

Faculty of physics

Mathematical complement to Quantum Field Theory

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Disclaimer: these notes are not official and were not endorsed by professor. The content is almost entirely based on the lectures, but occasionally the notes were modified or extended slightly to improve clarity or add extra information. In particular, the mistakes are almost surely mine. If you find any, please let me know at simon.bukovsek@tum.de or Simon.Bukovsek@campus.lmu.de.

1 Orthogonal direct sums of Hilbert spaces

Definition 1.1. Let $\Lambda \neq \varnothing$ and for each $\lambda \in \Lambda$ $\mathscr{H}_\lambda$ a ($\mathbb{C}$-)Hilbert space. Then

$$\bigoplus_{\lambda\in\Lambda} \mathscr{H}_\lambda \coloneqq \left\{ \phi : \Lambda \to \bigcup_{\lambda\in\Lambda} \mathscr{H}_\lambda : \phi(\lambda) \in \mathscr{H}_\lambda \forall \lambda \in \Lambda, \sup\left(\sum_{\lambda\in E} \|\phi(\lambda)\|^2_{\mathscr{H}_\lambda} : E \subseteq \Lambda, E \text{ finite} \right) < \infty \right\}$$

is the *direct orthogonal sum* of the Hilbert spaces $\mathscr{H}_\lambda$, $\lambda \in \Lambda$.

One can define the set of all finite subsets $\mathcal{C}(\Lambda) \coloneqq \{E \subseteq \Lambda : E \text{ finite}\}$. With this, a limit over all elements in Λ can be defined

$$a = \lim_{E\in\mathcal{C}(\Lambda)} \sum_{\lambda\in E} \phi(\lambda) \iff \forall \varepsilon > 0 \exists H(\varepsilon) \in \mathcal{C}(\Lambda) : \left\| a - \sum_{\lambda\in E} \phi(\lambda) \right\| < \varepsilon \forall E \supseteq H(\varepsilon).$$

Theorem 1.1. *A direct orthogonal sum $\bigoplus_{\lambda\in\Lambda} \mathscr{H}_\lambda$ is a Hilbert space equipped with scalar product*

$$\langle \cdot, \cdot \rangle_{\bigoplus_{\lambda\in\Lambda} \mathscr{H}_\lambda} : \left(\bigoplus_{\lambda\in\Lambda} \mathscr{H}_\lambda \right) \times \left(\bigoplus_{\lambda\in\Lambda} \mathscr{H}_\lambda \right) \to \mathbb{C}$$

and

$$((\phi(\lambda))_{\lambda\in\Lambda}, (\psi(\lambda))_{\lambda\in\Lambda}) \mapsto \lim_{E\in\mathcal{C}(\Lambda)} \sum_{\lambda\in E} \langle \phi(\lambda), \psi(\lambda) \rangle_{\bigoplus_{\lambda\in\Lambda} \mathscr{H}_\lambda} = \lim_{\lambda\in\Lambda} \langle \phi(\lambda), \psi(\lambda) \rangle_{\bigoplus_{\lambda\in\Lambda} \mathscr{H}_\lambda}.$$

Hints for the proof: if

$$\phi = (\phi(\lambda))_{\lambda\in\Lambda}, \psi(\psi(\lambda))_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda} \mathscr{H}_\lambda$$

and $\alpha, \beta \in \mathbb{C}$ and $E \subseteq \Lambda$ is finite, then

$$\sum_{\lambda\in E} \|\alpha\phi(\lambda) + \beta\psi(\lambda)\|^2 \leq \sum_{\lambda\in E} \left(2|\alpha|^2 \|\phi(\lambda)\|^2 + 2|\beta|^2 \|\psi(\lambda)\|^2 \right).$$

Taking supremum on each side:

$$\begin{aligned} \sup_{E\in\mathcal{C}(\Lambda)} \sum_{\lambda\in E} \|\alpha\phi(\lambda) + \beta\psi(\lambda)\|^2 &\leq \sup_{E\in\mathcal{C}(\Lambda)} \sum_{\lambda\in E} \left(2|\alpha|^2 \|\phi(\lambda)\|^2 + 2|\beta|^2 \|\psi(\lambda)\|^2 \right) \\ &\leq 2|\alpha|^2 \sup_{E\in\mathcal{C}(\Lambda)} \sum_{\lambda\in E} \|\phi(\lambda)\|^2 + 2|\beta|^2 \sup_{E\in\mathcal{C}(\Lambda)} \sum_{\lambda\in E} \|\psi(\lambda)\|^2 \\ &< \infty. \end{aligned}$$

It is also useful to use the *Cauchy-Schwarz inequality*:

$$\sum_{\lambda\in E} |\langle \phi(\lambda), \psi(\lambda) \rangle| \leq \left(\sum_{\lambda\in E} \|\phi(\lambda)\|^2 \right)^{\frac{1}{2}} \left(\sum_{\lambda\in E} \|\psi(\lambda)\|^2 \right)^{\frac{1}{2}}.$$

2 Tensor products

Definition 2.1. If V_i, $i \in I$ (I is finite) are K-vector spaces (K field), then the *tensor product* of V_i, $i \in I$ is

$$\left(\bigotimes_{i \in I} V_i, \tau \right)$$

where $\bigotimes_{i \in I} V_i$ is a K-vector space and

$$\tau : \prod_{i \in I} V_i \to \bigotimes_{i \in I} V_i$$

is a multilinear map fulfilling the following statement. For each multilinear map $\phi : \prod_{i \in I} V_i \to W$ (W being a K-vector space) there is a unique K-linear map $f : \bigotimes_{i \in I} V_i \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} \prod_{i \in I} V_i & \xrightarrow{\phi} & W \\ & {\scriptstyle \tau} \searrow \quad \nearrow {\scriptstyle f} & \\ & \bigotimes_{i \in I} V_i & \end{array}$$

or equivalently that $\phi = f \circ \tau$.

Example 1. Let's check that $\mathbb{R}^m \otimes \mathbb{R}^m = M(m \times n, \mathbb{R})$. Define

$$E_{ij} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & 1 & \vdots \\ 0 & \dots & 0 \end{bmatrix},$$

where the only 1 is in i-th column and j-th row. Matrices $(E_{ij})_{ij}$ form a base for $M(m \times n, \mathbb{R})$. Let $\tau : \mathbb{R}^m \times \mathbb{R}^n \to M(m \times n, \mathbb{R})$ be a multilinear map with

$$\left(\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right) \mapsto \begin{bmatrix} x_1 y_1 & \dots & x_1 y_n \\ \vdots & & \vdots \\ x_m y_1 & \dots & x_m y_n \end{bmatrix}.$$

Then $f(E_{ij}) = \phi(\mathbf{e}_i, \mathbf{e}_j)$, where $\mathbf{e}$ is a standard base element.

2.1 Tensor product of Hilbert spaces

Let $\mathscr{H}_1, \mathscr{H}_2$ be $\mathbb{C}$-Hilbert spaces and $x_1 \in \mathscr{H}_1$, $x_2 \in \mathscr{H}_2$ given. Define a bilinear map $\mathscr{H}_1 \times \mathscr{H}_2 \to \mathbb{C}$ with the prescription $(y_1, y_2) \mapsto \langle x_1, y_1 \rangle_{\mathscr{H}_1} \langle x_2, y_2 \rangle_{\mathscr{H}_2}$. There exists exactly one linear map $\phi_{x_1,x_2} : \mathscr{H}_1 \otimes \mathscr{H}_2 \to \mathbb{C}$ such that $\phi_{x_1,x_2}(y_1 \otimes y_2) = \langle x_1, y_1 \rangle_{\mathscr{H}_1} \langle x_2, y_2 \rangle_{\mathscr{H}_2}$.

$$Z = \sum_{j=1}^{m} y_{1,j} \otimes y_{2,j} \in \mathscr{H}_1 \otimes \mathscr{H}_2 \implies \phi_Z : \mathscr{H}_1 \otimes \mathscr{H}_2 \to \mathbb{C}$$

such that

$$\psi_Z(x_1 \otimes x_2) = \phi_{x_1,x_2}(Z) \implies \langle \cdot, \cdot \rangle_{\mathscr{H}_1 \otimes \mathscr{H}_2} : (\mathscr{H}_1 \otimes \mathscr{H}_2) \times (\mathscr{H}_1 \otimes \mathscr{H}_2) \to \mathbb{C}, (Z, Y) \mapsto \psi_Z(Y).$$

In special case

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle_{\mathscr{H}_1} \langle y_1, y_2 \rangle_{\mathscr{H}_2}$$

The Hilbert space tensor product $\mathscr{H}_1 \hat{\otimes} \mathscr{H}_2$ is the completion of $\mathscr{H}_1 \otimes \mathscr{H}_2$ with respect to the norm given by the scalar product

$$\mathscr{H}_1 \hat{\otimes} \ldots \hat{\otimes} \mathscr{H}_n = \left(\left(\mathscr{H}_1 \hat{\otimes} \mathscr{H}_2 \right) \ldots \hat{\otimes} \mathscr{H}_n \right).$$

Definition 2.2. The full *Fock space* over $\mathscr{H}$ is defined as

$$\mathcal{F}(\mathscr{H}) := \bigoplus_{n=0}^{\infty} \mathscr{H}^{\hat{\otimes} n},$$

where $\mathscr{H}^{\hat{\otimes} 0} := \mathbb{C}$ and

$$\mathscr{H}^{\hat{\otimes} n} = \underbrace{\mathscr{H} \hat{\oplus} \ldots \hat{\oplus} \mathscr{H}}_{n\text{-times}}.$$

Lemma 2.1. *Let $\mathscr{H}$ be a Hilbert space and*

$$\pi = \begin{pmatrix} 1 & \ldots & n \\ \pi(1) & \ldots & \pi(n) \end{pmatrix} \in \gamma_n$$

a permutation then there is a unitary map $\hat{p}_\pi : \mathscr{H}^{\hat{\otimes} n} \to \mathscr{H}^{\hat{\otimes} n}$ such that $\hat{p}_\pi(x_1 \otimes \cdots \otimes x_n) = x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(n)}$. This map has the following properties:

1. $(\hat{p}_\pi)^{-1} = (\hat{p}_\pi)^* = \hat{p}_{\pi^{-1}}$,
2. $\hat{p}_{\mathrm{id}} = \mathrm{id}_{\mathscr{H}^{\hat{\otimes} n}}$,
3. $\hat{p}_\pi \circ \hat{p}_\sigma = \hat{p}_{\sigma \circ \pi}$.

