

Faculty of physics

Advanced Quantum Field Theory

Notes of lectures by Prof. Lorenzo Tancredi, Ph.D.

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Disclaimer: these notes are not official and were not endorsed by professor. The content is almost entirely based on the lectures, but occasionally the notes were modified or extended slightly to improve clarity or add extra information. In particular, the mistakes are almost surely mine. If you find any, please let me know at simon.bukovsek@tum.de or Simon.Bukovsek@campus.lmu.de.

Overview

Core topics:

- Consequences of unitarity of QFT,
- On-shell methods.
- Non-Abelian gauge theories,
- Spontaneous symmetry breaking (Quantum theory),
- Anomalies,

Chapter 1

Unitarity in QFT

1.1 Optical theorem

The crucial consequence of unitarity is that the probability is conserved. Given a *Hamiltonian* $\mathcal{H}$, the evolution of the *S-matrix* is $S = e^{i\mathcal{H}t}$. The state in Schrödinger picture evolves as

$$|\psi, t\rangle = e^{-i\mathcal{H}t}\,|\psi, 0\rangle = S\,|\psi, 0\rangle\,.$$

As the probability is conserved, we have

$$\langle\psi, t|\psi, t\rangle = \langle\psi, 0|S^\dagger S|\psi, 0\rangle = \langle\psi, 0|\psi, 0\rangle\,.$$

It follows that $S^\dagger S = 1$ (unitarity of the S-matrix) or equivalently

$$(e^{-i\mathcal{H}t})^\dagger(e^{i\mathcal{H}t}) = e^{i(\mathcal{H}^\dagger - \mathcal{H})t} = \mathbb{1} \quad \Longrightarrow \quad \mathcal{H}^\dagger = \mathcal{H}$$

(Hamiltonian is Hermitian). However, we are most interested in the T-matrix (or *transfer matrix*), given by $S = \mathbb{1} - iT$. The T-matrix is related to the *scattering amplitude* $\mathcal{M}$ by the relation

$$\langle f|T|i\rangle = (2\pi)^4\delta^{(4)}(p_i - p_f)\mathcal{M}(i \to f).$$

The matrix $\mathcal{M}$ is calculated using Feynman diagrams. The unitarity of the S-matrix implies the following relation for the T-matrix:

$$i(T^\dagger - T) = T^\dagger T.$$

Manipulating this equation gives us the following on the right-hand side:

$$\begin{aligned}\langle f|i(T^\dagger - T)|i\rangle &= i\,\langle f|T^\dagger|i\rangle - i\,\langle f|T|i\rangle = i(\langle i|T|f\rangle^* - \langle i|T|f\rangle)\\ &= i(2\pi)^4\delta^{(4)}(p_i - p_f)[\mathcal{M}^*(f \to i) - \mathcal{M}(i \to f)]\end{aligned}$$

and on the left-hand side:

$$\begin{aligned}\langle f|T^\dagger T|i\rangle &= \sum_x \int \mathrm{d}x\ \langle f|T^\dagger|x\rangle\,\langle x|T|i\rangle\\ &= \sum_x \int \mathrm{d}x\ \delta^{(4)}(p_f - p_x)\delta^{(4)}(p_i - p_f)\mathcal{M}(i \to x)\mathcal{M}^*(f \to x).\end{aligned}$$

We inserted the identity operator

$$\sum_x \int \mathrm{d}x \ |x\rangle \langle x| = 1,$$

where $|x\rangle$ is a complete set of (multiparticle) states and the measure is given by

$$\mathrm{d}x = \prod_f \frac{\mathrm{d}^3\mathbf{p}_f}{(2\pi)^3} \frac{1}{2E_f}.$$

Equating the left-hand side and the right-hand side gives

$$\mathcal{M}(i \to f) - \mathcal{M}^*(f \to i) = i \sum_x \int \mathrm{d}x \ (2\pi)^4 \delta^{(4)}(p_i - p_x) \mathcal{M}(i \to x) \mathcal{M}^*(f \to x).$$

This is called the *optical theorem*. Specializing to the case where the initial state equals the final state ($|i\rangle = |f\rangle = |y\rangle$), we get

$$\mathcal{M}(y \to y) - \mathcal{M}^*(y \to y) = 2\mathrm{Im}\left(\mathcal{M}(y \to y)\right) = i \sum_x \int \mathrm{d}x (2\pi)^4 \delta^{(4)}(p_y - p_x) |\mathcal{M}(y \to x)|^2.$$

Let's consider two distinct cases:

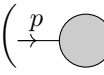
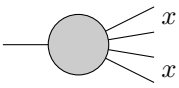
1) $|y\rangle$ is a one particle state. In this case we are talking about the decay width

$$\Gamma(y \to x) = \frac{1}{2m_y} \int \mathrm{d}x \ (2\pi)^4 \delta^{(4)}(p_y - p_x) |\mathcal{M}(y \to x)|^2.$$

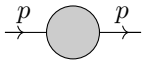
The optical theorem gives can then be rewritten as

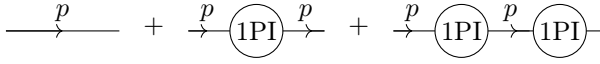
$$\mathrm{Im}\left(\mathcal{M}(y \to y)\right) = m_y \sum_x \Gamma(y \to x) = m_y \Gamma_{\mathrm{tot}}.$$

The left-hand side is the imaginary part of the two point correlation function put on shell. Diagrammatically, this reads

$\mathrm{Im}\Big($  $\Big) = m_y \sum_x \Big|$  $\Big|^2$

The two-point function is given as a sum of all possible diagrams with two external legs, which can be summed as a geometric series of *one-particle-irreducible* (1PI) diagrams:

$i\Pi(p) =$ 

$=$ 

$$= \frac{i}{p^2 - m_y^2 + \Sigma(p^2) + i\varepsilon}.$$

The mass m_y is the bare mass of the Lagrangian, while $\Sigma(p^2)$ is the self-energy correction and refers to the 1PI. The defining (on-shell) relation for the physical mass $\overline{m}_y$ of the particle is

$$(\overline{m}_y)^2 - m_y^2 + \Sigma(\overline{m}_y^2) = 0.$$

But this only works for real Σ. In general, it holds $\Sigma(p^2) = \text{Re}\ \Sigma(p^2) + i\text{Im}\ \Sigma(p^2)$. The total decay rate is

$$\begin{aligned}\Gamma_{\text{tot}} &= \frac{1}{m_y}\text{Im}\left[\mathcal{M}(y \to y)\right] = \frac{1}{m_y}\text{Im}\ \left[i\Pi(p^2)\right] \\ &= \frac{1}{m_y}\text{Im}\left(\begin{array}{c}p \\ \rightarrow \\ \end{array}\!\!\boxed{\text{1PI}}\!\!\begin{array}{c}\rightarrow \\ p \\ \end{array}\right) + \mathcal{O}(\Gamma_{\text{tot}}/m_y) \\ &\approx \frac{1}{m_y}\text{Im}\left(\Sigma(\overline{m}_y)\right).\end{aligned}$$

The assumption $\Gamma_{\text{tot}}/m_y \ll 1$ is that the particle is unstable but long-lived. The condition for the mass of the particle is then

$$(\overline{m}_y)^2 - m_y^2 + \text{Re}\ \Sigma(\overline{m}_y^2) = 0.$$

This is called the *real pole mass scheme*. The propagator can be simplified to

$$i\Pi(p^2 \sim \overline{m}_y^2) \approx \frac{i}{p^2 - (\overline{m}_y^2) + i\overline{m}_y\Gamma_{\text{tot}}}.$$

It is the *Breit-Wigner propagator*. The cross-section $xx \to y \to zz$ can be approximated by

$$\sigma \sim \left|\begin{array}{ccc} x & & z \\ & \diagdown\!\!\diagup\!\!\!

Symbolically, it reads

$$\mathrm{Im}\,(\mathcal{M}(p_1p_2 \to p_1p_2)) = 2\sqrt{s}|\mathbf{p}_{\mathrm{CM}}| \sum_x \sigma(p_1p_2 \to x) = 2\sqrt{s}|\mathbf{p}_{\mathrm{CM}}|\sigma(p_1p_2 \to \text{anything}).$$

Assume we have a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

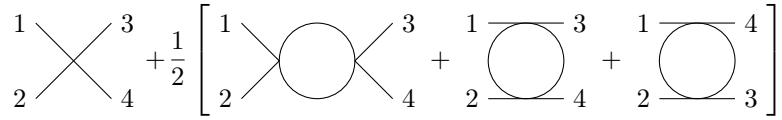
and consider the scattering process

$$\phi(p_1) + \phi(p_2) \to \phi(p_3) + \phi(p_4).$$

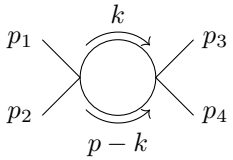
We will use the *Mandelstam variables*

$$a = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_2 - p_3)^2.$$

It also holds $u + t + s = 4m^2$ and in order for the reaction to happen, we need $s \geq 4m^2$ and $t, u < 0$. The scattering amplitude is given by Feynman diagrams:

$i\mathcal{M}^{(1)}(1\,2 \to 3\,4) =$  $+\mathcal{O}(\lambda^3).$

One particular part is (denoting $p = p_1 + p_2$, $s = p^2$)

$i\mathcal{M}_s(p^2) =$  $= -\lambda^2 \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{i^2}{(k^2 - m^2 - i\delta)((k+p)^2 - m^2 + i\delta)}.$

Now we use the *Feynman parametrization* to combine the two propagators into one:

$$\begin{aligned}
i\mathcal{M}_s(p^2) &= -i^2\lambda^2 \int \frac{\mathrm{d}^D k}{(2\pi)^D} \int_0^1 \mathrm{d}x\, \frac{1}{[(k^2 - m^2)(1-x) + (k^2 + p^2 + 2kp)x - m^2x + i\delta]^2} \\
&= -i^2\lambda^2 \int \frac{\mathrm{d}^D k}{(2\pi)^D} \int_0^1 \mathrm{d}x\, \frac{1}{[(k+px)^2 + p^2x(1-x) - m^2 + i\delta]^2} \\
&= \lambda^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{[k^2 + p^2x(1-x) - m^2 + i\delta]^2} \\
&\overset{\text{Wick}}{=} \lambda^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k_{\mathrm{E}}}{(2\pi)^D} \frac{1}{[k_{\mathrm{E}}^2 + p_{\mathrm{E}}^2x(1-x) - m^2]^2} \\
&= i\lambda^2 \frac{\Gamma(\varepsilon)}{(4\pi)^{D/2}} \int_0^1 \mathrm{d}x\, \left(p_{\mathrm{E}}^2x(1-x) + m^2\right)^{-\varepsilon} \\
&= \frac{i\lambda^2}{(4\pi)^{D/2}}\Gamma(1+\varepsilon)\left[\frac{1}{\varepsilon} - \int_0^1 \mathrm{d}x\, \ln\left(p_{\mathrm{E}}^2x(1-x) + m^2\right)\right] + \mathcal{O}(\varepsilon)
\end{aligned}$$

In the calculation above, we used a Wick rotation and introduced $p_{\mathrm{E}}^2 = p^2 - i\delta$ and the dimensional regularization $\varepsilon = 2 - D/2$. We also used the fact that $\Gamma(1+\varepsilon) = \varepsilon\Gamma(\varepsilon)$. The question is, when

is the expression in the integral divergent. It happens, when the argument of the logarithm is zero, $m^2 + p^2 x(1-x) = 0$. This happens for

$$x_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4m^2}{p^2}}.$$

We now denote $\Delta(x) = m^2 - p^2 x(1-x)$ and evaluate the integral in three pieces:

$$\begin{aligned}\int_0^1 \mathrm{d}x\, \ln(\Delta(x) - i\delta) &= \int_0^{x_-} \mathrm{d}x\, \ln(\Delta(x)) + \int_{x_+}^1 \mathrm{d}x\, \ln(\Delta(x)) + \int_{x_-}^{x_+} \mathrm{d}x\, \ln(-[-\Delta(x)] - i\delta) \\ &= \int_0^{x_-} + \int_{x_+}^1 \mathrm{d}x\, \ln(\Delta(x)) + \int_{x_-}^{x_+} \mathrm{d}x\, \ln(-\Delta(x)) + \ln(-1 - i\delta) \int_{x_-}^{x_+} \mathrm{d}x \\ &= \int_0^{x_-} \mathrm{d}x\, \ln(\Delta(x)) + \int_{x_+}^1 \mathrm{d}x\, \ln(\Delta(x)) + \int_{x_-}^{x_+} \mathrm{d}x\, \ln(-\Delta(x)) - i\pi(x_1 - x_2)\end{aligned}$$

Hence, the imaginary part of the scattering amplitude is given by

$$\mathrm{Im}\,[\mathcal{M}_s(s)] = \frac{\lambda^2}{16\pi^2}\sqrt{1 - \frac{4m^2}{s}}\theta(s - 4m^2).$$

The branch cut discontinuity of the amplitude is

$$\mathrm{Disc}[i\mathcal{M}_s(s)] = \lim_{\delta\to 0}\left[i\mathcal{M}_s(s + i\delta) - i\mathcal{M}_s(s - i\delta)\right] = 2i\mathrm{Im}\,\mathcal{M}_s(s).$$

There is no imaginary part for u and t channel, as they have the same structure (with s swapped for appropriate letter) and both u and t are not greater than $4m^2$ (there is also no imaginary contribution from the tree level). Hence,

$$\mathrm{Im}\,[\mathcal{M}^{(1)}(1\,2 \to 3\,4)] = \frac{\lambda^2}{16\pi^2}\sqrt{1 - \frac{4m^2}{s}}\theta(s - 4m^2) = \mathrm{Im}\,[\mathcal{M}^{(1)}(1\,2 \to 1\,2)] \propto \sum_x \sigma(1\,2 \to x).$$

