gauge redundancy:

$$\overline{\Psi}_0 \not{D}_0 \Psi_0 = \overline{\Psi}_0(\not{\partial} - ie_0 \not{A}_0)\Psi_0 = Z_2 \overline{\Psi}(\not{\partial} - iZ_e e\sqrt{Z_3}\not{A})\Psi.$$

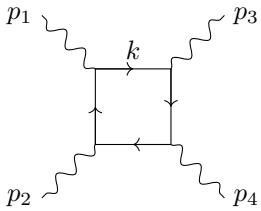
So we need

$$\boxed{Z_e\sqrt{Z_3} = 1}$$

which is called the *Ward identity* (and we will show it now). From the terms $A_\mu A^\mu$, $\partial_\mu A^\nu A^\mu A_\nu$, $(A^\mu A_\mu)^2$ none of them carries gauge redundancy. Consider the following.



For example



$$\sim \int \frac{\mathrm{d}^4 k}{(\not{k} - m)^4} \sim \int \frac{\mathrm{d}^4 k\, p^4}{k^8} = \text{finite.}$$

When we replace $A_\mu(p) \to A_\mu(p) + p_\mu \alpha$, we get $p_\mu M^\mu = 0$.

On-shell renormalization

First consider the electron 2-point function

$$S_0^{(2)}(p) = \int \mathrm{d}^4(x-y) e^{ip(x-y)} \langle \Omega | \Psi_\alpha^0(x) \overline{\Psi}_\beta^0(y) | \Omega \rangle \xrightarrow{p^2 \to m^2} Z_2 \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon} + \text{non-singuar terms.}$$

Summing over all one-particle irreducible (1PI) diagrams we get

$$S^{(2)}(p) = \frac{i}{\not{p} - m + i\varepsilon} + \frac{i}{\not{p} - m + i\varepsilon} \underbrace{(-i\Sigma(p,m))}_{\text{1PI}} \frac{i}{\not{p} - m + i\varepsilon} + \cdots = \frac{i}{\not{p} - m - \Sigma(p,m)}$$
$$\xrightarrow{p^2 \to m^2} \frac{i}{\not{p} - m + i\varepsilon}$$

where $m = 0{,}511\,\text{MeV}$ is the mass of the electron. In order for residue to be one it must hold

$$\left.\frac{\partial \Sigma}{\partial \not{p}}\right|_{p^2 = m^2} = 0, \quad \Sigma(p,m)|_{p^2=m^2} = 0.$$

For the photon 2-point function we get

$$\begin{aligned} G^{(2)}(k) &= \frac{i}{k^2 + i\varepsilon}\left(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}(1-\xi)\right) + \\ &\quad + \frac{i}{k^2 + i\varepsilon}\left(-\eta_{\mu\rho} + \frac{k_\mu k_\rho}{k^2}(1-\xi)\right) i\Pi_{\rho\sigma}(k) \frac{i}{k^2 + i\varepsilon}\left(-\eta_{\sigma\nu} + \frac{k_\sigma k_\nu}{k^2}(1-\xi)\right) + \dots \\ &= \frac{i}{k^2 + i\varepsilon}\left[\left(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1 - \Pi(k^2)} - \xi \frac{k_\mu k_\nu}{k^2}\right] \end{aligned}$$

where we assumed $\Pi_{\mu\nu}(k^2) = (k^2\eta_{\mu\nu} - k_\mu k_\nu)\Pi(k^2) + k_\mu k_\nu \Pi_2(k^2)$. We can show (using Ward identity) that $\Pi_2(k^2) = 0$ at any loop order. We determine the value of Z_3 from

$$G^{(2)}(k) \xrightarrow{k^2\to 0} Z_3 \frac{i}{k^2+i\varepsilon}\left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) + \text{non-singular terms.}$$

In terms of bare fields, the photon 2-point function is

$$G_0^{(2)} = \frac{i}{k^2+i\varepsilon}\left[\left(-\eta_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1-\Pi_0(k^2)} - \xi_0\frac{k_\mu k_\nu}{k^2}\right] = \frac{i}{k^2+i\varepsilon}\left[\left(-\eta_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1-\Pi_0(k^2)} - \xi Z_\xi \frac{k_\mu k_\nu}{k^2}\right]$$
$$\xrightarrow{k^2\to 0} Z_3\frac{i}{k^2+i\varepsilon}\left(-\eta_{\mu\nu}+(1-\xi)\frac{k_\mu k_\nu}{k^2}\right) + \text{non-singular terms.}$$