Proof. We define a map $\phi_\pi : \mathscr{H}^n \to \mathscr{H}^{\otimes n}$ with $\phi_\pi(x_1, \ldots, x_n) := x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(n)}$, which is multilinear. Using universal property it can be show that there exists a unique linear map $p_\pi : \mathscr{H}^{\otimes n} \to \mathscr{H}^{\otimes n}$ such that $\phi = p_\pi \circ \tau$, i.e.

$$\phi(x_1, \ldots, x_n) = p_\pi(\tau(x_1, \ldots, x_n)) = x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(n)}.$$

It holds

$$\begin{aligned} \langle p_\pi(x_1 \otimes \cdots \otimes x_n), p_\pi(y_1 \otimes \cdots \otimes y_n) \rangle &= \langle x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(n)}, y_{\pi^{-1}(1)} \otimes \cdots \otimes y_{\pi^{-1}(n)} \rangle \\ &= \langle x_{\pi^{-1}(1), y_{\pi^{-1}(1)}} \rangle_{\mathscr{H}} \ldots \langle x_{\pi^{-1}(n), y_{\pi^{-1}(n)}} \rangle_{\mathscr{H}} \\ &= \langle x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n \rangle. \end{aligned}$$

It follows that p_π is bounded and $\|p_\pi(x_1 \otimes \cdots \otimes x_n)\| = \|x_1 \otimes \cdots \otimes x_n\|$ and also for linear combinations. Then take a continuous extension $\hat{p}_\pi$ of p_π to $\mathscr{H}^{\hat{\otimes} n}$. The map $\hat{p}_\pi$ has the same properties as p_π. If $x, y \in \mathscr{H}^{\hat{\otimes} n}$ then there are $(x_m)_{m \in \mathbb{N}}$, $(y_m)_{m \in \mathbb{N}}$ in $\mathscr{H}^{\otimes n}$ such that $x_m \to x$ and $y_m \to y$. Then

$$\langle \hat{p}_\pi(x), \hat{p}_\pi(y) \rangle = \lim_{m \to \infty} \langle p_\pi(x), p_\pi(y) \rangle = \lim_{m \to \infty} \langle x_m, y_m \rangle = \langle x, y \rangle.$$

We used the fact that $\langle p_\pi(z), p_\pi(w) \rangle = \langle z, w \rangle$ for all $z, w \in \mathscr{H}^{\otimes n}$. □

Lemma 2.2. *Let $\mathscr{H}$ be a $\mathbb{C}$-Hilbert space. Then*

$$\hat{S}_n := \frac{1}{n!}\sum_{\pi\in\gamma_n}\hat{p}_\pi : \mathscr{H}^{\hat{\otimes}n} \to \mathscr{H}^{\hat{\otimes}n} \quad \text{and} \quad \hat{A}_n := \frac{1}{n!}\sum_{\pi\in\gamma_n}\operatorname{sign}(\pi)\hat{p}_\pi : \mathscr{H}^{\hat{\otimes}n} \to \mathscr{H}^{\hat{\otimes}n}$$

are orthogonal projections and $\hat{S}_n[\mathscr{H}^{\hat{\otimes}n}]$ and $\hat{A}_n[\mathscr{H}^{\hat{\otimes}n}]$ are Hilbert spaces.

Proof. It holds

$$\hat{S}_n \circ \hat{S}_n = \left(\frac{1}{n!}\sum_{\pi\in\gamma_n}\hat{p}_\pi\right)\circ\left(\frac{1}{n!}\sum_{\sigma\in\gamma_n}\hat{p}_\sigma\right) = \left(\frac{1}{n!}\right)^2\sum_{\pi\in\gamma_n}\sum_{\sigma\in\gamma_n}\hat{p}_{\sigma\circ\pi} = \frac{1}{n!}\sum_{\tau\in\gamma_n}\hat{p}_\tau = \hat{S}_n$$

and

$$(\hat{S}_n)^* = \frac{1}{n!}\sum_{\pi\in\gamma_n}\hat{p}_\pi^* = \frac{1}{n!}\sum_{\pi\in\gamma_n}\hat{p}_{\pi^{-1}} = \hat{S}_n.$$

The last equals sign holds because $\{\pi^{-1}|\pi\in\gamma_n\} = \gamma_n$. These two facts ($\hat{S}_n\circ\hat{S}_n = \hat{S}_n$ and $(\hat{S}_n)^* = \hat{S}_n$) determine that $\hat{S}_n$ is indeed an orthogonal projection. The same holds for $\hat{A}_n$ where we have to use $\operatorname{sign}(\pi)\operatorname{sign}(\sigma) = \operatorname{sign}(\pi\circ\sigma)$. Since $\hat{S}_n$ is an orthogonal projection $\mathrm{id}_{\mathscr{H}^{\hat{\otimes}n}} - \hat{S}_n$ is also an orthogonal projection on $(\hat{S}_n[\mathscr{H}^{\hat{\otimes}n}])^\perp$. The symbol $\perp$ denotes an orthogonal complement. If T is a densely defined operator and T^* exists, then $\operatorname{Ker}(T^*) = (\operatorname{Ran}(T))^*$ is a closed subspace. Since

$$\hat{S}_n[\mathscr{H}^{\hat{\otimes}n}] = \operatorname{Ker}(\mathrm{id}_{\mathscr{H}^{\hat{\otimes}n}} - \hat{S}_n) = \operatorname{Ker}((\mathrm{id}_{\mathscr{H}^{\hat{\otimes}n}} - \hat{S}_n)^*) = (\operatorname{Ran}(\mathrm{id}_{\mathscr{H}^{\hat{\otimes}n}} - \hat{S}_n))^\perp$$

is closed subset of $\mathscr{H}^{\hat{\otimes}n}$ it is also complete. □

Definition 2.3. Let $\mathscr{H}$ be a Hilbert space. Then

$$\mathcal{F}_b(\mathscr{H}) := \bigoplus_{n=0}^{\infty}\hat{S}_n[\mathscr{H}^{\hat{\otimes}n}]$$

with $\hat{S}_0[\mathscr{H}^{\hat{\otimes}0}] := \mathbb{C}$ is called the *bosonic Fock space* and

$$\mathcal{F}_f(\mathscr{H}) := \bigoplus_{n=0}^{\infty}\hat{A}_n[\mathscr{H}^{\hat{\otimes}n}]$$

with $\hat{A}_0[\mathscr{H}^{\hat{\otimes}0}] := \mathbb{C}$ is called the *fermionic Fock space*.

Lemma 2.3.

$$\mathcal{F}_{b,e} := \{\tilde{\phi} = (\phi_n)_{n\in\mathbb{N}} \in \mathcal{F}_b(\mathscr{H}) : \text{ There is a } N_{\tilde{\phi}} \in \mathbb{N} \text{ such that } \phi_n = 0\ \forall n \geq N_{\tilde{\phi}}\}$$

is a dense subspace of $\mathcal{F}_b(\mathscr{H})$.

Proof. Exercise! □

3 Creation and annihilation operators

Remark. Let $\mathscr{H}$ be a Hilbert space, $n \in \mathbb{N}$, $\psi \in \mathscr{H}$. Then

$$b^+_{n+1} : \mathscr{H}^{\hat{\oplus} n} \to \mathscr{H}^{\hat{\oplus}(n+1)}, \quad b^+_{n+1}(\psi)[\varphi_1 \otimes \cdots \otimes \varphi_n] \coloneqq \sqrt{n+1}\psi \otimes \varphi_1 \otimes \cdots \otimes \varphi_n$$

$$b^-_{n-1} : \mathscr{H}^{\hat{\oplus} n} \to \mathscr{H}^{\hat{\oplus}(n-1)}, \quad b^-_{n-1}(\psi)[\varphi_1 \otimes \cdots \otimes \varphi_n] \coloneqq \sqrt{n}\langle \psi, \varphi_1\rangle \varphi_2 \otimes \cdots \otimes \varphi_n$$

define operators because

$$\|b^+_{n+1}(\psi)[\varphi_1 \otimes \cdots \otimes \varphi_n]\| \leq \sqrt{n+1}\|\psi\| \, \|\varphi_1 \otimes \ldots \varphi_n\|$$

implies $\|b^+_{n+1}(\psi)\| \leq \sqrt{n+1}\|\psi\|$ and

$$\|b^-_{n-1}(\psi)[\varphi_1 \otimes \cdots \otimes \varphi_n]\| \leq \sqrt{n}\|\psi\| \, \|\varphi_1\| \, \|\varphi_2 \otimes \ldots \varphi_n\|$$

implies $\|b^-_{n-1}(\psi)\| \leq \sqrt{n}\|\psi\|$.

Lemma 3.1. *It holds* $(b^+_{n+1}(\psi))^* = b^-_n(\psi)$.