At tree level to the lowest order, the scattering amplitude is simply $i\mathcal{M}^{\mathrm{tree}}(1\,2 \to 3\,4) = i\lambda$. The cross-section is

$$\begin{aligned}\sigma(1\,2 \to 3\,4) &= \frac{1}{4\sqrt{s}|\mathbf{p}_{\mathrm{CM}}|}\int \frac{\mathrm{d}^3 p_3}{(2\pi)^3}\frac{1}{2E_{\mathbf{p_3}}}\int \frac{\mathrm{d}^3 p_4}{(2\pi)^3}\frac{1}{2E_{\mathbf{p_4}}}(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p4)\lambda^2 \\ &= \frac{1}{2\sqrt{s}|\mathbf{p}_{\mathrm{CM}|}}\frac{\lambda^2}{4}\Pi_2.\end{aligned}$$

The Lorentz invariant phase space is given by

$$\Pi_2 = \int \frac{\mathrm{d}^3 p_3}{(2\pi)^4}(2\pi)\delta(p_3^2 - m^2)\int \frac{\mathrm{d}^3 p_4}{(2\pi)^4}(2\pi)\delta(p_4^2 - m^2)(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4) = \frac{1}{8\pi}\sqrt{1 - \frac{4m^2}{s}}.$$

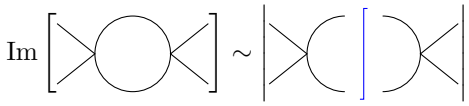
Therefore,

$$\sigma(1\,2 \to 3\,4) = \frac{1}{2\sqrt{s}|\mathbf{p}_{\mathrm{CM}}|}\frac{\lambda^2}{4}\frac{1}{8\pi}\sqrt{1 - \frac{4m^2}{s}}\theta(s - 4m^2)$$

and

$$\mathrm{Im}\,(\mathcal{M}(1\,2 \to 1\,2)) = 2\sqrt{s}|\mathbf{p}_{\mathrm{CM}}|\sum_x \sigma(1\,2 \to x) = \frac{\lambda^2}{32\pi^2}\sqrt{1 - \frac{4m^2}{s}}.$$

We are now going to show for this example (but it works in general), that we can cut the loop and replace the cuts with delta functions:



1.2 Cutkosky rules

Consider the scattering process

$$p_1 + p_2 \to p_3 + \cdots + p_N.$$

These could have any spin, but ultimately we always compute scalar quantities, so we can consider spin zero. The Feynman propagator is

$$\Pi_{\mathrm{F}}(k) = \frac{i}{k^2 - m^2 + i\varepsilon}.$$

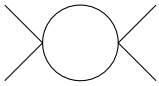
The discontinuity of the propagator is given by

$$\mathrm{Disc}[\Pi_{\mathrm{F}}(k)] = \lim_{\varepsilon\to 0}\left[\frac{i}{k^2-m^2+i\varepsilon} - \frac{i}{k^2-m^2-i\varepsilon}\right] = \lim_{\varepsilon\to 0}\left[\frac{2\varepsilon}{(k^2-m^2)^2+\varepsilon^2}\right] = 2\pi\delta(k^2-m^2).$$

The Feynman propagator can also be written using $\omega_k = \sqrt{\mathbf{k}^2 + m^2}$

$$\begin{aligned}
\Pi_{\mathrm{F}}(k) &= \frac{1}{2\omega_k}\left[\frac{1}{k_0-\omega_k+i\varepsilon} - \frac{1}{k_0+\omega_k-i\varepsilon}\right] \\
&= \underbrace{\frac{i}{2\omega_k}\left[\frac{1}{k_0-\omega_k-i\varepsilon} - \frac{1}{k_0+\omega_k-i\varepsilon}\right]}_{\Pi_{\mathrm{R}}(k)} + \frac{i}{2\omega_k}\underbrace{\left[\frac{1}{k_0+\omega_k-i\varepsilon} - \frac{1}{k_0-\omega_k-i\varepsilon}\right]}_{-2i\varepsilon/((k_0-\omega_k)^2+\varepsilon^2)} \\
&= \Pi_{\mathrm{R}}(k) + \frac{\pi}{\omega_k}\delta(k_0-\omega_k).
\end{aligned}$$

The scattering amplitude is

$i\mathcal{M}(s) =$ 

$$\begin{aligned}
&= -\lambda^2\int\frac{\mathrm{d}^D k}{(2\pi)^D}\Pi_{\mathrm{F}}(k)\Pi_{\mathrm{F}}(k-p) \\
&= -\lambda^2\left[\int\frac{\mathrm{d}^D k}{(2\pi)^D}\Pi_{\mathrm{R}}(k)\Pi_{\mathrm{R}}(k-p) + \int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\pi^2}{\omega_k\omega_{k-p}}\delta(k_0-\omega_k)\delta(k_0-p_0-\omega_{k-p}) + \right. \\
&\qquad \left. + \int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\pi}{\omega_k}\delta(k_0-\omega_k)\Pi_{\mathrm{R}}(k-p) + \int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\pi}{\omega_{k-p}}\delta(k_0-p_0-\omega_{k-p})\Pi_{\mathrm{R}}(k)\right].
\end{aligned}$$

The first term is zero, because we can close the contour without any poles, since all are on the same side of the plane. The second term is also zero, as we can always go in the special frame $p^\mu = (\sqrt{s}, 0)$, where $\omega_k = \omega_{k-p} = \sqrt{\mathbf{k}^2 + m^2}$. In the last two terms, we can exchange the retarded propagators with the Feynman propagators (the additional terms would be the same

as the second term in the expression, which we have already showed is zero). The discontinuity of the scattering amplitude is

$$\begin{aligned}\text{Disc}\left[i\mathcal{M}(s)\right] = -\lambda^2 \Bigg[&\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\pi}{\omega_k} \delta(k_0 - \omega_k)(2\pi)\delta((k-p)^2 - m^2) + \\ &+ \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\pi}{\omega_{k-p}} \delta(k_0 - p_0 - \omega_{k-p})(2\pi)\delta(k^2 - m^2) \Bigg] .\end{aligned}$$

The delta functions of the squares can be expanded as

$$\delta(k^2 - m^2) = \delta((k_0 - \omega_k)(k_0 + \omega_k)) = \frac{1}{2\omega_k} \left[\delta(k_0 - \omega_k)\theta(k_0) + \delta(k_0 + \omega_k)\theta(-k_0)\right]$$

and the discontinuity becomes

$$\text{Disc}(i\mathcal{M}(s)) = -i\lambda^2 \left[\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\pi}{\omega_k} \delta(k_0 - \omega_k) \frac{\pi}{\omega_{k-p}} \delta(p_0 - k_0 - \omega_{k-p}) \right] .$$

The delta functions don't look particularly Lorentz invariant, but we can express them covariantly if we assure that the energy is positive:

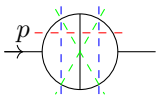
$$\delta^{(+)}(k^2 - m^2) = \delta(k^2 - m^2)\theta(k_0) = \frac{\delta(k_0 - \omega_k)}{2\omega_k},$$

$$\delta^{(+)}((p-k)^2 - m^2) = \delta((p-k)^2 - m^2)\theta(p_0 - k_0) = \frac{\delta(p_0 - k_0 - \omega_{k-p})}{2\omega_{k-p}}.$$

The discontinuity is now expressible covariantly as

$$\text{Disc}(i\mathcal{M}(s)) = -\lambda^2 \int \frac{\mathrm{d}^D k}{(2\pi)^D} 2\pi\delta^{(+)}\left(k^2 - m^2\right) 2\pi\delta^{(+)}\left((p-k)^2 - m^2\right) .$$

The cuts can only cut the propagators where the particles can go on shell. If the diagram is more complicated, we have to sum over all permissible cuts. In the diagram below, we can cut green and blue lines, but not the red one.



1.3 Poles of the Green function

Assume scalar field ϕ. The n-point Green function is given by

$$G_n(p_1, \ldots, p_n) = \int \mathrm{d}^4 x_1 \, e^{ip_1 x_1} \cdots \int \mathrm{d}^4 x_n \, e^{ip_n x_n} \, \langle \Omega | T\{\phi(x_1) \ldots \phi(x_n)\} | \Omega \rangle \, .$$

Let r of these particles be incoming and $n - r$ outgoing and denote

$$p = p_1 + \cdots + p_r = p_{r+1} + \cdots + p_n.$$

For the theorem, we need to assume that there exists a state $|\psi\rangle$ with mass m_ψ in the Hilbert space, such that $\langle \psi | \phi(x_1) \ldots \phi(x_r) | \Omega \rangle \neq 0$. Then the following theorem holds.

Theorem 1.1. *The Green function has a pole at $p^2 = m_\psi^2$.*
As the momentum p approaches the pole, the Green function behaves as

$$G(p_1,\dots,p_n) = (2\pi)^4\delta^{(4)}(p_1+\dots+p_r-p_{r+1}-\dots-r_n)\times\mathcal{M}_\psi^{1,r}\frac{i}{p^2-m_\psi^2+i\varepsilon}\mathcal{M}_\psi^{r+1,n}$$
$$+ \textit{non-divergent terms}$$

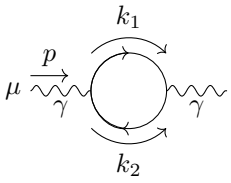
where

$$(2\pi)^4\delta^{(4)}(p_1+\dots+p_r-p)\mathcal{M}_\psi^{1,r} = \int \mathrm{d}^4x_1\, e^{ip_1x_1}\cdots\int \mathrm{d}^4x_r\, e^{ip_nx_r}\,\langle\Omega|T\{\phi(x_1)\dots\phi(x_r)\}|\Omega\rangle\,.$$

The theorem is very technical, and we will omit the proof. The reader can find it in Weinberg or Schwartz. However, we will use the theorem several times during this course.

1.4 Polarization sums and propagators

Let's consider a QED example:



$$= A = C(-1)\int\frac{\mathrm{d}^Dk_1}{(2\pi)^D}\frac{\mathrm{Tr}[\gamma^\mu(\not{k}_2-m)\gamma^\nu(\not{k}_1+m)]}{(k_1^2-m^2)(k_2^2-m^2)}(2\pi)^D\delta^{(D)}(k_1+k_2-p).$$

We have moved all the couplings to the overall constant C. The factor (-1) comes from the fermion loop. The discontinuity is

$$\mathrm{Disc}(A) = C\int\frac{\mathrm{d}^Dk_1}{(2\pi)^D}\,\mathrm{Tr}\left[\gamma^\mu(\not{k}_2-m)\gamma^\nu(\not{k}_1+m)\right](2\pi)^D\delta^{(D)}(k_1+k_2-p)$$
$$(2\pi)\delta^+(k_1^2-m^2)(2\pi)\delta^+(k_2^2-m^2)$$

On the other hand, we can also calculate the decay rate of the photon (that is not on-shell):

$$\Gamma(\gamma^*\to e^+e^-) = \left|\ \gamma^* \rightsquigarrow \begin{matrix} e^+ \\ e^- \end{matrix}\ \right|^2$$
$$= \frac{C}{p^2}\sum_{s,s'}\int\frac{\mathrm{d}^{D-1}k_1}{(2\pi)^{D-1}}\frac{1}{2\omega_{k_1}}\int\frac{\mathrm{d}^{D-1}k_2}{(2\pi)^{D-1}}\frac{1}{2\omega_{k_2}}$$
$$(2\pi)^D\delta^{(D)}(k_1+k_2-p)\overline{v}_s(k_2)\gamma^\mu u_{s'}(k_1)\overline{u}_{s'}(k_1)\gamma^\nu v_s(k_2)$$
$$= \frac{C}{p^2}\sum_{s,s'}\int\frac{\mathrm{d}^{D-1}k_1}{(2\pi)^{D-1}}\frac{1}{2\omega_{k_1}}\int\frac{\mathrm{d}^{D-1}k_2}{(2\pi)^{D-1}}\frac{1}{2\omega_{k_2}}$$
$$(2\pi)^D\delta^{(D)}(k_1+k_2-p)\,\mathrm{Tr}[\gamma^\mu(\not{k}_2-m)\gamma^\nu(\not{k}_1+m)].$$

The spin sums for massive spin 1 particles (Proca field) is given by

$$\sum_{j=1}^{3} \epsilon_j^\mu (\epsilon_j^\nu)^* = \frac{p^\mu p^\nu}{m^2} - g^{\mu\nu}$$

where the polarization is transversal to the momentum $p^\mu \epsilon_\mu = 0$. The propagator is

$$\Pi^{\mu\nu}(p) = -i \left(\frac{g^{\mu\nu} - p^\mu p^\nu / m^2}{p^2 - m^2 + i\varepsilon} \right).$$

For the massless spin 1 particles (photons) there is a slight complication. In addition to the transversal condition, we also have to impose the gauge condition, in this case we will choose the *axial gauge* $\epsilon^\mu n_\mu = 0$ for some light-like vector $n^\mu n_\mu = 0$. The spin sum is

$$\sum_{j=1}^{2} \epsilon_j^\mu (\epsilon_j^\nu)^* = -g^{\mu\nu} + \frac{p^\mu n^\nu + p^\nu n^\mu}{p^\sigma n_\sigma}.$$

The propagator is then given by

$$\Pi^{\mu\nu}(p) = i \frac{\sum_{j=1}^{2} \epsilon_j^\mu (\epsilon_j^\nu)^*}{p^2 - i\varepsilon}.$$

1.5 Spectral function

We start with the definition of the two-point correlation function

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \sum_X \int \mathrm{d}X \, \langle \Omega | \phi(x) | X \rangle \langle X | \phi(y) | \Omega \rangle .$$