From this it follows that $Z_3 = Z_\xi$ and

$$Z_3 = \frac{1}{1-\Pi_0(k^2=0)}.$$

To renormalize Z_e, we consider $|\mathbf{q}| \ll m_e$ to be the renormalization point. Consider the emission of a photon $e^- \to e^- + \gamma$:

$$\langle e^-(p',s')\gamma(q,\lambda);\text{out}|e^-(p,s);\text{in}\rangle = (2\pi)^4\delta^{(4)}(p'+q-p)ie\overline{u}(\mathbf{p},s)\gamma^\mu u(\mathbf{p},s)\epsilon_\mu^*(\mathbf{q},\lambda).$$

We will require that this holds for all orders for $q^\mu \to 0$. We can write the vertex as

$$(2\pi)^4\delta^{(4)}(p'+q-p)\sqrt{Z_3}Z_2 ie\overline{u}(\mathbf{p}',s')\Gamma_0^\mu(p',p)u(\mathbf{p},s)\epsilon_\mu^*(\mathbf{q},\lambda)$$

where

$$\Gamma_0^\mu(p,p') = \text{sum of all } ee\gamma \text{ diagrams}$$

is a 4×4 matrix in Dirac spinor space. The basis for Γ_0^μ is

$$\mathbb{1}, \gamma_\mu, \gamma_5, \gamma_\mu\gamma_5, \frac{2}{i}\sigma^{\mu\nu} \coloneqq [\gamma^\mu,\gamma^\nu].$$

One can check that there are exactly 16 degrees of freedom in this basis. Using the Ward identity $q_\mu\Gamma_0^\mu = 0$ we can show that we can write

$$\Gamma_0^\mu(p,p') = \gamma^\mu A + (p^\mu + p'^\mu)B + (p'^\mu - p^\mu)C.$$

We can simplify this further using the *Gordon identity*

$$\overline{u}(p',s')\gamma^\mu u(p,s) = \overline{u}(p',s')\left(\frac{p'^\mu+p^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p_\nu - p'_\nu)}{2m}\right)u(p,s).$$

The final result is

$$\Gamma_0^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

The functions F_1 and F_2 are called *form factors*. For $q^\mu \to 0$ the second term vanishes and the on-shell renormalization condition involves F_1^0 only:

$$\frac{1}{Z_e} = \sqrt{Z_3}Z_2 F_1^0(q^2=0,m).$$

For the renormalized vertex we get $F_1(q^2=0,m) = 1$.

13.6 Ward identity

We will show that

$$F_1^0(q^2=0,m^2)=\frac{1}{Z_2}$$

which will imply $Z_e=Z_3^{-1/2}$ and that we can joint two of the counter-terms to $Z_2\overline{\Psi}\not{D}\Psi$.

Proof.

$$\begin{aligned}&\partial_\mu\langle\Omega|T\{\mathcal{J}_0^\mu\Psi_\alpha^0(z_1)\overline{\Psi}_\beta^0(z_2)\}|\Omega\rangle\\&\quad=\delta^{(4)}(x-z_1)\langle\Omega|T\{\Psi_\alpha^0(z_1)\overline{\Psi}_\beta^0(z_2)\}|\Omega\rangle-\delta^{(4)}(x-z_2)\langle\Omega|T\{\Psi_\alpha^0(z_1)\overline{\Psi}_\beta^0(z_2)\}|\Omega\rangle.\end{aligned}$$

When we Fourier transform the left-hand side, we get

$$(2\pi)^4\delta^{(4)}(q+p'-p)(-iq_\mu)S^0_{\alpha\alpha'}\hat{\Gamma}^{0\mu}_{\alpha'\beta'}S^0_{\beta'\beta}(p).$$

However, $\hat{\Gamma}$ is not yet Γ. Namely, $\hat{\Gamma}$ also includes the $\Pi_{\mu\nu}$ factors from the photon self-energy. But we can contract $q^\mu\Pi^{\mu\nu}=0$, so $q_\mu\hat{\Gamma}^\mu=q_\mu\Gamma^\mu$. The Fourier transformed right-hand side is

$$(2\pi)^4\delta^{(4)}(q+p'-p)(-1)(S^0_{\alpha\beta}(p)-S^0_{\alpha\beta}(p')).$$