Proof. For $\varphi_1, \ldots, \varphi_n, \phi_1, \ldots, \phi_{n+1} \in \mathscr{H}$

$$\begin{aligned}\langle b^+_{n+1}(\psi)[\varphi_1 \otimes \cdots \otimes \varphi_n], \phi_1 \otimes \cdots \otimes \phi_{n+1}\rangle &= \langle \sqrt{n+1}\psi \otimes \varphi_1 \otimes \cdots \otimes \varphi_n, \phi_1 \otimes \cdots \otimes \phi_{n+1}\rangle \\ &= \sqrt{n+1}\langle \psi, \phi_1\rangle \cdot \langle \varphi_1, \phi_2\rangle \cdot \cdots \cdot \langle \varphi_n, \phi_{n+1}\rangle \\ &= \langle \varphi_1 \otimes \cdots \otimes \varphi_n, b^-_n(\psi)[\phi_1 \otimes \cdots \otimes \phi_{n+1}]\rangle.\end{aligned}$$

Now take the linear combinations

$$\langle b^+_{n+1}(\psi)[x], y\rangle_{\mathscr{H}^{\hat{\otimes}(n+1)}} = \langle x, b^-_n(\psi)[y]\rangle_{\mathscr{H}^{\hat{\otimes} n}}$$

for $x \in \mathscr{H}^{\otimes n}$ and $y \in \mathscr{H}^{\otimes(n+1)}$ and extend them by continuity to $\mathscr{H}^{\hat{\otimes} n}$ and $\mathscr{H}^{\hat{\otimes}(n+1)}$. □

Remark. For $\psi \in \mathscr{H}$ and $n \in \mathbb{N}_0$

$$a^+_{n+1}(\psi) : \hat{S}_n[\mathscr{H}^{\hat{\otimes} n}] \to \hat{S}_{n+1}[\mathscr{H}^{\hat{\otimes}(n+1)}]$$

with $\psi_n \mapsto \sqrt{n+1}\hat{S}_{n+1}[\psi \otimes \psi_n]$ defines continuous linear operator. Since

$$\|\sqrt{n+1}\hat{S}_{n+1}[\psi \otimes \psi_n]\| \leq \sqrt{n+1}\|\psi \otimes \psi_n\| \leq \sqrt{n+1}\|\psi\| \, \|\psi_n\|$$

it holds $\|a^+_{n+1}(\psi)\| \leq \sqrt{n+1}\|\psi\|$. With this we can define a *creation operator*

$$a^+(\psi) : \mathcal{F}_{b,e} \to \mathcal{F}_{b,e}$$

$$(\psi_n)_{n\in\mathbb{N}_0} \mapsto (0, a^+_1(\psi)[\psi_0], a^+_2(\psi)[\psi_1], \ldots).$$

This operator is well-defined, since we have defined it on a subspace $\mathcal{F}_{b,e}$ (however it is not bounded).

The map $\phi : \mathscr{H}^n \to \hat{S}_{n-1}[\mathscr{H}^{\hat{\otimes}(n-1)}]$

$$(\psi_1, \ldots, \psi_n) \mapsto \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \langle \psi, \psi_j\rangle \hat{S}_{n-1}[\psi_1 \otimes \cdots \otimes \psi_{j-1} \otimes \psi_{j+1} \otimes \cdots \otimes \psi_n]$$

is multilinear and there exists exactly one $\mathbb{C}$-linear $\tilde{a}^+_{n-1}(\psi) : \hat{S}_n[\mathscr{H}^{\hat{\otimes}n}] \to \hat{S}_{n-1}[\mathscr{H}^{\hat{\otimes}(n-1)}]$ such that $\tilde{a}_{n-1}[\hat{S}_n[\psi_1 \otimes \cdots \otimes \psi_n]] = \phi(\psi_1, \dots, \psi_n)$. This implies $\|\tilde{a}_{n-1}(\psi)\| \leq \sqrt{n}\|\psi\|$. This can be extended to $a^-_{n-1} : \hat{S}_n[\mathscr{H}^{\hat{\otimes}n}] \to \hat{S}_{n-1}[\mathscr{H}^{\hat{\otimes}(n-1)}]$ with

$$a^-(\psi) : \mathcal{F}_{b,e}(\mathscr{H}) \to \mathcal{F}_{b,e}(\mathscr{H}),$$

$$(\psi_n)_{n\in\mathbb{N}_0} \mapsto a^-(\psi)[\psi] := (a^-_0(\psi)[\psi_1], a^-_1(\psi)[\psi_2], \dots).$$

It is called *annihilation operator*.

Lemma 3.2. *If $\psi, \varphi \in \mathscr{H}$ then*

$$a^+(\psi)[\mathcal{F}_{b,e}(\mathscr{H})] \subseteq \mathcal{F}_{b,e}(\mathscr{H}), \quad a^-(\psi)[\mathcal{F}_{b,e}(\mathscr{H})] \subseteq \mathcal{F}_{b,e}(\mathscr{H}).$$

So the commutators are well-defined and

$$[a^-(\psi), a^+(\varphi)][\tilde{\phi}] = \langle \psi, \varphi \rangle_{\mathscr{H}}[\tilde{\phi}]$$

and

$$[a^-(\psi), a^-(\varphi)][\tilde{\phi}] = [a^+(\psi), a^+(\varphi)][\tilde{\phi}] = 0.$$

Proof. If

$$\tilde{\phi} = (0, \dots, 0, \hat{S}_n[\varphi_1 \otimes \cdots \otimes \varphi_n], 0, \dots) \in \mathcal{F}_{b,e}(\mathscr{H})$$

then

$$\begin{aligned}
&[a^-(\psi), a^+(\varphi)][\tilde{\phi}] = \\
&= a^-(\psi)\left[\left(\underbrace{0, \dots, 0}_{(n+1) \text{ times}}, \hat{S}_{n+1}[\varphi \otimes \hat{S}_n[\varphi_1 \otimes \cdots \otimes \varphi_n]], 0, \dots\right)\right] - \\
&\quad - a^+(\varphi)\left[\left(\underbrace{0, \dots, 0}_{(n-1) \text{ times}}, \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\langle \psi, \varphi_j \rangle \hat{S}_{n-1}[\varphi_1 \otimes \cdots \otimes \varphi_{j-1} \otimes \varphi_{j+1} \otimes \cdots \otimes \varphi_n], 0, \dots\right)\right] \\
&= \left(0, \dots, 0, \sum_{j=1}^{n}\langle \psi, \varphi_j \rangle \hat{S}_n[\varphi \otimes \varphi_1 \otimes \cdots \otimes \varphi_{j-1} \otimes \varphi_{j+1} \otimes \cdots \otimes \varphi_n] + \langle \psi, \phi \rangle \hat{S}_n[\varphi_1 \otimes \cdots \otimes \varphi_n], 0, \dots\right) - \\
&\quad - \left(0, \dots, 0, \sum_{j=1}^{n}\langle \psi, \varphi_j \rangle \hat{S}_n[\varphi \otimes \varphi_1 \otimes \cdots \otimes \varphi_{j-1} \otimes \varphi_{j+1} \otimes \cdots \otimes \varphi_n], 0, \dots\right) \\
&= \langle \psi, \varphi \rangle \tilde{\phi}.
\end{aligned}$$

Now take linear combinations of $\hat{S}_n[\varphi_1 \otimes \cdots \otimes \varphi_n]$ and use extension by continuity, and we get

$[a^-(\psi), a^+(\varphi)][\tilde{\phi}] = \langle \psi, \varphi \rangle \tilde{\phi}$ for all $\tilde{\phi} \in \mathcal{F}_{b,e}(\mathscr{H})$. For the other commutators we have

$$\begin{aligned}
&[a^-(\psi), a^-(\varphi)][\tilde{\phi}] = \\
&= a^-(\psi) \left[\left(0, \dots, 0, \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \langle \varphi, \varphi_j \rangle \hat{S}_{n-1}[\varphi_1 \otimes \cdots \otimes \varphi_{j-1} \otimes \varphi_{j+1} \otimes \cdots \otimes \varphi_n], 0, \dots \right) \right] - \\
&\quad - a^-(\varphi) \left[\left(0, \dots, 0, \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \langle \psi, \varphi_j \rangle \hat{S}_{n-1}[\varphi_1 \otimes \cdots \otimes \varphi_{j-1} \otimes \varphi_{j+1} \otimes \cdots \otimes \varphi_n], 0, \dots \right) \right] \\
&= 0
\end{aligned}$$

The same holds for a^+ commutator. □

Lemma 3.3. *Operators $a^+(\psi)$ and $a^-(\psi)$ are closable and $a^+(\psi) \subseteq (a^-(\psi))^*$ and $a^-(\psi) \subseteq (a^+(\psi))^*$. What we mean by that is $\mathcal{D}(a^-(\psi)) \subseteq \mathcal{D}((a^+(\psi))^*)$ and $(a^+(\psi))|_{\mathcal{D}(a^-(\psi))} = a^-(\psi)$.*

A closable operator is an operator whose extension is closed. To define a closed operator, we first need to define a graph of an operator T:

$$\Gamma(T) \coloneqq \{(\psi, T[\psi]) : \psi \in \mathcal{D}(T)\} \subseteq \mathscr{H} \times \mathscr{H}.$$

An operator is closed if its graph is closed.

Proof. Let $\tilde{\phi} = (0, \dots, 0, \phi_n, 0, \dots)$ and $\tilde{\psi} = (0, \dots, 0, \psi_{n+1}, 0, \dots)$. Then

$$\begin{aligned}
\langle a^+(\psi)[\tilde{\psi}], \tilde{\psi} \rangle &= \langle a^+(\psi)[\phi_n], \psi_{n+1} \rangle \\
&= \langle \hat{S}_{n+1}[b^+_{n+1}(\psi)[\hat{S}_n[\psi_n]]], \psi_{n+1} \rangle \\
&= \langle \hat{S}_n[\phi_n], b^-_n(\psi) \rangle \\
&= \langle \phi_n, \hat{S}_n[b^-_n(\psi)] \hat{S}_{n+1}[\psi_{n+1}] \rangle \\
&= \langle \phi_n, a^-_n(\psi) \psi_{n-1} \rangle.
\end{aligned}$$

Considering all linear combinations we get

$$\langle a^+(\psi)[\tilde{\phi}], \tilde{\psi} \rangle = \langle \tilde{\phi}, a^-(\psi)[\tilde{\psi}] \rangle$$

for all $\tilde{\phi}, \tilde{\psi} \in \mathcal{F}_{b,e}(\mathscr{H})$. The domain of $(a^+(\psi))^*$ is

$$\mathcal{D}((a^-(\psi))^*) \coloneqq \{x \in \mathcal{F}_{b,e}(\mathscr{H}) | \tilde{\eta}_x : \mathcal{D}(a^-(\psi)) \to \mathbb{C};\ y \mapsto \langle x, a^-(\psi)[y] \rangle \text{ is continuous}\}.$$

For $x \in \mathcal{D}((a^-(\psi))^*)$ $\tilde{\eta}_x$ has continuous extension (unique as $D(a^-(\psi))$ is dense) $\eta_x : \mathcal{F}_b(\mathscr{H}) \to \mathbb{C}$. According to Riesz theorem there is a unique $z \in \mathcal{F}_b(\mathscr{H})$, such that $\langle z, w \rangle = \eta_x(w)$ for all $w \in \mathcal{F}_b(\mathscr{H})$. We define $(a^+(\psi))^*[x] \coloneqq z$. It is easy to see that $(a^+(\psi))^*$ is closed. The adjoint of densely defined operator is always closed. □

Remark. The domain of the closure is

$$\mathcal{D}\left(\overline{a^+(\psi)}\right) = \left\{ \tilde{\phi} = (\phi_n)_{n \in \mathbb{N}} \in \mathcal{F}_b(\mathscr{H}) : \sum_{n=0}^{\infty} \| a^+_{n+1}(\psi)[\phi_n] \|^2 < \infty \right\}.$$

If we want to work with limits, it is better to work with the closure, if we want to work with commutators, then it is better to work with the original operator.