We now write

$$\langle \Omega | \phi(x) | X \rangle = \langle \Omega | e^{i\hat{P}x} e^{-i\hat{P}x} \phi(x) e^{i\hat{P}x} e^{-i\hat{P}x} | X \rangle .$$

The momentum operator $\hat{P}^\mu$ acts on the state $|X\rangle$ as $\hat{P}^\mu |X\rangle = p_X^\mu |X\rangle$. It also holds $\hat{P}^\mu |\Omega\rangle = 0$. Using this, we can move the field $\phi(x)$ back to the origin:

$$e^{-i\hat{p}x} \phi(x) e^{i\hat{p}x} = \phi(0).$$

We can now write the two-point correlation function as

$$\begin{aligned} \langle \Omega | \phi(x) \phi(y) | \Omega \rangle &= \sum_X \int \mathrm{d}X e^{-i\hat{P}_X (x-y)} | \langle \Omega | \phi(0) | X \rangle |^2 \\ &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip(x-y)} \underbrace{\sum_X \int \mathrm{d}X \, (2\pi^4) \delta^{(4)}(p - p_X) | \langle \Omega | \phi(0) | X \rangle |^2}_{F(p^2),\ \text{real, positive}} . \end{aligned}$$

It is usual to parametrize the function $F(p^2)$ as

$$F(p^2) = 2\pi \theta(p_0) \rho(p^2)$$

where $\rho(p^2)$ is the *spectral function*. We can write

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)}(2\pi)\theta(p_0)\rho(p^2).$$

Next, we introduce the free particle propagator (not time ordered yet) in the coordinate space:

$$D(x,y,m^2) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{e^{-ip(x-y)}}{2\omega_p} = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)}(2\pi)\theta(p_0)\delta(p^2-m^2) = \langle\Omega|\phi_0(x)\phi_0(y)|\Omega\rangle\,.$$

We see that we can rewrite the two-point correlation function as

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \mathrm{d}q^2\delta(q^2-p^2)\int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)}(2\pi)\theta(p_0)\rho(p^2) = \int_0^\infty \mathrm{d}q^2\rho(q^2)D(x,y,q^2).$$

In the free theory, it holds $\rho(q^2) = \delta(q^2-m^2)$. The last step is to time-order the two-point correlation function. We can write

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \int \mathrm{d}q^2\,\rho(q^2)\underbrace{\left[D(x,y,q^2)\theta(x_0-y_0) + D(y,x,q^2)\theta(y_0-x_0)\right]}_{\Delta(x,y,q^2)}$$

where

$$\Delta(x,y,q^2) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-q^2+i\varepsilon}e^{ip(x-y)}$$

is called the free Feynman propagator. Writing the propagator in the full form, we have

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \int \frac{d^4p}{(2\pi)^4}e^{ip(x-y)}i\int \mathrm{d}q^2\,\frac{\rho(q^2)}{p^2-q^2+i\varepsilon}$$

and

$$\Pi(p^2) = i\int \mathrm{d}q^2\,\frac{\rho(q^2)}{p^2-q^2+i\varepsilon}.$$

The discontinuity of the propagator is

$$\mathrm{Disc}[\Pi(p^2)] = 2\pi\rho(p^2).$$

Until now, we didn't know much about the spectral function, but now we can use the theorem from the previous section

$$\rho(p^2)\frac{1}{2\pi}\,\mathrm{Disc}\left[\Pi(p^2)\right].$$

On the other hand, we can also use the general definition of the spectral function:

$$(2\pi)\rho(q^2)\theta(q_0) = \sum_X \int \mathrm{d}X\,(2\pi)^4(q-p_X)|\,\langle\Omega|\phi(0)|X\rangle\,|^2.$$

Assume that there exists a single particle state ($X=1$) with p_1 and $p_1^2 = m^2$. We can then split the measure in two parts

$$\sum_X \int \mathrm{d}X = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\delta^+(p_1^2-m^2) + \sum_{X\neq 1}\int \mathrm{d}X.$$

We do the same with the spectral function:

$$\rho(q^2)\theta(q_0) = \left[\rho_1(q^2) + \rho_{X\neq 1}(q^2)\right]\theta(q_0).$$

Furthermore, for the one-particle state, we can write

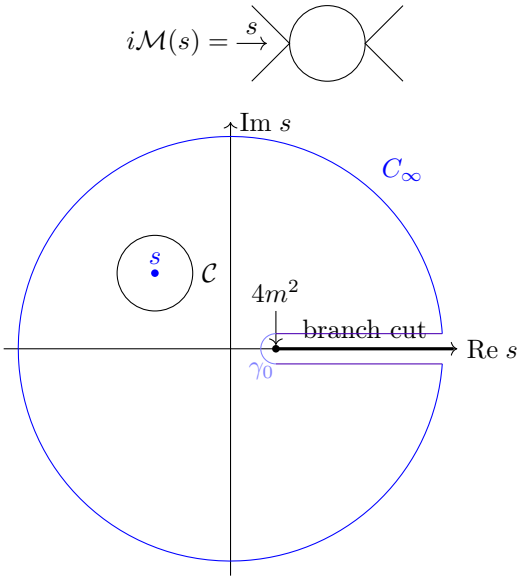
$$\rho_1(q^2)\theta(q_0) = \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4}\delta^+(p_1^2 - m^2)|\langle\Omega|\phi(0)|X\rangle|^2(2\pi)^4\delta^4(q-p_1) = \delta(q^2-m^2)\theta(q_0)|\langle\Omega|\phi(0)|1q\rangle|^2.$$

The propagator is

$$\pi(p^2) = i\frac{|\langle\Omega|\phi(0)|1p\rangle|^2}{p^2 - m^2 + \varepsilon} + i\int_0^\infty \mathrm{d}q^2\,\frac{\rho_{X\neq 1}(q^2)}{p^2 - q^2 + i\varepsilon} = \frac{iZ}{p^2 - m^2 + i\varepsilon} + \text{multiparticle states}.$$

1.6 Dispersion relations

We study the scattering amplitude



We perform the clever trick of evaluating the contour integral in the s complex plane around a particular point s:

$$\begin{aligned}
i\mathcal{M}_s(s) &= \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{\mathrm{d}s'}{s'-s}(i\mathcal{M}_s(s')) \\
&= \frac{1}{2\pi i}\left[\int_{C_\infty} + \int_{\gamma_0}\right]\frac{\mathrm{d}s'}{s'-s}(i\mathcal{M}_s(s')) + \frac{1}{2\pi i}\left[\int_{4m^2+i\varepsilon}^{\infty} - \int_{4m^2-i\varepsilon}^{\infty}\right]\frac{\mathrm{d}s'}{s'-s}(i\mathcal{M}_s(s')) \\
&= \frac{1}{2\pi i}\left[\int_{C_\infty}\frac{\mathrm{d}s'}{s'-s}(i\mathcal{M}_s(s')) + \int_{4m^2}^{\infty}\frac{\mathrm{d}s'}{s'-s}\,\mathrm{Disc}\left[i\mathcal{M}_s(s')\right]\right].
\end{aligned}$$

We have derived the *dispersion relation* for the scattering amplitude:

$$i\mathcal{M}_s(s) = \frac{1}{2\pi i}\int_{C_\infty}\frac{\mathrm{d}s-}{s'-s}(i\mathcal{M}_s(s-)) + \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{\mathrm{d}s'}{s'-s}\mathrm{Im}\,(i\mathcal{M}_s(s')).$$

The first term is zero unless there is a UV divergence. Furthermore, we can calculate the difference between two amplitudes:

$$i\mathcal{M}_s(s) - i\mathcal{M}_s(s_0) = \\ = \frac{1}{2\pi i}\int_{C_\infty} \mathrm{d}s' \, (i\mathcal{M}_s(s-)) \frac{s-s_0}{(s'-s)(s'-s_0)} + \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}s' \, \frac{s-s_0}{(s'-s)(s'-s_0)} \mathrm{Im}\,(i\mathcal{M}_s(s')).$$

Now, the terms fall as $1/s'^2$, so they are less divergent. We might need more than one subtraction, so that we can suppress the divergences in the scattering amplitude. These subtractions are closely related to the counter-terms in the renormalization theory.

Chapter 2

On-shell method

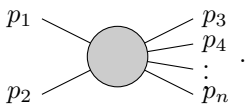
This is a huge topic, that will draw a lot of results from the previous chapter.

2.1 Spinor-helicity formalism

This formalism has been developed in the last 25 years. It will simplify calculating scattering amplitudes for massless particles and will be very useful in the later chapters.

Consider the scattering process

$$S = |\psi^{\text{out}}_{p_3\sigma_3\ldots p_n\sigma_n}|\psi^{\text{in}}_{p_1\sigma_1\, p_2\sigma_2}\rangle =$$



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From QFT1 we recall that single state particle states are related to the irreducible representations of the Poincaré group:

$$|\psi_i\rangle \xrightarrow{\text{Poincaré}} P_{ij}\,|\psi_j\rangle\,.$$

How do we classify them? Poincaré irreducible representations are associated to the particles through the *little group*. We have three general options:

- $p^\mu = 0$: vacuum;
- $p^\mu \neq 0$, $p^2 = 0$: little group ISO(2), reference momentum $p^\mu = (E, 0, 0, E)$. The translations in the plane are ignored, so the effective little group is $\text{SO}(2) \sim \text{U}(1)$. The representations are $e^{ih\phi}$ and the condition is $e^{4\pi ih} = 1$ (the 4 in the exponent is due to Lorentz group having a double cover). The helicity is $h = n/2$ ($n \in \mathbb{Z}$). Sometimes, we require the theory to be parity invariant (QED) and then the particles with exactly opposite helicity coincide (photons).
- $p^\mu \neq 0$, $p^2 = m^2$: little group SO(3), reference momentum $p^\mu = (m, 0, 0, 0)$. The spin is $j = n/2$ ($n \in \mathbb{N}_0$) and for each spin there are $2j + 1$ states.

In the example above, assume that all external particles are photons. The S-matrix is

$$S = (\epsilon_1^{\mu_1})^*(\epsilon_2^{\mu_2})^*\epsilon_3^{\mu_3}\ldots\epsilon_n^{\mu_n}\mathcal{A}_{\mu_1\mu_2\ldots\mu_n}.$$

The factor $\mathcal{A}$ is a Lorentz tensor and is given by Feynman diagrams. The polarization vectors contract the Lorentz indices to ultimately give a Lorentz scalar. The same would work with spinors and Dirac indices.

2.1.1 Massless spinors

We will start with *Weyl spinors*. As we have seen in QFT1 the Lorentz group $\mathrm{SO}(1,3)$ is isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$ (almost), so we can classify all types of particles by two numbers (j_1, j_2). The simplest non-trivial representations are

- $(1/2, 0)$: left-handed Weyl spinor ψ_L,
- $(0, 1/2)$: right-handed Weyl spinor ψ_R.

The rotation of the spinor is given by the *Pauli matrices* $\boldsymbol{\sigma}$:

$$\psi_{\mathrm{L,R}} \to \psi'_{\mathrm{L,R}} = e^{i\boldsymbol{\theta}\boldsymbol{\sigma}} \psi_{\mathrm{L,R}}$$

and the boosts are

$$\psi_{\mathrm{L,R}} \to \psi'_{\mathrm{L,R}} = e^{\pm\boldsymbol{\phi}\boldsymbol{\sigma}} \psi_{\mathrm{L,R}}.$$

Here, we denoted the vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can also use the barred versions of spinors:

$$\overline{\psi}_\mathrm{L} = i\sigma_2 \psi_\mathrm{L}^\dagger \qquad \text{transforms as } \psi_\mathrm{R},$$

$$\overline{\psi}_\mathrm{R} = -i\sigma_2 \psi_\mathrm{R}^\dagger \qquad \text{transforms as } \psi_\mathrm{L}.$$

Barring the spinor twice gives us back the original spinor. In order to disambiguate left- and right-handed spinors we sometimes use the dotted and undotted indices:

$$\psi_\mathrm{L} = \psi_a, \qquad \psi_\mathrm{R} = \psi^{\dot{a}}.$$

For example,

$$i\sigma_2(\psi_\mathrm{L})^* = (i\sigma_2)^{\dot{a}\dot{b}}(\psi_b)^* = (i\sigma_2)^{\dot{a}\dot{b}}\psi_{\dot{b}} = \psi^{\dot{a}},$$

$$-i\sigma_2(\psi_\mathrm{R})^* = (-i\sigma_2)_{ab}(\psi^{\dot{b}})^* = (i\sigma_2)_{ab}\psi^b = \psi_a.$$

The conjugation changes the dotted and undotted indices and $i\sigma_2$ raises and lowers them (so it acts as a metric). It is completely real, so it holds $(i\sigma_2)^* = -i\sigma_2$:

$$(i\sigma_2)^{\dot{a}\dot{b}} = (i\sigma_2)^{ab}, \qquad (-i\sigma_2)_{\dot{a}\dot{b}} = (-i\sigma_2)_{ab}.$$

They also act as the Levi-Civita tensors:

$$(i\sigma_2)^{\dot{a}\dot{b}} = (i\sigma_2)^{ab} = \varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$(-i\sigma_2)_{\dot{a}\dot{b}} = (-i\sigma_2)_{ab} = \varepsilon_{ab} = \varepsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, $\varepsilon^{ab}\varepsilon_{bc} = \delta^a_c$. We can use this formalism to do spinor products:

$$\psi^b\chi_b = \psi_a\chi_b\varepsilon^{ab}.$$

We have to be very careful with the order of the indices, because the spinors don't commute:

$$\psi^a\chi_a = \psi_a\chi_b\varepsilon^{ab} = -\psi_a\chi_b\varepsilon^{ba} = -\chi^a\psi_a.$$

Since the spinors are anticommuting, we have a very useful identity: $\psi_a\psi^a = 0$. The same works with the right-handed spinors:

$$\psi_{\dot{b}}\chi^{\dot{b}} = \psi^{\dot{a}}\chi^{\dot{b}}\varepsilon_{\dot{a}\dot{b}}.$$