Combining both sides gives

$$(p-p')_\mu\Gamma_0^\mu(p',')=iS_0^{-1}(p')-iS_0^{-1}(p).$$

In the $q\to0$ limit, we get

$$\boxed{\Gamma^\mu(p,p)=i\frac{\partial}{\partial p^\mu}S_0^{-1}(p).}$$

We take the on-shell limit $p^2\to m^2$ and $S_2(p)\to iZ_2/(\not{p}-m+i\varepsilon)$. Then

$$\gamma^\mu F_1^0(q^2=0,m)=\frac{i}{iZ_2}\frac{\partial}{\partial p_\mu}(\not{p}-m)=\frac{\gamma^\mu}{Z_2}.$$

It follows that $Z_e=Z_3^{-1/2}$. □

13.6.1 Scale dependence of QED coupling

The charge correction is

$$e^2_{\text{eff}}(q^2)=\frac{e^2}{1-\Pi(q^2)}.$$

The one-loop correction is

$$i\Pi_{\mu\nu}(q^2)=(-1)(ie)^2\tilde{\mu}^{2\epsilon}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\operatorname{tr}(\gamma_\mu i(\not{k}-m)\gamma_\nu i(\not{k}+\not{q}+m))}{(k^2-m^2)((k+q)^2-m^2)}+(-i)(Z_3-1)(q^2\eta_{\mu\nu}-q_\mu q_\nu).$$

After a long calculation, we get

$$\Pi(q^2)=-\frac{2\alpha_{\text{em}}}{\pi}\Gamma(\epsilon)\int_0^1\mathrm{d}x\,x(1-x)\left(\frac{m^2-x(1-x)q^2-i\varepsilon}{4\pi\tilde{\mu}^2}\right)^\epsilon-(Z_3-1).$$

In the on-shell renormalization it holds

$$Z_3 = 1 - \frac{2\alpha_{\text{em}}}{3\pi}\left(\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2}\right)$$

and hence

$$\Pi(q^2) = \frac{2\alpha_{\text{em}}}{\pi}\int_0^1 \mathrm{d}x\, x(1-x)\ln\left(\frac{m^2 - x(1-x)q^2 - i\varepsilon}{m^2}\right).$$

Let's consider scattering two electrons with momenta $\mathbf{p}, \mathbf{p}' \ll m$. Then $q^2 = -\mathbf{q}^2$ and $\Pi(-\mathbf{q}^2)$ is positive and increasing with $\mathbf{q}^2$. The effective interaction $e_{\text{eff}}(\mathbf{q}^2)$ increases with $\mathbf{q}^2$. The classical description of the electron potential is

$$V(\mathbf{r}) = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{r}\cdot\mathbf{q}} \frac{e^2}{-\mathbf{q}^2} = \frac{\alpha_{\text{em}}}{|\mathbf{r}|}.$$

Since α changes with energy, the electric potential also changes when we come very close to the electron. The interpretation of this is that the vacuum is polarized and screens the charge.

The QED renormalization also describes the anomalous magnetic moment. The QM Hamiltonian is

$$H = \frac{p^2}{2m} + V(r) + \frac{e}{2m}\mathbf{B}\cdot(\mathbf{L} + g\mathbf{S}).$$

Using Dirac equations, we can show that $g = 2$. We also have the following relations

$$\mathbf{S} = \frac{\boldsymbol{\sigma}}{2}, \quad H = -\boldsymbol{\mu}\cdot\mathbf{B}, \quad \boldsymbol{\mu} = -g\frac{e}{m}\mathbf{S}.$$

The magnetic moment is $\overline{\Psi} g\frac{e}{4} F_{\mu\nu}\sigma^{\mu\nu}\Psi$ and combining this with the expression for Γ_0^μ we get

$$g = 2 + 2F_2(q^2 \to 0).$$

After a long calculation we get $F_2(0) = \alpha/(2\pi)$ and

$$g = 2 + \frac{\alpha}{\pi} = 2{,}002\,32.$$

This is called the *Landé g-factor*, and it is calculated up to fifth order in α. The measurement is

$$g = 2{,}002\,319\,304\,361\,7 \pm 0{,}000\,000\,000\,000\,3.$$

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