4 Point-wise versions of creation and annihilation operators

Lemma 4.1. *The space*

$$\mathcal{F}_{b,e,C_0}(L^2(\mathbb{R}^3)) \coloneqq \{\tilde{\phi} = (\phi_n)_{n\in\mathbb{N}} \in \mathcal{F}_{b,e}(L^2(\mathbb{R}^3)) : \phi_n \textit{ continuous with compact support}\}$$

is dense in $\mathcal{F}_b(L^2(\mathbb{R}^3))$.

Proof. We use the well known fact that the space of continuous functions with compact support $\mathcal{C}_c^\infty$ is dense in $L^2(\mathbb{R}^3)$. Let $\phi_n \in \hat{S}_n[L^2(\mathbb{R}^3)^{\otimes n}]$. There is $\varphi_n \in \mathcal{C}_c(\mathbb{R}^{3n})$ such that $\|\varphi_n - \phi_n\| < \varepsilon/2^n$. □

Definition 4.1. Let $f : \mathbb{R}^3 \to \mathbb{C}$ be a continuous function with compact support. Then

$$\not{a}^{\pm}(f) : \mathcal{F}_{b,e,C_0}(L^2(\mathbb{R}^3)) \to \mathcal{F}_{b,e,C_0}(L^2(\mathbb{R}^3))$$

by $(\not{a}^+(f)[\tilde{\psi}])_0 \coloneqq 0$ and

$$(\not{a}^+(f)[\tilde{\psi}])_n(k_1,\dots,k_n) \coloneqq \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f(k_j)\psi_{n-1}(k_1,\dots,k_{j-1},k_{j+1},\dots,k_n).$$

Furthermore,

$$(\not{a}^-(f)[\tilde{\psi}])_0 \coloneqq \int_{\mathbb{R}^3} f(k)\psi_1(k)\,\mathrm{d}^3k$$

and

$$(\not{a}^-(f)[\tilde{\psi}])_n(k_1,\dots,k_n) = \sqrt{n+1}\int_{\mathbb{R}^3} f(k)\psi_{n+1}(k,k_1,\dots,k_n)\,\mathrm{d}^3k.$$

For $k \in \mathbb{R}^3$

$$a^-(k) : \mathcal{F}_{b,e,C_0}(L^2(\mathbb{R}^3)) \to \mathcal{F}_{b,e,C_0}(L^2(\mathbb{R}^3))$$

by $(a^-(k)[\tilde{\psi}])_0 \coloneqq \psi_1(k)$ and

$$(a^-(k)[\tilde{\psi}])_n(k_1,\dots,k_n) \coloneqq \sqrt{n+1}\psi_{n+1}(k,k_1,\dots,k_n)$$

and a *sesquilinear form*

$$a^+(k) : \mathcal{F}_{b,e,C_0}(L^2(\mathbb{R}^3)) \times \mathcal{F}_{b,e,C_0}(L^2(\mathbb{R}^3)) \to \mathbb{C}, \quad (\tilde{\psi},\tilde{\phi}) \mapsto \langle a^-(l)[\tilde{\psi}],\tilde{\phi}\rangle$$

This is well-defined because $f\psi_n$ is continuous with compact support.

5 Spectral calculus and its applications

Definition 5.1. Let $\mathfrak{A}$ be a σ-algebra on set X and $\mathscr{H}$ a Hilbert space then $E : \mathfrak{A} \to L(\mathscr{H})$ is a *spectral measure* if

a) for each $B \in \mathfrak{A}$ the operator $E(B) \in L(\mathscr{H})$ is orthogonal projection $(E(B))^* = E(B) = (E(B))^2$,
b) $E(\varnothing) = 0$, $E(X) = \mathrm{id}_{\mathscr{H}}$,
c) if $(A_n)_{n\in\mathbb{N}}$ is s sequence of pairwise disjoint $A_n \in \mathfrak{A}$ and $\varphi \in \mathscr{H}$, then

$$E\left(\bigcup_{n\in\mathbb{N}} A_n\right)[\varphi] = \lim_{N\to\infty} \sum_{n=1}^{N} E(A_n)[\varphi].$$

Remark. If $E : \mathfrak{A} \to L(\mathscr{H})$ is a spectral measure and $\phi, \varphi \in \mathscr{H}$, then

$$\begin{aligned} \mu_{\varphi,\varphi} : \mathfrak{A} &\to [0,\infty) \\ B &\mapsto \langle \varphi, E(B)[\varphi]\rangle = \|E(B)[\varphi]\|^2 \end{aligned}$$

is a finite measure ($\mu_{\varphi,\varphi}(X) = \|\varphi\|^2$) and

$$\begin{aligned} \mu_{\phi,\varphi} : \mathfrak{A} &\to \mathbb{C} \\ B &\mapsto \langle \phi, E(B)[\varphi]\rangle \end{aligned}$$

is a *complex measure*.

Theorem 5.1. *If $\mu : \mathfrak{A} \to \mathbb{C}$ is a complex measure, Then*

$$|\mu| : \mathfrak{A} \to [0,\infty)$$
$$A \mapsto |\mu|(A) := \sup\left\{\sum_{n=1}^{N} |\mu(A_n)| : (A_n)_{n\in\mathbb{N}} \text{ pointwise disjoint } A_n \in \mathfrak{A}, A = \bigcup_{n=1}^{N} A_n\right\}$$

$|\mu|$ is a finite measure. There is an $h : X \to \mathbb{C}$ such that $|h(x)| = 1$ for each $x \in X$ and $\mathrm{d}\mu = h\mathrm{d}|\mu|$, i.e.

$$\int_A \mathrm{d}\mu = \int_A h\mathrm{d}|\mu|$$

for each $A \in \mathfrak{A}$. If $E : \mathfrak{A} \to L(\mathscr{H})$ is a spectral measure, then

$$|\mu_{\phi,\varphi}|(A) \le \sqrt{\mu_{\phi,\phi}(A)}\sqrt{\mu_{\varphi,\varphi}(A)}.$$

Theorem 5.2. *If $E : \mathfrak{A} \to L(\mathscr{H})$ is a spectral measure and $f : X \to \hat{\mathbb{C}}$ a measurable function $E(\{x \in X : f(x) = \infty\}) = 0$, then*

$$\mathcal{D}(\tilde{\psi}_E(f)) := \left\{ \varphi \in \mathscr{H} : \int_X |f|^2 \, \mathrm{d}\mu_{\varphi,\varphi} < \infty \right\}$$

is a dense subspace of $\mathscr{H}$ and for all $\phi \in \mathscr{H}$, $\varphi \in \mathcal{D}(\tilde{\psi}_E(f))$

$$\int_X |f|, \mathrm{d}|\mu_{\phi,\varphi}| \leq \|\phi\| \left(\int_X |f|^2 \, \mathrm{d}\mu_{\varphi,\varphi} \right)^{1/2} < \infty.$$

So by

$$\langle \phi, \tilde{\psi}_E(f)[\varphi] \rangle = \int_X f \, \mathrm{d}\mu_{\phi,\varphi}$$

we can define the operator $\tilde{\psi}(f) : \mathcal{D}(\tilde{\psi}_E(f)) \to \mathscr{H}$.

Properties:

a)
$$\|\tilde{\psi}_E(f)[\varphi]\|^2 = \int_X |f|^2 \, \mathrm{d}\mu_{\varphi,\varphi}, \quad \text{for } \varphi \in \mathcal{D}(\tilde{\psi}_E(f)),$$

b) $(\tilde{\psi}_E(f))^* = \tilde{\psi}_E(\overline{f})$, $\tilde{\psi}_E(f) \circ (\tilde{\psi}_E(f))^* = \tilde{\psi}_E(|f|^2) = (\tilde{\psi}_E(f))^* \circ (\tilde{\psi}_E(f))$,

c) if $g : X \to \hat{\mathbb{C}}$ measurable and $E(\{x \in X : g(x) = \infty\}) = 0$, then $\tilde{\psi}_E(f) \circ \tilde{\psi}_E(g) \subseteq \tilde{\psi}_E(fg)$ and $\tilde{\psi}_E(f) \circ \tilde{\psi}_E(g) = \tilde{\psi}_E(fg)$ on $\mathcal{D}(\tilde{\psi}_E(g)) \cap \mathcal{D}(\tilde{\psi}_E(fg))$,

d) $\tilde{\psi}_E(\alpha f + \beta g) = \alpha \tilde{\psi}_E(f) + \beta \tilde{\psi}_E(g)$,

e) $\tilde{\psi}_E(f) \in L \iff f$ is bounded E-almost everywhere,

f) if $E(\{x \in X : f(x) = 0\}) = 0$, then $(\tilde{\psi}_E(f))^{-1} = \tilde{\psi}_E(1/f)$,

g) if $p : \mathbb{C} \to \mathbb{C}$ is a polynomial, then $\tilde{\psi}_E(pf) = p(\tilde{\psi}_E(f))$.

Theorem 5.3 (*Spectral theorem*)**.** *For self-adjoint operator $A : \mathcal{D}(A) \to \mathscr{H}$ there is a unique spectral measure $E_A : \mathfrak{B}(\mathbb{R}) \to L(\mathscr{H})$ such that $A = \tilde{\psi}_{E_A}(\mathrm{id}_{\mathbb{R}})$.*

$$\mathcal{D}(A) = \left\{ \varphi \in \mathscr{H} : \int_{\mathbb{R}} |\,\mathrm{id}_{\mathbb{R}}\,|^2 \, \mathrm{d}\mu_{\varphi,\varphi} < \infty \right\}, \quad \langle \phi, A[\varphi] \rangle = \int_{\mathbb{R}} \mathrm{id}_{\mathbb{R}} \, \mathrm{d}\mu_{\phi,\varphi}.$$

Furthermore, $E_A(\sigma(A)) = \mathrm{id}_{\mathscr{H}}$.