The (massless) *Dirac spinor* is given by the direct sum of the left- and right-handed Weyl spinors $(1/2, 0) \oplus (0, 1/2)$. It is useful to define the projection operators

$$P_{\mathrm{L}} = \frac{1-\gamma_5}{2} = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_{\mathrm{R}} = \frac{1+\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix}.$$

Let's introduce some new notation:

$$u(p) = N\begin{pmatrix}\psi_{\mathrm{L}}(p)\\ \psi_{\mathrm{R}}(p)\end{pmatrix}, \quad u_{\mathrm{L}}(p) = N\begin{pmatrix}\psi_{\mathrm{L}}(p)\\ 0\end{pmatrix} = N\begin{pmatrix}\psi_a(p)\\ 0\end{pmatrix}, \quad u_{\mathrm{R}}(p) = N\begin{pmatrix}0\\ \psi_{\mathrm{R}}(p)\end{pmatrix} = N\begin{pmatrix}0\\ \psi^{\dot{a}}(p)\end{pmatrix}.$$

The barred spinors are given with the rows

$$\overline{u}_{\mathrm{L}}(p) = u^\dagger_{\mathrm{L}}\gamma^0 = N^*(0, \psi_{\dot{a}}), \qquad \overline{u}_{\mathrm{R}}(p) = u^\dagger_{\mathrm{R}}\gamma^0 = N^*(\psi^a, 0).$$

The last bit of the new notation is the redesign of the bra-ket:

$$u_{\mathrm{L}}(p) = |p], \qquad u_{\mathrm{R}}(p) = |p\rangle,$$

$$\overline{u}_{\mathrm{L}}(p) = \langle p|, \qquad \overline{u}_{\mathrm{R}}(p) = [p|.$$

The magic of this notation is that if it looks wrong, then it is zero:

$$\overline{u}_{\mathrm{L}}(p)u_{\mathrm{L}}(q) = \langle p|q] = 0, \qquad \overline{u}_{\mathrm{R}}(p)u_{\mathrm{R}}(q) = [p|q\rangle,$$

$$\overline{u}_{\mathrm{L}}(p)u_{\mathrm{R}}(q) = \langle p|q\rangle \neq 0, \qquad \overline{u}_{\mathrm{R}}(p)u_{\mathrm{L}}(q) = [p|q] \neq 0.$$

We will often omit the middle vertical lines in brakets. Moreover, it holds

$$\langle pq\rangle = -\langle qp\rangle, \quad [pq] = -[qp], \quad \langle pp\rangle = [pp] = 0.$$

There is much elegance with this notation:

- Let's consider complex conjugating the spinor product:

$$\langle pq\rangle^* = (\overline{u}_{\mathrm{L}}(p)u_{\mathrm{R}}(q))^* = [qp], \qquad [pq]^* = \langle qp\rangle.$$

- Using this notation, the completeness relations are expressed very elegantly

$$\sum_\lambda u_\lambda(q)\overline{u}_\lambda(q) = \not{q} = |q\rangle[q| + |q]\langle q|.$$

- Another sleek identity is
$$\langle pq\rangle\,[pq] = |\,\langle pq\rangle\,|^2 = \langle p|\not{q}|p].$$

- One more identity:
$$\langle p|\not{q}|p] = \sum_\lambda \overline{u}_\lambda(p)\frac{1+\gamma_5}{2}\not{q}u_\lambda(p) = \mathrm{Tr}(\not{p}\not{q}P_\mathrm{L}) = 2p\cdot q = |\,\langle pq\rangle\,|^2 = |[pq]|^2.$$

  This only works for massless spinors. From this we can extract
$$\langle pq\rangle = \sqrt{|2p\cdot q|}\,e^{i\phi_{pq}}, \qquad [qp] = \sqrt{|2p\cdot q|}\,e^{-i\phi_{pq}}.$$

A few more properties of the new notation:

- The Dirac equation is
$$\not{p}\,|p\rangle = \not{p}|p] = \langle p|\,\not{p} = [p|\not{p} = 0.$$

- $$|p\rangle\,[p| = \frac{1+\gamma_5}{2}\not{p}, \qquad |p]\,\langle p| = \frac{1-\gamma_5}{2}\not{p}.$$

- *Gordon's identity*:
$$\langle p|\gamma^\mu|p] = [p|\gamma^\mu|p\rangle = 2p^\mu.$$

- Conjugation:
$$[p|\gamma^\mu|q\rangle = \langle q|\gamma^\mu|p].$$

- *Fierz identities*:
$$[p|\gamma^\mu|q\rangle[k|\gamma_\mu|l\rangle = 2[pk]\,\langle lq\rangle\,,$$
$$\langle p|\gamma^\mu|q]\langle k|\gamma_\mu|l] = 2\,\langle pk\rangle\,[lq].$$

- *Schouten's identities*:
$$\langle pq\rangle\,\langle kl\rangle = \langle pk\rangle\,\langle ql\rangle + \langle pl\rangle\,\langle kq\rangle\,,$$
$$[pq][kl] = [pk][ql] + [pl][kq].$$

- $$\langle p|\gamma^{\mu_1}\dots\gamma^{\mu_{2n}}|q] = 0, \qquad [p|(\gamma^{\mu_1}\dots\gamma^{\mu_{2n}})|q\rangle = 0,$$
$$\langle p|\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}|q\rangle = 0, \qquad [p|\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}|q] = 0.$$

- $$\langle p|\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}|q] = [q|\gamma^{\mu_{2n+1}}\dots\gamma^{\mu_1}|p\rangle.$$

Let's quickly prove the second Schouten identity. There can be at most two independent terms, say $|k]$ and $|l]$. We can express $|q] = A|k] + B|l]$, where the coefficients are given by the scalar products:
$$A = -\frac{[lq]}{[kl]}, \qquad B = \frac{[kq]}{[kl]}.$$
We operate with $[p|$ on $|q] = A|k] + B|l]$ and multiply with $\langle kl\rangle$ to get
$$[pq][kl] = [pk][ql] + [pl][kq].$$

Next, we want to analytically continue all momenta in order to make some of them outgoing instead of ingoing. We make the substitution $p \to -p$ and in order for that to work, we have to change the spinor-helicity formalism:

$$\langle -p| = i\,\langle p|\,, \quad [-p| = i[p|, \quad |-p\rangle = i\,|p\rangle\,, \quad |-p] = i|p].$$

With that, all expressions containing actual momenta p acquire a minus sign:

$$\not{p} = |p]\,\langle p| + |p\rangle\,[p| \longrightarrow -\not{p},$$
$$\langle pq\rangle\,]qp[= 2pq \longrightarrow -2pq.$$

We have seen in the homework that $p_{a\dot{b}} = p_\mu \sigma^\mu = |p\rangle\,[p|$ and $p^{\dot{a}b} = p_\mu \overline{\sigma}^\mu = |[\rangle\, p][p|$. Using the chiral representation of the Dirac matrices we can write

$$\not{p} = p_\mu \gamma^\mu = p_\mu \sigma^\mu + p_\mu \overline{\sigma}^\mu = p_{a\dot{b}} + p^{\dot{a}b} = |p]_a\,\langle p|_{\dot{b}} + |p\rangle^{\dot{a}}\,[p|^b.$$

The determinant vanishes

$$\det(p_{a\dot{b}}) = \det \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix} = p_0^2 - \sum_{i=1}^{3} p_i^2 = p^2 = 0$$

as it should for the massless particles. Every 2×2 matrix can be written as a tensor product of two vectors, so we can always write $p^{\dot{a}b} = \lambda^{\dot{a}} \tilde{\lambda}^b$.

A Lorentz scalar such as $\not{p}$ should remain unchanged under Lorentz transformations. Under the little group transformations, even the momentum itself should remain the same. One way to achieve this is to scale bras and kets with the same complex number $z \in \mathbb{C}$ (that depends on the transformation):

$$|p\rangle \to z\,|p\rangle\,, \qquad |p] \to \frac{1}{z}|p],$$
$$\langle p| \to z\,\langle p|\,, \qquad [p| \to \frac{1}{z}[p|.$$

If the momentum is real, we can use the conjugacy relations $|p\rangle = |p]^*$ to show that $z = 1/z^*$. In other words, $|z|^2 = 1$ so the little group transformations only change the phase of bras and kets.

2.1.2 Massless spin 1 particles

For spin 1 particles we have the polarization vectors $\varepsilon_\pm^\mu(p) = \varepsilon_\lambda^\mu(p)$, where λ denotes the helicity. There are three relations that polarization vectors obey:

$\varepsilon_\lambda^\mu p_\mu = 0$	transversality
$(\varepsilon_\lambda^\mu)^* \varepsilon_{\lambda\mu} = -1$	normalization
$\varepsilon_\lambda^\mu \varepsilon_{\lambda\mu} = 0$	light-like

One explicit representation of these vectors Is

$$p^\mu = (E, 0, 0, E)$$
$$\varepsilon_+^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0)$$
$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0)$$

To be able to express the polarization vectors in more general terms, define another light-like vector r^μ, called the *reference momentum*. It can be mostly arbitrary, but it shouldn't be parallel to p^μ, i.e. $p_\mu r^\nu \neq 0$. In practice, we often take r to be some other light-like vector that we have in the problem (a momentum of another particle, perhaps). Vectors p and r define a plane, and we can impose that the polarization vectors are orthogonal this plane, i.e. $\varepsilon^\mu_\pm r_\mu = 0$. We can build two new vectors

$$\eta_1^\mu = [r\gamma^\mu p\rangle, \qquad \eta_2^\mu = \langle r\gamma^\mu p].$$

It can be shown using the spinor-helicity formalism that these two vectors are orthogonal to p^μ and r^μ, $\eta^\mu_{1,2} p_\mu = \eta^\mu_{1,2} r_\mu = 0$. Having four orthogonal vectors, we can factorize the polarization vectors into components:

$$\varepsilon^\mu_\pm = A_\pm p^\mu + B_\pm r^\mu + C_\pm \eta_1^\mu + D_\pm \eta_2^\mu,$$

where $A_\pm$, $B_\pm$, $C_\pm$ and $D_\pm$ are some coefficients. From the orthogonality conditions it directly follows that $A_\pm = 0$ and $B_\pm = 0$. The other two are fixed by the normalization condition:

$$\varepsilon^\mu_+ = -\frac{[r\gamma^\mu p\rangle}{\sqrt{2}[rp]} = (\varepsilon^\mu_-)^*, \qquad \varepsilon^\mu_- = \frac{\langle r\gamma^\mu p]}{\sqrt{2}\langle rp\rangle} = (\varepsilon^\mu_-)^*.$$

The polarization vectors scale with the little group transformation ($p \to \Lambda p = p$) as

$$\varepsilon^\mu_+(\Lambda p, r) \to z^2 \varepsilon^\mu_+(p,r), \qquad \varepsilon^\mu_-(\Lambda p, r) \to \frac{1}{z^2}\varepsilon^\mu_-(p,r),$$

$$\varepsilon^\mu_+(p, \Lambda r) \to \varepsilon^\mu_+(p,r), \qquad \varepsilon^\mu_-(p, \Lambda r) \to \varepsilon^\mu_-(p,r).$$

We see that they scale with double the phase as the spinors (very much anticipated), but the reference momentum doesn't scale at all. We can calculate the difference between two polarization vectors with difference reference momenta:

$$\varepsilon^-_\mu(p,r) - \varepsilon^-_\mu(p,q) = \cdots = -\frac{1}{\sqrt{2}}\frac{2\langle rq\rangle}{\langle rp\rangle\langle qp\rangle} p^\mu = Cp^\mu.$$

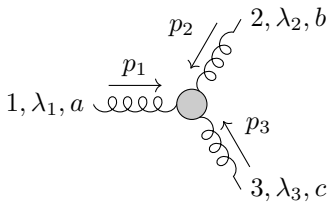
The difference is proportional to the momentum p^μ, which shows that changing the reference momentum doesn't change the physics. The scattering amplitude can always be expressed in terms of the polarization vectors $\varepsilon_\mu \mathcal{M}^\mu$. Say, we have two polarization vectors ε_1^μ and ε_2^μ with differently chosen reference momenta. The difference between the two amplitudes is

$$\varepsilon_1^\mu \mathcal{M}_\mu - \varepsilon_2^\mu \mathcal{M}_\mu = (\varepsilon_1 - \varepsilon_2)^\mu \mathcal{M}_\mu \propto p^\mu \mathcal{M}_\mu = 0$$

by *Ward's identity*.

2.2 Uniqueness of Yang-Mills theory

We want to extend our knowledge of the QED to more general massless spin-1 particles (for now); we will call them *gluons*. We will use what we learned about the on-shell methods to draw some general conclusions about such theory. The simplest amplitude we could consider is the 2-point amplitude. Sadly, this is just the propagator, which is by definition off-shell. The next simplest amplitude is the 3-point amplitude.



The symbols $\lambda_i = \pm$ represent the helicities. The letters a, b and c stand for internal degrees of freedom. Since all the particles are massless and all the momenta are considered to be incoming, it holds $p_1+p_2+p_3 = 0$ and $p_i^2 = 0$. Unfortunately, this immediately implies that the amplitude is zero (kinematically forbidden). However, this is only true for real momenta, and we can actually learn a lot by extending momenta to the complex plane. The conservation of momentum in spinor-helicity formalism reads

$$|1\rangle [1| + |2\rangle [2| + |3\rangle [3| = 0,$$

$$|1] \langle 1| + |2] \langle 2| + |3] \langle 3| = 0,$$

where we have replaced all the ps of momenta by just their indices. The left- and right-handed representations are completely independent. Contracting the top equation with $\langle 1|$ and $\langle 2|$ gives

$$\langle 12\rangle [2| + \langle 13\rangle [3| = 0, \qquad \langle 21\rangle [1| + \langle 23\rangle [3| = 0.$$

This implies one of two things. Either the square spinors are all proportional to each other and

$$[12] = [23] = [31] = 0$$

or the coefficients are all zero, so

$$\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0.$$

If the momenta are real, it holds $]ij[^* = \langle ji\rangle$ and everything is zero – also the whole amplitude. But since scattering amplitudes are analytic functions, we can analytically continue the momenta into the complex plane. In that case, just one type of brackets must be zero. Most generally, we can write the 3-point amplitude as

$$\mathcal{M}_3(1_a^{\lambda_1} 2_b^{\lambda_2} 3_c^{\lambda_3}) = \begin{cases} C^{abc} \langle 12\rangle^A [23]^B [31]^C, & [ij] = 0, \\ C^{abc} [12]^A \langle 23\rangle^B \langle 31\rangle^C, & \langle ij\rangle = 0. \end{cases}$$

Looking at the little group scaling, we know that the amplitude should scale as $z^{2\lambda_1}$ for p_1. This implies that $A + C = 2\lambda_1$ for the first case. Writing out all the other momentum little group scaling gives us a system of equations with the solution:

$$\mathcal{M}_3(1_a^{\lambda_1} 2_b^{\lambda_2} 3_c^{\lambda_3}) = \begin{cases} C^{abc} \langle 12\rangle^{\lambda_1+\lambda_2-\lambda_3} \langle 23\rangle^{\lambda_2+\lambda_3-\lambda_1} \langle 31\rangle^{\lambda_3+\lambda_1-\lambda_2}, \\ C^{abc} [12]^{\lambda_3-\lambda_1-\lambda_2} [23]^{\lambda_1-\lambda_2-\lambda_3} [31]^{\lambda_2-\lambda_3-\lambda_1}. \end{cases} .$$

Now we have to choose which type of brackets we want to use (and which one to set to zero). We choose that one that gives a non-divergent (e.g. zero) amplitude for real momenta. For zero or one negative helicity, we set square brackets to zero and in the other two cases we set the angle brackets to zero. For example,

$$\mathcal{M}(1^+2^+3^+) = C^{abc} \langle 12\rangle \langle 23\rangle \langle 31\rangle .$$

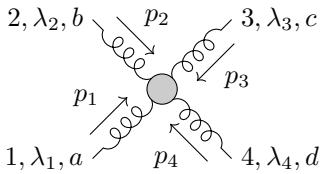
Counting the dimensions, an n-point amplitude has the dimension $[\mathcal{M}_n] = 4 - n$ and every bra-ket has the dimension $[\langle ij \rangle] = 1$. A conclusion in the case of a 3-point amplitude with all same helicities is that the constant C^{abc} needs to have the dimension $[C^{abc}] = -2$. The constant can only depend on the coupling constant and under the assumption that the theory is renormalizable, the only admissible solution is $C = 0$. This means that this amplitude *vanishes to all orders* in perturbation theory even for complex momenta: $\mathcal{M}(1^+2^+3^+) = \mathcal{M}(1^-2^-3^-) = 0$. Now we discuss the case of one negative helicity. We set the square brackets to zero. The amplitudes are

$$\mathcal{M}(1^+2^+3^-) = C^{abc} \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$
$$\mathcal{M}(1^+2^-3^+) = C^{abc} \frac{\langle 31 \rangle^3}{\langle 12 \rangle \langle 23 \rangle}$$
$$\mathcal{M}(1^-2^+3^+) = C^{abc} \frac{\langle 23 \rangle^3}{\langle 12 \rangle \langle 31 \rangle}$$

By exchanging the particles $1 \leftrightarrow 2$, we expect the same result because gluons are bosons and should be symmetric under the exchange. But spinors are antisymmetric, this means that the constants C^{abc} *must be fully antisymmetric*:

$$C^{abc} = -C^{bac}.$$

We can learn more about the structure of the amplitudes by looking at the 4-point amplitude.



The 4-point amplitude is a function of the Mandelstam variables:

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_2 + p_3)^2,$$

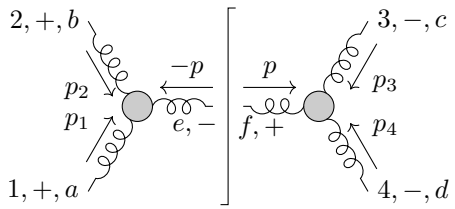
where $s + t + u = 0$. We will consider the *maximally helicity-violating* (MHV) amplitudes,

$$\mathcal{M}_4(1^+2^+3^-4^-) = \langle 12 \rangle^2 [34]^2 F^{abcd}(s, t, u),$$

where we used the little group scaling. It is called maximally helicity violating, because it can be shown that for general N-point amplitudes, if all or all but one particle have the same helicity, the amplitude is automatically zero. Hence, having two particles with different helicity is the maximally allowed violation. Another option for parametrization is called the *Parke-Taylor* parametrization:

$$\mathcal{M}_4(1^+2^+3^-4^-) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \tilde{F}^{abcd}.$$

We won't use it here, but it also works in general for N-point amplitudes. The dimension of the function F is $[F^{abcd}] = -4$. First, we will look at the s channel:



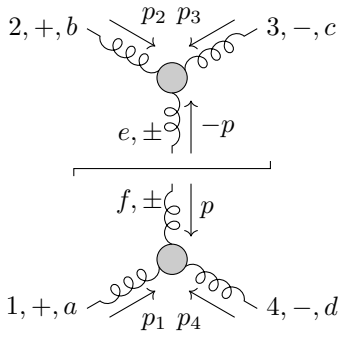
Generally, we should sum over all possible intermediate single-particle states, that means over both helicities. However, we have shown before that three-point amplitude vanishes for all same helicities so f must have positive helicity and e negative. We have, using the Cutcosky rules

$$\begin{aligned}\lim_{s\to 0}\mathcal{M}_4(1^+2^+3^-4^-) &= \frac{\delta_{ef}}{p^2}\mathcal{M}_3(1^+2^+p^-)\mathcal{M}_3(3^-4^-(-p)^+)\\ &= \frac{C^{abe}C^{cde}}{s}\frac{\langle 12\rangle^3}{\langle 2p\rangle\langle p1\rangle}\frac{[34]^3}{[3(-p)][(-p)4]}\\ &= -\frac{C^{abe}C^{cde}}{s}\frac{\langle 12\rangle^3[34]^3}{\langle 2p\rangle[p4][3p]\langle p1\rangle}\\ &= -\frac{C^{abe}C^{cde}}{s}\frac{\langle 12\rangle^3[34]^3}{u}.\end{aligned}$$

In the last step, we used the definition of $p = -p_1 - p_2 = p_3 + p_4$, so $\langle 2p\rangle[p4] = -\langle 21\rangle[14] = \langle 12\rangle[14]$ and $[3P]\langle P1\rangle = [34]\langle 41\rangle$. We can then express

$$\lim_{s\to 0}\left[suF^{abcd}(s,t,u)\right] = -C^{abe}C^{cde}.$$

Moving to the u-channel, we use the same procedure.



Now, however, both helicities of the u gluon are possible,

$$\begin{aligned}\lim_{u\to 0}\mathcal{M}_4(1^+2^+3^-4^-) &= \frac{C^{ade}C^{bce}}{u}\left[\frac{\langle 1p\rangle^3[3p]^3}{\langle 14\rangle\langle 3p\rangle[32][2p]} + \frac{[2p]^3\langle 2p\rangle^3}{[41][1p]\langle 23\rangle\langle 3p\rangle}\right]\\ &= \text{tedious work}\\ &= \frac{C^{ade}C^{bce}}{su}\langle 12\rangle^2[34]^2.\end{aligned}$$

In the (omitted) computation, one uses that $p^2 = u = 0$, which corresponds to $\langle 14\rangle[41] = 0$. But since for complex momenta $\langle ij\rangle^* \neq [ji]$, we can use that only one of the two terms is zero.

We choose $[14] = 0$ and it cancels the second term. The final result is

$$\lim_{u\to 0}\left[usF^{abcd}(s,t,u)\right] = C^{ade}C^{bce}.$$

Doing the same for the t-channel gives us the last relation:

$$\lim_{t\to 0}\left[stF^{abcd}(s,t,u)\right] = C^{ace}C^{bde}.$$

From these calculations and the momentum conservation $s+t+u=0$, we can find a convenient (but far from unique) way to parametrize the function F:

$$F^{abcd}(s,t,u) = \frac{1}{su}f_1^{abcd}(s/u) + \frac{1}{tu}f_2^{abcd}(t/u).$$

The Taylor expansion of the functions f_1 and f_2 gives

$$F^{abcd}(s,t,u) = \frac{1}{su}\sum_{n=0}^{\infty} a_n^{abcd}(s/u)^n + \frac{1}{tu}\sum_{n=0}^{\infty} b_n^{abcd}(t/u)^n.$$

The locality requires that the poles in s and t are at most order 1, but the order of u pole can be higher. In the limit $s\to 0$ we have

$$\lim_{s\to 0} F^{abcd} = \frac{1}{su}a_0^{abcd} + \mathcal{O}(1).$$

Combining this with the calculation of the s-channel gives $a_0^{abcd} = -C^{abe}C^{cde}$. Similarly, for $t\to 0$, we have

$$\lim_{t\to 0} F^{abcd} = \frac{1}{tu}b_0^{abcd} + \mathcal{O}(1), \quad \Longrightarrow \quad b_0^{abcd} = -C^{ace}C^{bde}.$$

The pole in u is more complicated:

$$\begin{aligned}\lim_{u\to 0}(suF^{abcd}) = C^{ade}C^{bce} &= \sum_{n=0}^{\infty} a_n^{abcd}(s/u)^n - \sum_{n=1}^{n} b_n^{abcd}(-s/u)^n \\ &= \sum_{n=0}^{\infty}\left(a_n^{abcd} - (-1)^n b_n^{abcd}\right)(s/u)^n + \mathcal{O}(1).\end{aligned}$$

Since we know that the coefficients C must be finite, there should be no poles left in the expression above. So, $a_n^{abcd} = (-1)^n b_n^{abcd}$ for $n>0$ and $C^{ade}C^{bce} = a_0^{abcd} - b_0^{abcd}$. Using previous results gives

$$\boxed{C^{abe}C^{cde} + C^{cae}C^{bde} + C^{ade}C^{bce} = 0.}$$

Everyone should recognize this, it is the *Jacobi identity*. This is an astonishing result! By only using the on-shell method and a set of very basic assumptions (Poincaré invariance, unitarity, locality and dimensional analysis), we have shown that any interaction must be a Lie group.

Chapter 3

Yang-Mills theory

3.1 Lie groups

A *group* is a set G with an operation, "$\cdot$", that satisfies the following properties:

- Closure: $g_1, g_2 \in G \implies g_1 \cdot g_2 \in G$.
- Associativity: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$.
- Identity: $\exists e \in G$ such that $g \cdot e = e \cdot g = g$ for all $g \in G$.
- Inverse: $\forall g \in G$, $\exists g^{-1} \in G$ such that $g \cdot g^{-1} = e = g^{-1} \cdot g$.

A group is *Abelian* if $ab = ba$ for all $a, b \in G$. Two groups G_1 and G_2 are *isomorphic* if there exists a bijective map $\phi : G_1 \to G_2$ that preserves the group structure:

$$\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2).$$

A subset $H \subset G$ is a *subgroup* if $\forall h_1, h_2 \in H \implies h_1 h_2 \in H, h_1^{-1} \in H$. A few examples of groups:

- *general linear group* $\mathrm{GL}(N, \mathbb{C})$: the set of all $N \times N$ invertible matrices with complex entries;
- *orthogonal group* $\mathrm{O}(N)$ is a subset of $\mathrm{GL}(N, \mathbb{R})$, with $OO^T = O^T O = \mathbb{1}$;
- *unitary group* $\mathrm{U}(N)$ is a subset of $\mathrm{GL}(N, \mathbb{C})$, with $UU^\dagger = U^\dagger U = \mathbb{1}$;
- *special unitary group* $\mathrm{SU}(N)$ is a subset of $\mathrm{U}(N)$, with $\det U = 1$;

A subgroup of $\mathrm{GL}(N, \mathbb{C})$ is called a *matrix* group. General linear group is $2N^2$-dimensional Euclidean space. Using the metric structure on $\mathbb{R}^{2N^2}$ we can define "nearby" matrices. We can also define the derivative of a matrix with respect to a parameter t, $M'(t) = \mathrm{d}M_{ij}/\mathrm{d}t$. A family of $N \times N$ matrices can depend on $m \leq 2N^2$ parameters. Such family defines a submanifold in $\mathbb{R}^{2N^2}$, in particular, for $m = 1$ it is a curve. A derivative of a curve $M'(t)$ is a tangent vector. *Lie group* is a family of matrices that defines a smooth manifold. For example, the group $\mathrm{SU}(2)$ is homeomorphic to the 3-dimensional sphere. In particular, for $\mathcal{U} \in \mathrm{SU}(2)$ we have

$$\mathcal{U} = \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}, \quad \det \mathcal{U} = 1.$$

Writing $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the determinant condition gives us the equation of the sphere: $x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$.