The notation $\mathfrak{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$, i.e. an algebra that is generated by all open subsets of $\mathbb{R}$.

Example 2. Let $H : \mathcal{D}(H) \to \mathscr{H}$ be self-adjoint and $f : \mathbb{R} \to \mathbb{C}$, $x \mapsto e^{itx}$, then $\tilde{\psi}_{E_H}(f) := f(H) = e^{itH}$. We see that $|e^{itx}| = 1$ for $t, x \in \mathbb{R}$, so f is bounded and $\tilde{\psi}_{E_A}(f) = e^{itH}$ is bounded and defined on $\mathscr{H}$. From $e^{-itx} e^{itx} = 1$ it follows

$$(\tilde{\psi}_E(f)^{-1}) = \tilde{\psi}_E(1/f) = e^{-itH} = (\tilde{\psi}_{E_A}(f))^* = (\tilde{\psi}_{E_A}(\overline{f})).$$

So $e^{itH} : \mathscr{H} \to \mathscr{H}$ is unitary.

Definition 5.2. Let $A : \mathcal{D}(A) \to \mathscr{H}$ and $T : \mathcal{D}(T) \to \mathscr{H}$ be self-adjoint operators with spectral measures $E_A : \mathfrak{B}(\mathbb{R}) \to L(\mathscr{H})$ and $E_T : \mathfrak{B}(\mathbb{R}) \to L(\mathscr{H})$. Then A and T *strongly commute*, if $E_A(B)E_T(C) = E_T(C)E_A(B)$ for all $B, C \in \mathfrak{B}(\mathbb{R})$.

Remark.

$$\begin{aligned} A \text{ and } T \text{ strongly commute} &\iff e^{itA}e^{isT} = e^{isT}e^{itA} \text{ for all } s,t \in \mathbb{R} \\ &\iff (A-z)^{-1}(T-\omega)^{-1} = (T-\omega)^{-1}(A-z)^{-1} \text{ for all } z \in \rho(A), \omega \in \rho(T) \\ &\iff E_A(B)(T-\omega)^{-1} = (T-\omega)^{-1}E_A(B) \text{ for all } B \in \mathfrak{B}(\mathbb{R}), \omega \in \rho(T). \end{aligned}$$

The notation $\rho(A)$ denotes the *resolvent set* of A.

Example 3. Let $A : \mathcal{D}(A) \to \mathscr{H}$ be self-adjoint $z \in \rho(A)$. Then

$$(A-z)^{-1} = \begin{cases} -i\int_0^\infty e^{-itz}e^{itA}\,\mathrm{d}t & \text{if } \operatorname{Im} z > 0, \\ i\int_0^\infty e^{-itz}e^{itA}\,\mathrm{d}t & \text{if } \operatorname{Im} z < 0. \end{cases}$$

Proof. For $\operatorname{Im} z > 0$, $x \in \mathbb{R}$ the map $f \in L^1([0,\infty))$, $f : [0,\infty) \to [0,\infty)$, $t \mapsto |e^{-itz}e^{ixt}| = e^{t\operatorname{Im} z}$ we have

$$-i\int_0^\infty e^{ixt}e^{-izt}\,\mathrm{d}t = -i\lim_{R\to\infty}\int_0^R e^{i(x-z)t}\,\mathrm{d}t = -i\lim_{R\to\infty}\frac{e^{i(x-z)t}}{i(x-z)}\bigg|_0^R = \frac{1}{x-z}$$

and

$$\begin{aligned} \int_{\mathbb{R}}\int_0^\infty |e^{ixt}e^{-izt}|\,\mathrm{d}t\,\mathrm{d}|\mu_{\phi,\varphi}|(x) &= \int_{\mathbb{R}}\int_0^\infty e^{t\operatorname{Im} z}\,\mathrm{d}t\,\mathrm{d}|\mu_{\phi,\varphi}|(x) \\ &= \int_{\mathbb{R}}\lim_{R\to\infty}\frac{e^{t\operatorname{Im} z}}{\operatorname{Im} z}\bigg|_0^R \mathrm{d}\|\mu_{\phi,\varphi}\|(x) \\ &= -\frac{|\mu_{\phi,\varphi}|(\mathbb{R})}{\operatorname{Im} z} < \infty. \end{aligned}$$

Using Fubini and spectral calculus we get

$$\begin{aligned} \langle\phi, (A-z)^{-1}[\varphi]\rangle &= \int_{\mathbb{R}}\frac{1}{x-z}\,\mathrm{d}\mu_{\phi,\varphi} \\ &= \int_{\mathbb{R}}\left(-i\int_0^\infty e^{ixt}e^{-izt}\,\mathrm{d}t\right)\mathrm{d}\mu_{\phi,\varphi} \\ &\overset{\text{Fubini}}{=} -i\int_0^\infty\left(\int_{\mathbb{R}} e^{ixt}\mathrm{d}\mu_{\phi,\varphi}\right)e^{-izt}\,\mathrm{d}t \\ &= -i\int_0^\infty \langle\phi, e^{itA}[\varphi]\rangle e^{-izt}\,\mathrm{d}t \end{aligned}$$

□

6 Groups, Lie groups, Lie algebras and representations

Definition 6.1. Let $G \neq \varnothing$ be a set and let $* : G \times G \to G$ be a function, such that:
a) $*$ is associative, i.e. $(g * h) * k = g * (h * k)$ for all $g, h, k \in G$,
a) there is an element $e \in G$ such that $g * e = e * g = g$ for all $g \in G$ (e is called "*neutral element*" of G),
a) for any $g \in G$ there is an element $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$ (g^{-1} is called "*inverse element*" of g).
Then $(G, *)$ is called a *group*. If in addition $*$ is commutative, i.e. $g * h = h * g$ for all $g, h \in G$, then $(G, *)$ is called an *Abelian group* or *commutative group*.

Example 4.

a) Let $X \neq \varnothing$ and $\gamma(X) := \{f : X \to X | f \text{ is bijective}\}$ with $\circ$ composition of maps, then $(\gamma(X), \circ)$ is a group. A special case is $X = \{1, \dots, n\}$ and $\gamma_n = \{\pi : \{1, \dots, n\} \to \{1, \dots, n\} | \pi \text{ bijective}\}$.

b) If V is a K-vector space, then $(V, +)$ is an Abelian group.

c) If K is a field, then $(K, +)$ and $(K \setminus \{0\}, \cdot)$ are commutative groups.

d) If
$$M_n(\mathbb{C}) = \left\{ A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} : a_{11}, \dots, a_{nn} \in \mathbb{C} \right\},$$
then $(M_n(\mathbb{C}), +)$ is an Abelian group, but $(M_n(\mathbb{C}), \cdot)$ is not a group (as every element doesn't have an inverse).

e) If $\mathrm{GL}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) : A \text{ invertible}\}$ (called *general linear group*), then $(\mathrm{GL}(n, \mathbb{C}), \cdot)$ is a group: it is associative,
$$E_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in \mathrm{GL}(n, \mathbb{C})$$
is the neutral element and A^{-1} is the inverse element of $A \in \mathrm{GL}(n, \mathbb{C})$.

f) $(\{z \in \mathbb{C} : |z| = 1\}, \cdot)$ is a group.

Definition 6.2. If $(G, *)$ and $(H, \odot)$ are groups, then $\varphi : G \to H$ is called *group homomorphism* if $\varphi(g_1 * g_2) = \varphi(g_1) \odot \varphi(g_2)$ for all $g_1, g_2 \in G$.
If $S \subseteq G$ such that $e_g \in S$ and for all $g_1, g_2 \in S$ $g_1 * g_2, g_1^{-1} \in S$, then S is called a *subgroup* of G.

Lemma 6.1. *Let $\varphi : G \to H$ be a group homomorphism. Then*
a) for each subgroup $S \subseteq G$, $\varphi(S)$ is a subgroup of H,
b) for each subgroup $T \subseteq H$ then $\varphi^{-1}(T)$ is a subgroup of G. In particular, it also holds for $\varphi^{-1}(\{e_H\})$.

Proof. If $\varphi : G \to H$ is a group homomorphism, then $\varphi(e_G) = \varphi(e_G * e_G) = \varphi(e_G) \odot \varphi(e_G)$, so $\varphi(e_G) = e_H$.

a) $e_H = \varphi(e_G) \in \varphi(S)$. For any $w_1, w_2 \in \varphi(S)$ there are $g_1, g_1 \in S$ such that $w_1 = \varphi(g_1)$, $w_2 = \varphi(g_2)$ and $w_1 \odot w_2 = \varphi(g_1) \odot \varphi(g_2) = \varphi(g_1 * g_2) \in \varphi(S)$. If $w = \varphi(g) \in \varphi(S)$, then $e_H = \varphi(e_G = \varphi(g * g^{-1})) = \varphi(g) \odot \varphi(g^{-1})$, hence $\varphi(g^{-1})$ is the inverse of $\varphi(g)$.

b) Exercise.