Since Lie groups are smooth manifolds, we can associate the *Lie algebra* to the group as its tangent space at the unit element. For any curve $g(t)$ in the group that passes through the identity at $t = 0$, we can express it as a Taylor expansion:

$$g(t) = \mathbb{1} + tA + \mathcal{O}(t^2).$$

$A = g'(0)$ is a matrix that belongs to the Lie algebra. The set of all matrices needed to span all curves at the identity are a basis of the Lie algebra. Lie algebra of a group G is sometimes denoted as AG, but we will denote it with the lowercase Fraktur letter of the group, $\mathfrak{g}$. Since the group manifold is embedded in a real Euclidean space, the tangent space is also a real vector space and hence also the Lie algebra. This means that $\forall A_1, A_2 \in \mathfrak{g}$ it holds

$$cA_1 \in \mathfrak{g}\ \forall c \in \mathbb{R}, \qquad A_1 + A_2 \in \mathfrak{g}.$$

For the first statement, consider the curve $g_1(t) = \mathbb{1} + tA_1 + \mathcal{O}(t^2)$. Then $\tilde{g}_1(ct) = \mathbb{1} + cA_1 t + \mathcal{O}(t)$. For the second statement, we can use the curve $g_2(t) = \mathbb{1} + tA_2 + \mathcal{O}(t^2)$ and write

$$g_1(t)g_2(t) = \tilde{g}(t) = 1 + (A_1 + A_2)t + \mathcal{O}(t^2).$$

Having the Lie algebra, we can define the *Lie bracket* (the commutator) of any two elements $A_1, A_2 \in \mathfrak{g}$:

$$[A_1, A_2] = A_1A_2 - A_2A_1.$$

This commutator is again an element of the Lie algebra, $[A_1, A_2] \in \mathfrak{g}$. To see this, choose the same $g_1(t)$ and $g_2(t)$ as before. Then

$$g(t) = \underbrace{g_2(\sqrt{t})g_1(\sqrt{t})g_1^{-1}(\sqrt{t})g_2^{-2}(\sqrt{t})}_{\in G} = 1 + \underbrace{[A_1, A_2]}_{\in \mathfrak{g}} t + \mathcal{O}(t^2).$$

Of course, mathematicians would first define the Lie bracket and then construct the Lie algebra from it.

Example 1.

1. Let $\mathcal{U} \in \mathrm{U}(N)$. Then $\mathcal{U} = \mathbb{1} + At + \mathcal{O}(t^2)$ when $\mathcal{U}$ is close to the identity. By the unitarity condition, we have
$$\mathcal{U}\mathcal{U}^\dagger = \mathcal{U}^\dagger\mathcal{U} = \mathbb{1} = \mathbb{1} + (A + A^\dagger)t + \mathcal{O}(t^2).$$
It follows that $A = -A^\dagger$, so $\mathfrak{u}(N)$ is the Lie algebra of all $N \times N$ anti-Hermitian matrices.
2. For $\mathrm{SU}(N)$ we add the condition $\det\mathcal{U} = 1$. In the algebra, this gives
$$\det(\mathbb{1} + At + \mathcal{O}(t^2)) = 1 + \mathrm{tr}(A)t + \mathcal{O}(t^2) \equiv 1,$$
where we used the famous formula $\det(e^A) = e^{\mathrm{tr}(A)}$. Hence, $\mathrm{tr}(A) = 0$ and the Lie algebra is $\mathfrak{su}(N)$, the set of all $N \times N$ anti-Hermitian traceless matrices. In general, adding an S to the group (making it special) imposes the Lie algebra matrices to be traceless. In physics, however, we don't like anti-Hermitian matrices, so we replace A with iB,
$$\mathcal{U} = \mathbb{1} + iBt + \mathcal{O}(t^2).$$
The matrix B is then Hermitian.

3. Consider the group $O = \mathrm{O}(N)$ (and $\mathrm{SO}(N)$). The orthogonality condition gives

$$OO^T = O^T O = \mathbb{1} = (\mathbb{1} + At + \mathcal{O}(t^2))(\mathbb{1} + A^T t + \mathcal{O}(t^2)) = \mathbb{1} + (A + A^T)t + \mathcal{O}(t^2).$$

Hence, $A = -A^T$ and the Lie algebra $\mathfrak{o}(N)$ is the set of all $N \times N$ antisymmetric matrices. The algebra $\mathfrak{so}(N)$ of the special orthogonal group $\mathrm{SO}(N)$ is the set of all $N \times N$ antisymmetric traceless matrices, but since antisymmetric already implies traceless, we see that $\mathfrak{so}(N)$ and $\mathfrak{o}(N)$ are the same algebras.

The *dimension* of a Lie algebra is the dimension of the tangent space at the identity of the Lie group.

Example 2.

1. For $\mathfrak{su}(N)$, the dimension can be counted as the number of free parameters in traceless anti-Hermitian complex matrices. Generally, there are $2N^2$ parameters in $N \times N$ complex matrices. The traceless condition gives us one constraint and the anti-Hermitian condition gives us N^2 constraints. Hence, the dimension of $\mathfrak{su}(N)$ is $N^2 - 1$.
2. For $\mathfrak{so}(N)$, we have $N(N-1)/2$ free (real) parameters in the antisymmetric matrices. From the dimension counting only we can already see that $\dim(\mathfrak{so}(3)) = \dim(\mathfrak{su}(2))$ and we will later see that these two algebras are isomorphic.

As Lie algebras are real vector spaces, we can choose a basis T_j. We will call this basis the *generators* of the Lie algebra. We showed that the commutator of two elements of the Lie algebra is again an element of the Lie algebra and the same must hold for the generators. In the most general case, we can write

$$[T_i, T_j] = \sum_k C_{ijk} T_k.$$

From now on, we will omit the summation sign and employ the Einstein summation convention. The factors C_{ijk} are called the *structure constants*. All we can say about structure constants now is that they are real numbers and that they are antisymmetric in the first two indices. If we use the expansion $g(t) = \mathbb{1} + iAt + \mathcal{O}(t^2)$ (e.g., in $\mathfrak{su}(N)$), we have

$$[T_i, T_j] = iC_{ijk} T_k.$$

Example 3.

1. In $\mathfrak{su}(2)$ one choice of generators is $T_i = \sigma_i/2$, where σ_i are Pauli matrices:

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The structure constants are the *Levi-Civita symbol*: $[T_i, T_j] = i\varepsilon_{ijk} T_k$.

2. In $\mathfrak{so}(3)$, we need 8 generators. The canonical choice is $T_i = \lambda_i/2$, where λ_i are the *Gell-Mann matrices*:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is very important to note that the structure constants are *not unique* and depend on the choice of generators (the basis of the algebra). From the commutator $[A, B] = AB - BA$, *Jacobi identity* follows:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

In the language of structure constants, this means

$$C_{ijk}C_{jkl} + C_{jkl}C_{kij} + C_{kij}C_{ijk} = 0.$$

This is the defining property of the Lie algebra and some people start with this equation and then define the Lie algebra.

A convenient way to relate the elements of the Lie algebra to the elements of the group is to use the *exponential map* $g(t) = e^{At}$ for $g \in G$ and $A \in \mathfrak{g}$. The algebra element A can be expressed as a sum of the generators $A = \theta^a T^a$. For $\mathfrak{su}(n)$ we have to add another i, and we usually denote the parameter t as ϵ. So, for SU(N),

$$g = e^{i\epsilon\theta^a T^a}.$$

Another important construction is the *Lie subalgebra*. For $\mathfrak{a}$ and $\mathfrak{b}$, $\mathfrak{b} \subset \mathfrak{a}$ two Lie algebras, we can say that $\mathfrak{b}$ is a subalgebra of $\mathfrak{a}$ if it is closed under the Lie bracket. $\mathfrak{b}$ is an *ideal* (or *invariant subalgebra*) if $[A, B] \in \mathfrak{b}$ for all $A \in \mathfrak{a}$ and $B \in \mathfrak{b}$.

3.1.1 Representations of Lie algebras

Groups are abstract objects. We are interested to know how they act on different physical objects. Representations tell us exactly this. Let G be a group and V a vector space. A *representation* is a linear mapping $\varphi : G \to \mathrm{End}(V)$, $g \mapsto T(g)$ such that

$$T(e) = \mathbb{1}, \quad T(g_1 g_2) = T(g_1)T(g_2), \quad T(g^{-1}) = T(g)^{-1}.$$

We can do the same for Lie algebras. The representation of a Lie algebra is a linear mapping $\varphi : \mathfrak{g} \to \mathrm{End}(V)$, $A \mapsto T(A)$ such that

$$T(A + B) = T(A) + T(B), \quad T(\alpha A) = \alpha T(A), \quad T([A, B]) = [T(A), T(B)].$$

Take some representation R for the group G. Then for any $g \in G$, $T_R(g) \in \mathrm{End}(V)$. Given a representation R of the group G, we want to find a representation r of the Lie algebra $\mathfrak{g}$. We define

$$T_R(g) = T_R(\mathbb{1} + tA + \mathcal{O}(t^2)) = \mathbb{1} + itT_r(A) + \mathcal{O}(t^2).$$

The representation T_r is a good representation of the Lie algebra on V. The i that appears in the definition is a physicists' convention. If $T_R^\dagger T_R = T_R T_R^\dagger = \mathbb{1}$, we call the representation *unitary*. The equivalent algebra representation is then Hermitian $T_r(A) = T_r(A)^\dagger$.

The vector space V has some basis $e_i \in V$. The representation T_R acts on the basis vectors as

$$T_R(g)e_i = [T_R(g)]_{ij}e_j.$$

The dimension of $[T_R(g)]_{ij}$ is the same as the dimension of the vector space V, and it is called the *dimension* of the representation. Two representations R_1 and R_2 are *equivalent* if there exists an invertible operator S on V, such that for all $g \in G$,

$$T_{R_2}(g) = S T_{R_1}(g) S^{-1}.$$

Now take the subspace $W \in V$. We say that W is invariant under the representation T_R if for all $w \in W$ and $g \in G$, $T_R(g)w \in W$. There are always two trivial invariant subspaces: $W = \{0\}$ and $W = V$.

- If $T_R(g)$ has no non-trivial invariant subspaces, we say that the representation is *irreducible*.
- *Second Schur's lemma*: if T_R is finite dimensional irreducible representation of some group G acting on the vector space V and there exists an operator $A \in \mathrm{End}(V)$ such that $AT_R(g) = T_R(g)A$ for all $g \in G$, then $A = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{C}$.

One of the most important representations is the *fundamental representation*. Let G be a matrix group of size $N \times N$ and V a vectors space of $\dim V = N$. Let $\psi \in V$. The fundamental representation acts on V as

$$(T_F(g)\psi)_i = g_{ij}\psi_j.$$

Note that this is slightly different to the action of a group to the unit vector $T_{\mathrm{F}}(g)\mathbf{e}_i = g_{ji}\mathbf{e}_j$. It is common in literature, that the fundamental representation is denoted without any index. In $\mathrm{SU}(N)$ the fundamental representation is complex, so there exists the *conjugate representation* (anti-fundamental representation) F^* that acts on the row vectors:

$$(T_{\mathrm{F}^*}(g)\psi)_i = g^*_{ij}\psi_j = \psi_j g^\dagger_{ji}.$$

The next in importance is the *adjoint representation*. It acts on a vector space of the Lie algebra $\mathfrak{g}$. This is a real vector space of dimension $\dim \mathfrak{g}$. Let G be $N \times N$ matrix group and $A \in \mathfrak{g}$, which is also an $N \times N$ matrix. The action works as follows: for all $g \in G$ and $A \in \mathfrak{g}$,

$$T_{\mathrm{A}}(g)A = gAg^{-1}.$$

We will show that $T_{\mathrm{A}}(g)A \in \mathfrak{g}$. Denote $g_A(t) = \mathbb{1} + At + \mathcal{O}(t^2)$. Then

$$h(t) = \underbrace{gg_A(t)g^{-1}}_{\in G} = \mathbb{1} + t\underbrace{gAg^{-1}}_{\in \mathfrak{g}} + \mathcal{O}(t^2).$$

In order to show that the representation is consistent with the group structure, first note that $T_{\mathrm{A}}(e)A = eAe = A$. Then

$$T_{\mathrm{A}}(g_1g_2)A = g_1(g_2Ag_2^{-1})g_1^{-1} = T_{\mathrm{A}}(g_1)T_{\mathrm{A}}(g_2)A$$

and

$$T_{\mathrm{A}}(g^{-1})A = g^{-1}Ag = T_{\mathrm{A}}(g)^{-1}A.$$

Next, we want to find the representation generated by T_{A} in the Lie algebra. Let $g \in G$ be close to the identity. Write $g = \mathbb{1} + tN + \mathcal{O}(t^2)$ for some $B \in \mathfrak{g}$. Write (ignoring $\mathcal{O}(t^2)$)

$$T_{\mathrm{A}}(g)A = gAg^{-1} = (1 + tB)A(1 - tB) = A + t[B, A].$$

Hence, the action of the adjoint representation of the Lie algebra is

$$T_{\mathrm{a}}(B)A = [B, A].$$

It acts on the Lie algebra as a commutator! What is the explicit representation for generators of the adjoint representation as matrices? Let $T_j \in \mathfrak{g}$ be generators of the Lie algebra. A representation of a generator T_i acting on another generator of the Lie algebra T_j is given by

$$T_{\mathrm{a}}(T_i)T_j = C_{ijk}T_k,$$

but we can also write it explicitly as

$$T_{\mathrm{a}}(T_i)T_j = [T_{\mathrm{a}}]^{(i)}_{kj}T_k.$$

Hence, $[T_{\mathrm{a}}]^{(i)}_{kj} = C_{ijk}$: the adjoint representation generators *are* the structure constants!

3.1.2 Compact Lie groups and algebras

Compact groups are groups that are compact as manifolds (in topological sense). Most of the groups we physicists care about are compact. For example, SU(N), U(N) and SO(N) are all compact. An easy example is SU(2) which is homeomorphic to the 3-sphere S^3 (which we know is compact). A useful criterion for group compactness is the *Cartan-Killing criterion* for a compact group.

Theorem 3.1 (Cartan-Killing criterion)**.** *A finite-dimensional Lie group is compact, if and only if it has a positive definite scalar product (called the Killing form) which is invariant under the adjoint representation.*

The statement can be translated as follows. There exists a (not necessarily unique) bilinear form $\langle \bullet, \bullet \rangle$ such that $\forall A, B \in \mathfrak{g}$

$$\langle A, B\rangle = \langle T_{\mathrm{A}}(g)A, T_{\mathrm{A}}(g)B\rangle\,, \qquad \langle A, A\rangle \geq 0.$$

For matrix groups, the Killing form is given by the trace: $\langle A, B\rangle = -\operatorname{tr}(AB)$. All groups we deal with are compact. Thanks to the scalar product, we can orthogonalize the generators such that

$$\operatorname{tr}[T_i T_j] = -\frac{1}{2}\delta_{ij}.$$

The $1/2$ in front hold only for the fundamental representation, for other representations r we have to replace the $1/2$ with $C(r)$ (this number depends only on the representation). Using this normalization, we can see that structure constants become fully antisymmetric:

$$\operatorname{tr}([T_i, T_j]T_l) = C_{ijk}\operatorname{tr}(T_k T_l) \quad \Longrightarrow \quad \operatorname{tr}(T_i T_j T_l - T_j T_i T_l) = -\frac{1}{2}C_{ijl},$$

$$\operatorname{tr}([T_i, T_l]T_j) = C_{ilk}\operatorname{tr}(T_k T_j) \quad \Longrightarrow \quad \operatorname{tr}(T_i T_l T_j - T_l T_i T_j) = -\frac{1}{2}C_{ilj}.$$

By the cyclicity of the trace, we can write

$$\operatorname{tr}(T_i T_j T_l - T_j T_l T_i) = +\frac{1}{2}C_{ilj}.$$

Another very important statement about representations of compact groups is the following.

> *Any finite dimensional representation of a compact group is equivalent to a unitary representation. Any finite dimensional representation of a compact Lie algebra is equivalent to an (anti-)Hermitian representation (depending on the i convention).*

Lastly, we will discuss simple and semisimple Lie algebras. A Lie algebra is *semisimple* if it does not contain an Abelian invariant subalgebra. It is *simple* if it does not contain any Abelian invariant subalgebra. Any compact Lie algebra (i.e. an algebra of a compact Lie group) can be *uniquely* represented as a direct sum of U(1) subalgebras and simple subgroups:

$$\mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1) \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n.$$

This implies that Lie groups can be *locally* decomposed as

$$G = \mathrm{U}(1) \oplus \cdots \oplus \mathrm{U}(1) \oplus G_1 \oplus \cdots \oplus G_n$$

where G_i are simple groups. The global decomposition can be more tricky (remember the famous double cover). For compact simple Lie groups the Killing form is also unique. For example, for SU(N) the only non-trivial invariant bilinear form is the trace. In a great mathematical effort of the 19th century, people have classified all simple compact Lie groups. There are three infinite families SO(N), SU(N), Sp(N) and five exceptional groups G_2, F_4, E_6, E_7 and E_8.

The final convention for SU(N) will be the following.

- The generators are Hermitian, $\forall A \in \mathfrak{su}(N)$, $A = iA^a T^a_{\text{R}}$ (where T^a_R are the generators in a given representation R).
- The scalar product is normalized as $\operatorname{tr}(T^a T^b) = T(R)\delta^{ab}$, where $T(R)$ is called the *index of representation*.
- In the fundamental representation, $\operatorname{tr}[T^a_{\text{F}} T^b_{\text{F}}] = \delta^{ab}/2$ and in the adjoint representation $\operatorname{tr}[T^a_{\text{A}} T^b_{\text{A}}] = N\delta^{ab}$. Both normalizations imply each other.
- The structure constants are given by $[T^a_R, T^b_R] = if^{abc}T^c_R$.

Basis independent quantities that characterize the representation are the *Casimir operators*. For SU(2) it is

$$\mathbf{J}^2 = \sum_a T^a_R T^a_R = j(j+1).$$

The quadratic Casimirs for SU(N) are (by Schur's lemma)

$$\sum_a T^a_R T^a_R = C_2(R)\mathbb{1}.$$

Taking the trace gives

$$\sum_a \operatorname{tr}[T^a_R T^a_R] = C_2(R)\operatorname{tr}(\mathbb{1}) = C_2(R)d(R)$$

where we denoted the dimension of the representation as $d(R)$. This trace looks familiar to the relation $\operatorname{tr}(T^a T^b) = T(R)\delta^{ab}$. Taking $a = b$ gives $\operatorname{tr}(T^a T^b) = T(R)d(G)$ where $d(G)$ is the dimension of the group. Combining these two results gives

$$T(R)d(G) = C_2(T)d(R).$$

This is very important and useful relation. For SU(N) it holds $d(G) = N^2 - 1$, $d(\text{F}) = N$, $d(\text{A}) = N^2 - 1$, $T(\text{F}) = T_{\text{F}} = 1/2$ and $T(\text{A}) = T_{\text{A}} = N$. The quadratic Casimir operators are given by

$$C_2(\text{F}) = C_{\text{F}} = \frac{1}{2}\frac{N^2-1}{N} = \frac{N^2-1}{2N}, \quad \xrightarrow[\text{SU(3)}]{} C_{\text{F}} = \frac{4}{3},$$

$$C_2(\text{A}) = C_{\text{A}} = N\frac{N^2-1}{N^2-1} = N, \quad \xrightarrow[\text{SU(3)}]{} C_{\text{F}} = 3.$$

The structure constants can be expressed directly in terms of the commutators by considering

$$\operatorname{tr}([T^a, T^b]T^c) = if^{abd}\operatorname{tr}(T^d T^c) = if^{abc}T(R).$$

This gives us the relation

$$f^{abc} = \frac{i}{T_{\text{F}}}\operatorname{tr}([T^a, T^b]T^c).$$

Another relation between the generators in fundamental representation is

$$\sum_a T^a_{ij} T^a_{nk} = \frac{1}{2}\left(\delta_{ik}\delta_{nj} - \frac{1}{N}\delta_{ij}\delta_{nk}\right).$$

This is called the *Fierz identity* (again).

3.2 Abelian gauge theory

In this chapter, we will discuss *Yang-Mills theories*. These are gauge theories based on general Lie groups. Lie groups must be the basis for the interaction of spin-1 massless particles. However, we can limit ourselves to QFT with compact simple Lie groups. For example, QED is a gauge theory for spin-1 massless particles based on Abelian group U(1). The plan for this chapter is the following:

1. recap how gauge invariance is implemented in Abelian case,
2. generalize to the non-Abelian case,
3. quantize.

When we were first discussing fermion fields in QFT1, we introduced the fermion Lagrangian

$$\mathcal{L}_\Psi = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi.$$

We have quickly noticed that this Lagrangian invariant under *global* U(1) transformation $\Psi \to e^{i\alpha}\Psi$, $\alpha \in \mathbb{C}$. However, if we let α be a function of x things change slightly. The term $m\overline{\Psi}\Psi$ is still invariant, but the kinetic term is not:

$$\overline{\Psi}\partial\!\!\!/\Psi \to \overline{\Psi}'\partial\!\!\!/\Psi' = e^{i\alpha(x)}\overline{\Psi}\partial\!\!\!/\Psi + i\partial_\mu\alpha(x)\overline{\Psi}\gamma^\mu\Psi.$$

In order to preserve the local invariance, we modified the derivative operator and got the *covariant derivative*

$$\mathrm{D}_\mu = \partial_\mu - ieA_\mu(x).$$

The newly introduced field $A_\mu(x)$ has to transform according to

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x).$$

With this modification to the derivative operator the kinetic term also becomes locally U(1) invariant:

$$\overline{\Psi}\not{\mathrm{D}}\Psi \to \overline{\Psi}'\not{\mathrm{D}}\Psi' = \overline{\Psi}\not{\mathrm{D}}\Psi.$$

Hence, the new Lagrangian $\mathcal{L} = \overline{\Psi}(i\not{\mathrm{D}} - m)\Psi$ is invariant under the *local* U(1) transformation. But this is not the entire story – now that we have added a field, we should at very least also add a kinetic term for that field. From the Maxwell theory, we know that the term

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

should work. We have used the *field strength tensor*

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu F_\mu.$$

It is an easy exercise to check that $F_{\mu\nu}$ is gauge invariant. The geometrical intuition of gauge theory is that global internal symmetries don't influence transformations under Lorentz group. The main issue is the derivative, which requires a comparison of the field in two different points:

$$|\phi(x) - \phi(y)| \to |e^{i\alpha(x)}\phi(x) - e^{i\alpha(y)}\phi(y)| \neq |\phi(x) - \phi(y)|.$$

This is not a well-defined transformation properties. To fix this, we introduce the *Wilson line*, a bi-local operator $W(x, y)$ that transforms according to

$$W(x, y) \to e^{i\alpha(x)}W(x, y)e^{-i\alpha(y)}.$$

It also satisfies $W(x, x) = 1$. If we modify the difference of the fields, the transformation is well-behaved

$$W(x, y)\phi(y) - \phi(x) \to e^{i\alpha(x)}(W(x, y)\phi(y) - \phi(x)).$$

This expression has unambiguous transformation properties under U(1). Wilson line parallel-transports field from x to y. As the difference is well-defined, the derivative can be well-defined as well. Define

$$\mathrm{D}_\mu\phi(x) = \lim_{\delta x^\mu \to 0}\left(\frac{W(x, x + \delta x)\phi(x + \delta x) - \phi(x)}{\delta x^\mu}\right).$$

This is called the *covariant derivative*. It transforms as

$$\mathrm{D}_\mu\phi \to \mathrm{D}'_\mu\phi' = e^{i\alpha(x)}\phi(x).$$

We would like to know more about the Wilson line. We can expand it infinitesimally

$$W(x, x + \delta x) = \mathbb{1} + \beta_\mu\delta x^\mu + \mathcal{O}(\delta x^2).$$

Here, β_μ are Taylor coefficients. Plugging this expansion in the rule for Wilson line transformation gives

$$\begin{aligned} W(x, x + \delta x) \to e^{i\alpha(x)}W(x, x + \delta x)e^{-i\alpha(x+\delta x)} &= e^{i\alpha(x)}W(x, x + \delta x)e^{-i\alpha(x)}(1 - i\partial_\mu\alpha\delta x^\mu) \\ &= 1 + (\beta_\mu - i\partial_\mu\alpha(x))\delta x^\mu + \mathcal{O}(\delta x^2). \end{aligned}$$

On the other hand, we can also transform the line directly

$$W(x, x + \delta x) = \mathbb{1} + \beta_\mu\delta x^\mu + \mathcal{O}(\delta x^2) \to \mathbb{1} + \beta'_\mu\delta x^\mu + \mathcal{O}(\delta x^2).$$

Combining the two results implies the transformation property of the expansion function β

$$\beta_\mu \to \beta'_\mu = \beta_\mu - i\partial_\mu\alpha(x).$$

By convention, we choose $B_\mu = -ieA_\mu$. The covariant derivative expressed with the firs order approximation of the Wilson line is

$$\mathrm{D}_\mu\phi = \lim_{\delta x^\mu}\frac{(1 - ieA_\mu\delta x^\mu)\phi(x + \delta x) - \phi(x)}{\delta x^\mu} = (\partial_\mu - ieA_\mu)\phi(x).$$

The expansion $W(x, x + \delta x) = ieA_\mu(x)\delta x^\mu$ of course only holds to the first order in δx^μ. For a finite separator we would need to find a more general solution. The following solution has all the properties we need:

$$W_p(x, y) = \exp\left(-ie\int_{\gamma_p}\mathrm{d}z^\mu A_\mu(z)\right),$$

where γ_p is a path connecting x and y ($z^\mu(t)$ for $0 \leq t \leq 1$ and $z^\mu(0) = x^\mu$, $z^\mu(1) = y^\mu$). The index p denotes that we have to consider path ordering (similar to time ordering), but this is not particularly important for now. If the path is infinitesimally short, then $A_\mu(z) = A_\mu(x)$ for all $z^\mu \in [x^\mu, ^\mu + \delta x^\mu]$ and $z^\mu(t) = x^\mu t - \delta x^\mu(t-1)$. The solution is

$$W_p(x, x+\delta x) = e^{-ieA_\mu(x)\delta x^\mu}.$$

How does $W_p(x,y)$ transform? We already know the transformation of $A^\mu(x)$, so it is not hard to see that

$$W_p(x,y) \to W_p'(x,y) = \exp\left(-ie\int_{\gamma_p} A_\mu(z)\,\mathrm{d}z^\mu + i\int_{\gamma_p}\partial_\mu\alpha(z)\,\mathrm{d}z^\mu\right) = e^{i\alpha(x)}W_p(x,y)e^{-i\alpha(y)}.$$

If the path is closed, i.e. $x = y$, we can show that the *Wilson loop* is gauge invariant:

$$\begin{aligned}
W_p^{\text{loop}} &= \exp\left(ie\oint_{\gamma_p} A_\mu(z)\,\mathrm{d}z^\mu\right) \\
&= \exp\left(ie\int_\Sigma \mathrm{d}(A_\nu\,\mathrm{d}z^\nu)\right) \\
&= \exp\left(ie\int_\Sigma \partial_\mu A_\nu\,\mathrm{d}z^\nu\wedge\mathrm{d}z^\mu\right) \\
&= \exp\left(\frac{ie}{2}\int_\Sigma(\partial_\mu A_\nu - \partial_\nu A_\mu)\,\mathrm{d}z^\nu\wedge\mathrm{d}z^\mu\right) \\
&= \exp\left(\frac{ie}{2}\int_\Sigma F_{\mu\nu}\,\mathrm{d}\sigma^{\mu\nu}\right).
\end{aligned}$$

The integrand only depends on the gauge invariant field strength tensor $F_{\mu\nu}$, so the entire expression is gauge invariant. One last thing to note before moving to the non-Abelian case is the commutator of the covariant derivatives. It transforms in the same manner as the field

$$[\mathrm{D}_\mu, \mathrm{D}_\nu]\phi(x) \to e^{i\alpha(x)}[\mathrm{D}_\mu, \mathrm{D}_\nu]\phi(x).$$

It equals

$$[\mathrm{D}_\mu, \mathrm{D}_\nu] = [\partial_\mu, \partial_\nu] - ie[\partial_\mu, A_\nu] + ie[\partial_\nu, A_\mu] + e^2[A_\mu, A\nu] = -ie(\partial_\mu A_\nu - \partial_\nu A_\mu) = -ieF_{\mu\nu}.$$

The first term is zero by Schwartz theorem, the last one vanishes because A_μ commutes with itself. The terms in the middle cancel partially by the chain rule. What is left is a pure function of the position vector and doesn't operate on the field $\phi(x)$ at all (even though it is written as two derivatives). We will see a generalization of this commutator in the non-Abelian case.