□

Example 5.

a) The map $\det : M_n(\mathbb{K}) \to \mathbb{K}$ satisfies $\det(AB) = \det(A)\det(B)$ for all $A, B \in M_n(\mathbb{K})$. If we restrict the determinant to $\det : \mathrm{GL}(n, \mathbb{K}) \to \mathbb{K} \setminus \{0\}$ it defines a group homomorphism. Define the *special linear group*

$$\mathrm{SL}(n, \mathbb{R}) \coloneqq \det^{-1}(\{1\}) = \{A \in \mathrm{GL}(n, \mathbb{R}) : \det(A) = 1\}.$$

This is a subgroup of $\mathrm{GL}(n, \mathbb{R})$. In complex numbers we have $\mathrm{SL}(n, \mathbb{C}) \coloneqq \{A \in \mathrm{GL}(n, \mathbb{C}) : \det(A) = 1\}$, which is a subgroup of $\mathrm{GL}(n, \mathbb{C})$.

b) Let $B \in M_n(\mathbb{R})$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ the standard scalar product. Define *orthogonal group* as

$$\mathrm{O}(B) \coloneqq \{A \in \mathrm{GL}(n, \mathbb{R}) : \langle A\mathbf{x}, BA\mathbf{x} \rangle_{\mathbb{R}^n} = \langle \mathbf{x}, B\mathbf{y} \rangle_{\mathbb{R}^n} \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n\}.$$

$\mathrm{O}(B)$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$:

- $E_n \in \mathrm{O}(B)$,
- if $A, C \in \mathrm{O}(B)$ then $\langle C\mathbf{x}, BC\mathbf{y} \rangle_{\mathbb{R}^n} = \langle \mathbf{x}, B\mathbf{y} \rangle_{\mathbb{R}^n}$ and also $\langle AC\mathbf{x}, BAC\mathbf{y} \rangle_{\mathbb{R}^n} = \langle C\mathbf{x}, BC\mathbf{y} \rangle_{\mathbb{R}^n}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ so $AC \in \mathrm{O}(B)$.
- For $A \in \mathrm{O}(B)$

$$\langle A\mathbf{x}, BA\mathbf{y} \rangle_{\mathbb{R}^n} = \langle \mathbf{x}, B\mathbf{y} \rangle_{\mathbb{R}^n} = \langle A^{-1}(A\mathbf{x}), BA^{-1}(A\mathbf{y}) \rangle_{\mathbb{R}^n}.$$

We see that $A^{-1} \in \mathrm{O}(B)$, since $F_A : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{z} \mapsto A\mathbf{z}$ is bijective.

If we did the same procedure for complex matrices, we would get that *unitary group* $\mathrm{U}(B)$ is a subgroup of $\mathrm{GL}(n, \mathbb{C})$.

c) $\mathrm{O}(n) \coloneqq \{A \in M_n(\mathbb{R}) : A^T A = E_n\}$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$. This is a special case of $\mathrm{O}(B)$ for $B = E_n$ since $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^n} = \langle \mathbf{x}, A^T A\mathbf{y} \rangle_{\mathbb{R}^n} = \langle A\mathbf{x}, A\mathbf{y} \rangle_{\mathbb{R}^n}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

d) $\mathrm{U}(n) \coloneqq \{A \in M_n(\mathbb{C}) : A^* A = E_n\}$ is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ since $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{C}^n} = \langle \mathbf{x}, A^* A\mathbf{y} \rangle_{\mathbb{C}^n} = \langle A\mathbf{x}, A\mathbf{y} \rangle_{\mathbb{C}^n}$.

e) $\mathrm{SO}(n) \coloneqq \{A \in \mathrm{O}(n) : \det(A) = 1\}$ is called the *special orthogonal group*. It is a subgroup of $\mathrm{O}(n)$, since $\det : \mathrm{O}(n) \to \{-1, +1\}$ is a group homomorphism.

f) $\mathrm{SU}(n) \coloneqq \{A \in \mathrm{U}(n) : \det(A) = 1\}$ is called the *special unitary group*. It is a subgroup of $\mathrm{U}(n)$.

g) Let

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Define

$$\mathrm{O}(B) = \mathrm{O}(1,3) = \{A \in \mathrm{GL}(4,\mathbb{R}) : \langle A\mathbf{x}, BA\mathbf{y}\rangle_{\mathbb{R}^4} = \langle \mathbf{x}, B\mathbf{y}\rangle_{\mathbb{R}^4}, \text{ for all } \mathbf{x},\mathbf{y} \in \mathbb{R}^4\}.$$

It is called the *Lorentz group*. For any element $A \in \mathrm{O}(3,1)$ it holds $\det(A) \in \{-1,+1\}$ because $B = A^T B A$.

h) *Poincaré group*: for $x \in \mathbb{R}^d$ define translation $T_{\mathbf{x}} : \mathbb{R}^d \to \mathbb{R}^d$, $\mathbf{y} \mapsto \mathbf{y} + \mathbf{x}$. The Poincaré group is

$$\mathrm{P}(1,3) = \{T = T_{\mathbf{x}} \circ F_A : \mathbb{R}^4 \to \mathbb{R}^4 \mid x \in \mathbb{R}^4, A \in \mathrm{O}(1,3), F_A : \mathbb{R}^4 \to R^4, \mathbf{y} \mapsto A\mathbf{y}\}.$$

Element T is identified with $T \hat{=} (x, A)$. The operation $T_1 \circ T_2 : \mathbb{R}^4 \to \mathbb{R}^2$, $(x_1, A_1) * (x_2, A_2) = (x_1 + A_1 x_2, A_1 A_2)$ defines a group structure on $\mathrm{P}(1,3)$. Next define a map $\varphi : \mathrm{P}(1,3) \to \mathrm{GL}(5,\mathbb{R})$ with

$$T = (\mathbf{x}, A) \mapsto \begin{pmatrix} A & \mathbf{x} \\ 0 & 1 \end{pmatrix}.$$

We see that the determinant of this matrix is $\det(A)$, which is non-zero. So φ is a group homomorphism and injective. So we can identify $\mathrm{P}(1,3)$ with a subgroup

$$\varphi(\mathrm{P}(1,3)) = \left\{ \begin{pmatrix} A & \mathbf{x} \\ 0 & 1 \end{pmatrix} : A \in \mathrm{O}(1,3), \mathbf{x} \in \mathbb{R}^4. \right\}$$

of $\mathrm{GL}(5,\mathbb{R})$.

Definition 6.3. A subgroup $G \in \mathrm{GL}(n,\mathbb{C})$ is called a *matrix Lie group* if any sequence $(A_m)_{m\in\mathbb{N}}$ in G with $A_m \xrightarrow{m\to\infty} A \in \mathrm{GL}(n,\mathbb{C})$ implies $A \in G$ or A is not invertible. With other words, G is closed subgroup of $\mathrm{GL}(n,\mathbb{C})$.
A *Lie group* is a group $(G,*)$ that is also a differential manifold such that $* : G \times G \to G$, $(g,h) \mapsto g * h$ and $\iota : G \to G$, $g \mapsto g^{-1}$ are differentiable.

Remark.

a) $M_n(\mathbb{C})$ is an n^2 dimensional $\mathbb{C}$-vector space, every norm on $M_n(\mathbb{C})$ is equivalent so $A_m \xrightarrow{m\to\infty} A$ can be interpreted in any norm on $M_n(\mathbb{C})$.

b) If we consider $A_m = \frac{1}{m} E_n$, the limit of the sequence is $A = 0$, while each element of the sequence is invertible.

c) $\mathrm{GL}(n,\mathbb{C})$ is a matrix Lie group by definition.

d) $\mathrm{GL}(n,\mathbb{C})$ is also a Lie group: for each Banach space X the set $\mathrm{GL}(X) := \{T : X \to X \mid T \text{ bounded, linear, invertible}\}$ is an open subset of $L(X) := \{T : X \to X \mid T \text{ bounded, linear}\}$, because for $T \in \mathrm{GL}(X)$ and each $S \in L(X)$ such that $\|ST^{-1}\| < 1$:

$$(S+T)^{-1} = \sum_{k=0}^{\infty} (-1)^k (T^{-1})(ST^{-1})^k$$

Example 6.

a) Maps $\det : M_n(\mathbb{C}) \to \mathbb{C}$ and $\det : M_n(\mathbb{R}) \to \mathbb{R}$ are continuous. The preimage of $\{1\}$ (a closed set) is closed. So $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{R})$ are matrix Lie groups.

b) If $B \in M_n(\mathbb{R})$ or $B \in M_n(\mathbb{C})$ then $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ or $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, $(\mathbf{x}, \mathbf{y}) \mapsto \langle \mathbf{x}, B\mathbf{y}\rangle$ are continuous. Assume $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. If $(A_m)_{m\in\mathbb{N}}$ is a sequence in $\mathrm{O}(B) = \{A \in \mathrm{GL}(n,\mathbb{K}) : \langle \mathbf{x}, b\mathbf{y}\rangle_{\mathbb{K}^n} = \langle A\mathbf{x}, bA\mathbf{y}\rangle_{\mathbb{K}^n}\}$ such that $A_m \xrightarrow{m\to\infty} A$ in $M_n(\mathbb{K})$, then $\langle A_m\mathbf{x}, BA_m\mathbf{y}\rangle_{\mathbb{K}^n} \xrightarrow{m\to\infty} \langle A\mathbf{x}, BA\mathbf{y}\rangle_{\mathbb{K}^n}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{K}^n$. So if A is invertible, then $A \in \mathrm{O}(B)$.

c) $\mathrm{O}(n)$ is a matrix Lie group which is closed in $M_n(\mathbb{R})$. Also, $\mathrm{U}(n)$ is a matrix Lie group closed in $M_n(\mathbb{C})$. If $(A_m)_{m\in\mathbb{N}}$ is a sequence in $\mathrm{O}(n)$ (or $\mathrm{U}(n)$) such that $A_m \xrightarrow{m\to\infty} A$ in $M_n(\mathbb{K})$, then $A_m^* \xrightarrow{m\to\infty} A^*$ (use the supremum norm on $M_n(\mathbb{K})$). So $E_n = A_m^* A_m \xrightarrow{m\to\infty} A^*A = E_n$. So $A \in \mathrm{O}(n)$ or $\mathrm{U}(n)$.

Definition 6.4. Let $G \subseteq \mathrm{GL}(n,\mathbb{C})$ be a matrix Lie group.
a) G is *compact* if G is a compact subset of $M_n(\mathbb{C})$.
b) G is a *pointwise connected* if for any $A, B \in G$ there is a continuous path $g : [0,1] \to G$ such that $g(0) = A$ and $g(1) = B$.

Example 7.

a) $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are compact matrix Lie groups (by Example 6 c) for any sequence $(A_m)_{m\in\mathbb{N}}$ in $\mathrm{U}(n)$ such that $A_m \xrightarrow{m\to\infty} A \in M_n(\mathbb{C})$ it holds $A \in \mathrm{U}(n)$ so $\mathrm{U}(n)$ is closed in $M_n(\mathbb{C})$ and the row/column vectors of $A \in \mathrm{U}(n)$ form orthonormal base of $\mathbb{C}^n$ so $\mathrm{U}(n)$ is bounded).

b) $\mathrm{GL}(n,\mathbb{C})$ is not compact, because $\frac{1}{m}E_n \xrightarrow{m\to\infty} 0 \notin \mathrm{GL}(n,\mathbb{C})$.

c) $\mathrm{GL}(n,\mathbb{C})$ is connected.

Definition 6.5. Let G be a matrix Lie group. The *Lie algebra* of G is

$$\mathfrak{g} \coloneqq \{X \in M_n(\mathbb{K}) : e^{itX} \in G \text{ for any } t \in \mathbb{R}\}.$$

The expression

$$e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!}$$

is well-defined, because using the induced *supremum norm*

$$|||A||| \coloneqq \sup_{\mathbf{x}\in\mathbb{C}^n\setminus\{0\}} \frac{\|Ax\|}{\|x\|}$$

on $M_n(\mathbb{C})$ we can show $|||AB||| \le |||A||| \cdot |||B|||$ and $|||A^n||| \le |||A|||^n$.

If $A, B \in M_n(\mathbb{K})$ and $AB = BA$, then $e^{A+B} = e^A e^B$.

Theorem 6.2. *If $A, B \in M_n(\mathbb{C})$ then*
a)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\right) = Ae^{tA} = e^{tA}A,$$

b) (Trotter product formula)

$$e^{A+B} = \lim_{m\to\infty}\left(e^{A/m}e^{B/m}\right)^m.$$

The Trotter formula can be also written for infinite-dimensional operators, but then they and their sum must be self-adjoint.

Proof.

a) The power series for $e^{tA} = \sum_{m=0}^{\infty}(tA)^m/m!$ converges in norm (using an induced matrix norm because $\sum_{m=1}^{\infty}(|t|\,|||A|||)^m/m!$ is convergent majorant) and locally uniformly converges, so we can differentiate term by term, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{m=0}^{\infty}\frac{(tA)^m}{m!}\right) = \sum_{m=1}^{\infty} m\frac{(tA)^{m-1}}{m!}A = \sum_{m=1}^{\infty}\frac{(tA)^{m-1}}{(m-1)!}A = e^{tA}A = Ae^{tA}.$$

b) (We skipped this.)

□

Lemma 6.3. *Let G be a matrix Lie group and $\mathfrak{g}$ it is Lie algebra. If $A \in G$ and $X \in \mathfrak{g}$ then $AXA^{-1} \in \mathfrak{g}$.*

Proof. As a matrix Lie group, $G \subset \mathrm{GL}(n,\mathbb{C})$ so for $A \in G$ the inverse matrix A^{-1} exists. Due to exercise 31 we know

$$e^{tAXA^{-1}} = Ae^{tX}A^{-1}$$

for all $t \in \mathbb{R}$. We know $e^{tX} \in G$ as $X \in \mathfrak{g}$, so $AXA^{-1} \in \mathfrak{g}$ by definition of the Lie algebra $\mathfrak{g}$ of G. □

Example 8. Many examples of Lie algebras.

a) $\mathfrak{gl}(n,\mathbb{C})$ is Lie algebra of $\mathrm{GL}(n,\mathbb{C})$. As e^{tA} is invertible for all $t \in \mathbb{R}$, $A \in M_n(\mathbb{C})$, it holds

$$\mathfrak{gl}(n,\mathbb{C}) = M_n(\mathbb{C}).$$

b) The Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ of $\mathrm{GL}(n,\mathbb{R})$. As $e^{tA} \in \mathrm{GL}(n,\mathbb{R})$ for all $t \in \mathbb{R}$ and $A \in M_n(\mathbb{R})$ we get $\mathfrak{gl}(n,\mathbb{R}) \subseteq \mathfrak{gl}(n,\mathbb{R})$. If $B \in M_n(\mathbb{C})$ and $e^{tB} \in \mathrm{GL}(n,\mathbb{R})$, then

$$\frac{\mathrm{d}}{\mathrm{d}}\left(e^{tB}\right)\Big|_{t=0} = B.$$

Since the limit above exists and the entries are all real, it must hold $B \in M_n(\mathbb{R})$. From this it follows

$$\mathfrak{gl}(n,\mathbb{R}) = M_n(\mathbb{R}).$$

c) Let $\mathrm{SL}(n,\mathbb{C}) = \{A \in \mathrm{GL}(n,\mathbb{C}) : \det(A) = 1\}$. The corresponding Lie algebra is $\mathfrak{sl}(n,\mathbb{C}) = \{A \in M_n(\mathbb{C}) : e^{tA} \in \mathrm{SL}(n,\mathbb{C})\,\forall t \in \mathbb{R}\} = \{A \in M_n(\mathbb{C}) : \det(e^{tA}) = 1\,\forall t \in \mathbb{R}\}$. Each $A \in M_n(\mathbb{C})$ has a Jordan form $B = S^{-1}AS$ with $S \in \mathrm{GL}(n,\mathbb{C})$. So we can write

$$\det(e^{tA}) = \det(e^{tS^{-1}BS}) = \det(S^{-1}e^{tB}S) = \det(S^{-1})\det(e^{tB})\det(S) = \det(e^{tB}).$$

For a Jordan form, we know $\operatorname{tr}(B) = \sum_{k=1}^{n} \lambda_k$ where λ_k are eigenvalues of B (and A). Calculating the exponential, we get

$$\det(e^{tA}) = \det(e^{tB}) = \prod_{k=1}^{n} e^{t\lambda_k} = \exp\left(t \sum_{k=1}^{n} \lambda_k\right) = e^{t \operatorname{tr}(B)} = e^{t \operatorname{tr}(A)}.$$

So we see that

$$\mathfrak{sl}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) : \operatorname{tr}(A) = 0\}.$$

d) The Lie algebra $\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) : e^{tA} \in U(n) \, \forall t \in \mathbb{R}\}$ corresponds to the unitary group $\mathrm{U}(n)$. If $A^* = -A$, then $(e^{tA})^* = (e^{tA})^{-1}$, so $\{A \in M_n(\mathbb{C}) : A^* = -A\} \subseteq \mathrm{U}(n)$. If e^{iA} is unitary for all $t \in \mathbb{R}$, then $(e^{(}tA))^* = (e^{tA})^{-1}$. Differentiating with respect to t and setting $t = 0$ we get $A^* = -A$, so

$$\mathfrak{u}(n) \subseteq \{A \in M_n(\mathbb{C}) : A^* = -A\}.$$

e) Let

$$\begin{aligned}
\mathfrak{su}(n) &= \{A \in M_n(\mathbb{C}) : e^{tA} \in \mathrm{SU}(n)\} \\
&= \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C}) \, \forall t \in \mathbb{R} \\
&= \mathfrak{u}(n) \cap \mathfrak{sl}(n, \mathbb{C}) \\
&= \{A \in M_n(\mathbb{C}) : A^* = -A, \operatorname{tr}(A) = 0\}.
\end{aligned}$$

f) Let

$$\begin{aligned}
\mathfrak{o}(n) &= \{A \in M_n(\mathbb{C}) : e^{tA} \in \mathrm{O}(n) = \mathrm{U}(n) \cap \mathrm{GL}(n, \mathbb{R}) \, \forall t \in \mathbb{R}\} \\
&= \mathfrak{u}(n) \cap \mathfrak{gl}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : A^T = -A\}.
\end{aligned}$$

g)

$$\begin{aligned}
\mathfrak{so}(n) &= \{A \in M_n(\mathbb{C}) : e^{tA} \in \mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R}) \, \forall t \in \mathbb{R}\} \\
&= \{A \in M_n(\mathbb{R}) : A^T = -A\} \\
&= \mathfrak{o}(n).
\end{aligned}$$

h) Fix $B \in \mathrm{GL}(n, \mathbb{R})$ then $\langle A\mathbf{x}, BA\mathbf{y} \rangle_{\mathbb{R}^n} = \langle \mathbf{x}, A^T BA\mathbf{y} \rangle_{\mathbb{R}^n}$ so $A \in \mathrm{O}(B)$ if and only if $\langle \mathbf{x}, A^T BA\mathbf{y} \rangle_{\mathbb{R}^n} = \langle \mathbf{x}, B\mathbf{y} \rangle_{\mathbb{R}^n}$ for each $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ if and only if $A^T BA = B$. Define $\mathfrak{o}(B) = \{A \in M_n(\mathbb{C}) : e^{tA} \in \mathrm{O}(B) \subseteq \mathrm{GL}(n, \mathbb{R}) \, \forall t \in \mathbb{R}\} = \{A \in M_n(\mathbb{R}) : (e^{tA})^T B e^{tA} = B \, \forall t \in \mathbb{R}\}$. Since all matrices are invertible it holds

$$(e^{tA})^T B e^{ta} = B \iff B e^{tA} B^{-1} = ((e^{tA})^T)^{-1}.$$

As $B e^{tA} B^{-1} = e^{tBAB^{-1}} = ((e^{tA})^T)^{-1} = e^{-tA^T}$. Differentiating this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tBAB^{-1}}\right) = BAB^{-1} e^{tBAB^{-1}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-tA^T}\right) = -A^T e^{-tA^T}$$

and evaluating it at $t = 0$ gives $BAB^{-1} = -A^T$. So

$$\mathfrak{o}(B) = \{A \in M_n(\mathbb{R}) : BAB^{-1} = -A^T\}.$$

For

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathfrak{o}(1,3) = \{A \in M_4(\mathbb{R}) : BAB^{-1} = -A^T\}$ is the Lie algebra of the Lorentz group $\mathrm{O}(1,3)$. Inserting the explicit form of B we get

$$B \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} B^{-1} = \begin{pmatrix} a_{11} & -a_{12} & -a_{13} & -a_{14} \\ -a_{21} & a_{22} & a_{23} & a_{24} \\ -a_{31} & a_{32} & a_{33} & a_{34} \\ -a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

From this it follows that $BAB^{-1} = -A^T$:

$$\begin{pmatrix} a_{11} & -a_{12} & -a_{13} & -a_{14} \\ -a_{21} & a_{22} & a_{23} & a_{24} \\ -a_{31} & a_{32} & a_{33} & a_{34} \\ -a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} -a_{11} & -a_{21} & -a_{31} & -a_{41} \\ -a_{12} & -a_{22} & -a_{32} & -a_{42} \\ -a_{13} & -a_{23} & -a_{33} & -a_{43} \\ -a_{14} & -a_{24} & -a_{34} & -a_{44} \end{pmatrix}.$$

We get the following conditions:

$$a_{11} = a_{22} = a_{33} = a_{44} = 0$$

and

$$a_{12} = a_{21}, \quad a_{13} = a_{31}, \quad a_{14} = a_{41}, \quad a_{23} = -a_{32}, \quad a_{24} = -a_{42}, \quad a_{34} = -a_{43}.$$

So we can write

$$\mathfrak{o}(1,3) = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{12} & 0 & a_{23} & a_{24} \\ a_{13} & -a_{23} & 0 & a_{34} \\ a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} : a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34} \in \mathbb{R} \right\}.$$

i) Define

$$\begin{aligned} \mathfrak{so}(1,3) &= \{A \in M_4(\mathbb{C}) : e^{tA} \in \mathrm{SO}(1,3) = \mathrm{O}(1,3) \cap \mathrm{SL}(4,\mathbb{R}) \, \forall t \in \mathbb{R}\} \\ &= \mathfrak{o}(1,3) \cap \mathfrak{sl}(4,\mathbb{R}) \\ &= \mathfrak{o}(1,3). \end{aligned}$$

j) Remember the Poincaré group

$$\begin{aligned} \mathrm{P}(1,3) = \{T \coloneqq F_{\mathbf{x}} \circ F_A : \mathbb{R}^4 \to \mathbb{R}^4 | \\ \mathbf{x} \in \mathbb{R}^4, A \in \mathrm{O}(1,3), \ F_{\mathbf{x}} : \mathbb{R}^4 \to \mathbb{R}^4, \mathbf{y} \mapsto \mathbf{y} + \mathbf{x}, \ F_A : \mathbb{R}^4 \to \mathbb{R}^4, \mathbf{y} \mapsto A\mathbf{y}\} \end{aligned}$$

and

$$\varphi(\mathrm{P}(1,3)) = \left\{ \begin{pmatrix} A & \mathbf{x} \\ \mathbf{0}^T & 1 \end{pmatrix} : A \in \mathrm{O}(1,3), \mathbf{x} \in \mathbb{R}^4 \right\} \subset M_5(\mathbb{R}).$$

Let $B \in M_5(\mathbb{R})$ such that $e^{tB} \in \varphi(\mathrm{P}(1,3))$. The derivative of e^{tB} contains the last row $(\mathbf{0}^T, 1)$. By evaluating it at $t = 0$, we get

$$B = \begin{pmatrix} C & \mathbf{x} \\ \mathbf{0}^T & 0 \end{pmatrix}, \qquad C \in M_4(\mathbb{R}), \mathbf{x} \in \mathbb{R}^4.$$

It is easy to show that

$$B^l = \begin{pmatrix} C^l & lC^{l-1}\mathbf{x} \\ \mathbf{0}^T & 0 \end{pmatrix}.$$

We have

$$e^{tB} = E_5 + \sum_{l=1}^{\infty} \frac{(tB)^l}{l!} = \begin{pmatrix} \sum_{l=0}^{\infty} \frac{(tC)^l}{l!} & \sum_{l=1}^{\infty} \frac{t^l C^{l-1}}{l!}\mathbf{x} \\ \mathbf{0}^T & 1 \end{pmatrix} \in \varphi(\mathrm{P}(1,3)) = \begin{pmatrix} e^{tC} & e^{tC}C^{-1}\mathbf{x} \\ \mathbf{0}^T & 1 \end{pmatrix},$$

if $C \in \mathfrak{o}(1,3)$.

Definition 6.6. Let $\mathfrak{g}$ be the Lie algebra for the matrix Lie group G. The *exponential mapping* is

$$\begin{aligned} \exp : \mathfrak{g} &\to G \\ X &\mapsto e^X. \end{aligned}$$

Remark.

a) For all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$ it holds $e^{tX} \in G$, so $\exp : \mathfrak{g} \to G$ is well-defined, but in general not surjective. For example

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$$

but is not in the form e^X for any $X \in \mathfrak{sl}(2,\mathbb{C}) = \{A \in M_2(\mathbb{C}) : \mathrm{tr}(A) = 0\}$. For a traceless 2×2 matrices there are two options:

- 0 is the only eigenvalue of X, so X is similar to

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

  which have exponentials

$$\exp \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \exp \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- The eigenvalues are non-zero and have opposite signs $\lambda_1 = -\lambda_2$. Then X is diagonalizable and e^{tX} is diagonal matrix.

b) We know $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\} = \mathrm{GL}(1,\mathbb{C})$.

Theorem 6.4. *Let $B : \mathbb{R} \to \mathrm{GL}(n,\mathbb{C})$ be a continuous one-parameter group, i.e.*
a) B is continuous,
b) $B(0) = E_n$,
c) $B(t+s) = B(t)B(s)$ for all $t, s \in \mathbb{R}$.
Then there exists a unique element $A \in M_n(\mathbb{C})$, such that $B(t) = e^{tA}$.

Lemma 6.5. *Let $A \in M_n(\mathbb{C})$ such that $|||A - E_n||| < 1$. Then*

$$\log(A) \coloneqq \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-E_n)^m}{m}$$

is well-defined, $\log : \{A \in M_n(\mathbb{C}) : |||A - E_n||| < 1\} \to M_n(\mathbb{C})$, $A \mapsto \log(A)$ is continuous and $\exp(\log(A)) = A$ for $|||A - E_n||| < 1$. If $B \in M_n(\mathbb{C})$ satisfies $|||B||| < \ln(2)$ then $|||e^B - E_n||| < 1$ and $\log(e^B) = B$.

Proof sketch. The power series

$$\left(\sum_{m=1}^{M} (-1)^{m+1} \frac{z^m}{m} \right)_{M \in \mathbb{N}}$$

has 1 as a radius of convergence, so if $|||A - E_n||| < 1$. The series converges locally uniformly on $\{A \in M_n(\mathbb{C}) : |||A - E_n||| < 1\}$ and absolutely, hence log is well-defined continuous function. Consider the case of diagonalizable $A = SDS^{-1}$ with $D = \operatorname{diag}(z_1, \dots, z_n)$. Then

$$(A - E_n)^m = S \operatorname{diag}((z_1 - 1)^m, \dots, (z_n - 1)^m) S^{-1}$$

and

$$\log(A) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{S \operatorname{diag}((z_1 - 1)^m, \dots, (z_n - 1)^m) S^{-1}}{m} = S \operatorname{diag}(\log(z_1), \dots, \log(z_n)) S^{-1},$$

so

$$\begin{aligned} e^{\log(A)} &= \exp(S \operatorname{diag}(\log(z_1), \dots, \log(z_n)) S^{-1}) \\ &= S \exp(\operatorname{diag}(\log(z_1), \dots, \log(z_n))) S^{-1} \\ &= S \operatorname{diag}(z_1, \dots, z_n) S^{-1} \\ &= SDS^{-1} = A. \end{aligned}$$

□

Theorem 6.6. *For $\varepsilon \in (0, \ln 2)$ let $U_\varepsilon \coloneqq \{X \in M_n(\mathbb{C}) : |||X||| < \varepsilon\}$ and $V_\varepsilon \coloneqq \exp(U_\varepsilon)$. If $G \subseteq \mathrm{GL}(n, \mathbb{C})$ is a matrix Lie group with Lie algebra $\mathfrak{g}$, then there exists $\varepsilon \in (0, \ln 2)$ such that for all $A \in V_\varepsilon$ it holds $A \in G \iff \log(A) \in \mathfrak{g}$. So there is a neighbourhood U of $\mathbf{0}$ in $\mathfrak{g}$ and a neighbourhood V of E_n in G such that the restriction of $\exp$ to U $\exp|_U : U \to V$ is continuous, bijective and the inverse is continuous.*

Theorem 6.7. *Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and $\phi : G \to H$ be a Lie group homomorphism. Then there is a unique $\mathbb{R}$-linear map $\varphi : \mathfrak{g} \to G$ such that for $X \in \mathfrak{g}$*

$$\phi(e^X) = e^{\varphi(X)}.$$

These maps have some notable properties.

a) $\varphi(AXA^{-1}) = \phi(A)\varphi(X)\phi(A^{-1})$ for all $A \in G$ and $X \in \mathfrak{g}$.

b) $\varphi([X,Y]) = [\varphi(X), \varphi(Y)]$ for all $X, Y \in \mathfrak{g}$.

c) $\varphi(X) = (\mathrm{d}/\mathrm{d}t\, \phi(e^{tX}))|_{t=0}$ for all $X \in \mathfrak{g}$.

Proof. As $\phi : G \to \mathfrak{g}$ is continuous for each $X \in \mathfrak{g}$, it holds that $B : \mathbb{R} \to H$, $t \mapsto \phi(e^{tX})$ is a continuous one-parameter subgroup of H. By Theorem 6.4 there is a unique $Z_X \in M_n(\mathbb{C})$ such that $B(t) = \phi(e^{tX}) = e^{tZ_X}$ for all $t \in \mathbb{R}$. As $\phi(e^{tX}) = e^{tZ_X}$ for all $t \in \mathbb{R}$ it implies $Z_X \in G$, so $\varphi : \mathfrak{g} \to G$, $X \mapsto Z_X$ is well-defined and satisfies:

- for $t = 1$, $\phi(e^X) = e^{Z_X} = \varphi(X)$;
- for $t \to st$, $e^{ts\varphi(X)} = e^{stZ_X} = \phi(e^{stX}) = \phi(e^{s(tX)}) = e^{tZ_{sX}} = e^{t\varphi(sX)}$ for all $t \in \mathbb{R}$. So $s\varphi(X)$ and $\varphi(sX)$ are generators of the same continuous one-parameter subgroup, so $s\varphi(X) = \varphi(sX)$.

□

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