3.3 Non-Abelian gauge theory

Consider a Lagrangian of N complex scalar fields ϕ^i with $i = 1, \ldots, N$

$$\mathcal{L} = \partial_\mu\phi_i^\dagger\partial^\mu\phi_i - m^2\phi_i^\dagger\phi_i - \lambda(\phi_i^\dagger\phi_i)^2.$$

The Lagrangian is invariant under the global U(1) symmetry, but it is also invariant under the global SU(N) symmetry $\phi_i \to \omega_{ij}\phi_j$ for any $\omega \in \mathrm{SU}(N)$. This is only valid if all scalars have the

same mass. Remembering the knowledge from the chapter about Lie groups, we see that field ϕ transforms in fundamental representation. For simplicity, we will introduce vector notation with which the Lagrangian is a bit less cluttered:

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}, \qquad \mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi\phi^\dagger - \lambda(\phi\phi^\dagger)^2.$$

Now, we want to gauge the SU(N) symmetry. To do so, we require that the Lagrangian is invariant under the local SU(N) transformation $\phi \to \omega(x)\phi$. This means that the derivative has to transform covariantly, $(\mathrm{D}_\mu\phi)' = \omega(\mathrm{D}_\mu\phi)$. As in the Abelian case, we expect

$$\mathrm{D}_\mu\phi = \partial_\mu\phi + \mathbf{A}_\mu\phi,$$

however the field $\mathbf{A}_\mu$ now transforms differently than before:

$$(\mathrm{D}_\mu\phi)' = \partial_\mu\phi + \mathbf{A}'_\mu\phi' = \omega\partial_\mu\phi - (\partial_\mu\omega)\phi + \mathbf{A}'_\mu\omega\phi \equiv \omega\mathrm{D}_\mu\phi.$$

It implies $\omega\mathbf{A}_\mu = \partial_\mu\omega + \mathbf{A}'_\mu\omega$ or by rewriting

$$\boxed{\mathbf{A}'_\mu(x) = \omega(x)\mathbf{A}_\mu(x)\omega^{-1}(x) + \omega(x)\partial_\mu\omega^{-1}(x).}$$

A useful relation is

$$0 = \partial_\mu(\omega\omega^{-1}) = (\partial_\mu\omega)\omega^{-1} + \omega\partial_\mu\omega^{-1}.$$

The introduced field $\mathbf{A}_\mu$ is called the *gauge field* or *Yang-Mills field* and is a *connection* on the SU(N) principal bundle of the underlying Lorentzian manifold. If we expand the transformation function ω in terms of the generators of the Lie group $\omega = \mathbb{1} + i\varepsilon_a T^a$, the last term in the transformation of the gauge field becomes

$$\omega\partial_\mu\omega^{-1} = (\mathbb{1} + i\varepsilon_a T^a)\partial_\mu(\mathbb{1} - i\varepsilon_b T^b) = iT^a\partial_\mu\varepsilon_a(x) + \mathcal{O}(\varepsilon_a^2).$$

This means that the gauge field has to take values in the Lie algebra of the group and that it has to transform in *adjoint representation*. Instead of writing the gauge field as a general vector in the Lie algebra, we can write it as a linear combination of the generators of the Lie algebra:

$$\mathbf{A}_\mu(x) = -igA^a_\mu(x)T^a.$$

For the kinetic term of the gauge field, we expect something similar to the Maxwell Lagrangian, but not quite the same as the fields don't commute any more. We require the kinetic term to be gauge invariant and positively defined (in order to avoid negative energy solutions). We want $\mathbf{F}_{\mu\nu}$ to be built from the gauge field $\mathbf{A}_\mu$ in such a way that it transforms in the adjoint representation. If we manage to do that, the Cartan-Killing form will guarantee us the positive definiteness. We take the inspiration from the Abelian case and define

$$\mathbf{F}_{\mu\nu} = [\mathbf{D}_\mu, \mathbf{D}_\nu] = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu].$$

Writing it in terms of the components gives

$$\mathbf{F}_{\mu\nu} = -igT^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - g^2 A^a_\mu A^b_\nu [T^a, T^b] = -igT^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu).$$

The kinetic term of the gauge field is

$$\mathcal{L}_{\text{kin}} = \frac{1}{2g^2}\,\text{tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}) = \frac{(-ig)^2}{2g^2}F^a_{\mu\nu}F^{\mu\nu,a}\,\text{tr}(T^aT^b) = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a}.$$

Let's assume that a field ϕ transforms in the adjoint representation. We can write $\boldsymbol{\phi} = \phi^a T^a$. The derivative is

$$\mathrm{D}_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} + T_{\mathrm{A}}(\mathbf{A}_\mu)\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} + [\mathbf{A}_\mu, \boldsymbol{\phi}].$$

This is the standard action of the adjoint representation on the Lie algebra. Written in components, we have

$$[\mathrm{D}_\mu\phi]^a = (\partial_\mu\phi^a)T^a - igA^b_\mu\phi^c[T^b, T^c] = (\partial_\mu\phi^a + gf^{abc}A^b_\mu\phi^c)T^a.$$

To get the invariant Lagrangian we use the trace and the component form is

$$\mathcal{L} = \frac{1}{2}(\mathrm{D}_\mu\phi^a)(\mathrm{D}^\mu\phi^a) - \frac{m^2}{2}\phi^a\phi^a - \frac{\lambda}{4}(\phi^a\phi^a)^2.$$

If we have multiple fields that transform under one Abelian symmetry, they can have different charges. For example $\phi' = e^{i\alpha q_1}\phi$ and $\chi' = e^{i\alpha q_2}\chi$. The covariant derivatives for these fields would also be different

$$\mathrm{D}_\mu\phi = (\partial_\mu - ieq_1A_\mu)\phi, \quad \mathrm{D}_\mu\chi = (\partial_\mu - ieq_2A_\mu)\chi.$$

In the non-Abelian case there can only be one coupling per simple group. The interaction of the fields is uniquely determined by the representation they transform in.

The generalization of Yang-Mills theory to other types of fields is straightforward. Global symmetries have to commute with Lorentz symmetries, so we can simply start with the free Dirac Lagrangian and add the gauge field:

$$\mathcal{L}_{\text{YM}} = \overline{\Psi}(i\not{D} - m)\Psi + \frac{1}{2g^2}\,\text{tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}).$$

The covariant derivative of a field can be expanded as

$$\mathrm{D}_\mu\Psi = (\partial_\mu - igT^aA^a_\mu)\Psi$$

where T^a are the generators of the group in the representation of the field.

The last point of discussion in the classical picture of Yang-Mills theories are the equations of motion and conserved currents. For beginning, let's just consider the gauge kinetic term $\mathcal{L}_{\text{kin}} = -F^a_{\mu\nu}F^{\mu\nu,a}/4$. The variation of the action is

$$\begin{aligned}
\Delta S &= -\frac{1}{2}\int \mathrm{d}^4x\; F^a_{\mu\nu}\delta F^{\mu\nu,a} \\
&= -\frac{1}{4}\int \mathrm{d}^4x\; F^a_{\mu\nu}(\partial^\mu\delta A^{\nu,a} - \partial^\nu A^{\mu,a} + gf^{abc}(\delta A^{\mu,b}A^{\nu,c} + A^{\mu,b}\delta A^{\nu,c})) \\
&= \int \mathrm{d}^4x\; ((\partial^\mu F^a_{\mu\nu})\delta A^{\nu,a} - gf^{abc}F^a_{\mu\nu}A^b_\mu\delta A^c_\nu) \\
&= \int \mathrm{d}^4x\; (\partial^\mu F^a_{\mu\nu} - gf^{abc}A^{\mu,b}F^c_{\mu\nu})\delta A^{\nu,a} = 0
\end{aligned}$$

The equation of motion can be compactly written as

$$\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{\mu,b} F^c_{\mu\nu} = (\mathrm{D}^\mu F_{\mu\nu})^a = 0.$$

Adding fermion fields is then very simple. The equations of motion are

$$(\mathrm{D}^\mu F_{\mu\nu})^a = -g\overline{\Psi}_i \gamma_\nu T^a_{ij} \Psi_j,$$

$$(i\partial\!\!\!/ - m)\Psi_i = -g A\!\!\!/^a T^a_{ij} \Psi_j.$$

The conserved current associated with the gauge symmetry $\partial^\mu J^a_\mu = 0$ is

$$J^a_\mu = -\overline{\Psi}_i \gamma_\mu T^a_{ij} \Psi_j + f^{abc} A^{\nu,b} F^c_{\mu\nu}.$$

Sadly, this current is not gauge invariant (not even gauge covariant). However, the matter current alone

$$j^a_\mu = -\overline{\Psi}_i \gamma_\mu T^a_{ij} \Psi_j$$

is covariant (in fact, $\mathrm{D}^\mu j^a_\mu = 0$), but it is not conserved. For more details, see the exercise sheet.

Chapter 4

Quantization of Yang-Mills theories

4.1 Path integral formalism

First, we will recap the path integral formalism that we have thoroughly studied in QFT1. In non-relativistic quantum mechanics, the Hamiltonian takes the form of

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{Q})$$

where $\hat{P}$ and $\hat{Q}$ are momentum and position operators. For simplicity, we assume that the potential is time-independent. We are interested in the probability of initial state $|i\rangle = |x_i\rangle$ at $t = t_i$ changing to the final state $|f\rangle = |x_f\rangle$ at $t = t_f$. The transition amplitude $|x_i\rangle \to |x_f\rangle$ is given by

$$U(x_i, x_f; t_i, t_f) = \langle x_f | e^{-iH(t_f - t_i)} | x_i \rangle \,.$$

We can split the time evolution operator into infinitesimal steps $\delta t = (t_f - t_i)/n$,

$$U = \langle x_f | e^{-iH\delta t} \dots e^{-iH\delta t} | x_i \rangle = \int \prod_{k=1}^{n} \mathrm{d}x_k \, \langle x_f | e^{-iH\delta t} | x_{n-1} \rangle \dots \langle x_1 | e^{-iH\delta t} | x_i \rangle \,,$$

where we used the completeness relation

$$\mathbb{1} = \int \mathrm{d}x_k \; |x_k\rangle \langle x_k| \,, \qquad \langle x|y\rangle = \delta x - y.$$

The last expression for U actually also works for time-dependent Hamiltonians (in the previous case, we would need to time-order the exponential). Looking at one particular bra-ket, we can insert another "identity" from the momentum completeness relation

$$\mathbb{1} = \int \frac{\mathrm{d}p}{2\pi} \, |p\rangle \langle p| \,, \qquad \langle p|x\rangle = e^{-ipx},$$

to obtain

$$\begin{aligned}\langle x_a|e^{-iH\delta t}|x_b\rangle &= \int \frac{\mathrm{d}p}{2\pi}\,\langle x_a|p\rangle\,\langle p|e^{-iH(x_b,t_b)\delta t}|x_b\rangle \\ &= e^{-iV(x_b,t_b)\delta t}\int \frac{\mathrm{d}p}{2\pi} e^{-ip^2\delta t/(2m)+ip(x_a-x_b)} \\ &= e^{-iV(x_b,t_b)\delta t}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p\, e^{-\frac{1}{2}ap^2-Jp}\end{aligned}$$

with $a = i\delta t/m$ and $J = i(x_a - x_b)$. In the last line, we recognize the Gaussian integral

$$\int_{-\infty}^{\infty}\mathrm{d}p\, e^{-\frac{1}{2}ap^2-Jp} = \sqrt{\frac{2\pi}{a}}e^{J^2/(2a)}.$$

A single bra-ket can be written as

$$\langle x_a|e^{-iH\delta t}|x_b\rangle = \sqrt{\frac{im}{2\pi\delta t}}e^{iV(x_b,t_b)\delta t}e^{im\frac{(x_a-x_b)^2}{2\delta t}} = N\exp\left(i\delta t\left(\frac{m}{2}\left(\frac{x_a-x_b}{\delta t}\right)^2 - V(x_b,t_b)\right)\right),$$

where the prefactor was written with N. As $\delta t \to 0$ the first term becomes a time derivative, so we get

$$\frac{m}{2}\left(\frac{x_a-x_b}{2}\right)^2 - V(x_b,t_b) \to \frac{m}{2}\dot{x}_a^2 - V(x_a,t_a) = \mathcal{L}(x_a,\dot{x}_a).$$

The transition amplitude is

$$\begin{aligned}U(x_i,x_f;t_i,t_f) &= \left(\frac{im}{2\pi\delta t}\right)^{n/2}\int\prod_{k=1}^{n-1}\mathrm{d}x_k e^{i\mathcal{L}(x_n,\dot{x}_n)\delta t+\cdots+i\mathcal{L}(x_1,\dot{x}_1)\delta t} \\ &= \left(\frac{im}{2\pi\delta t}\right)^{n/2}\int\prod_{k=1}^{n-1}\mathrm{d}x_k\exp\left(i\sum_{k=1}^{n}\mathcal{L}(x_k,\dot{x}_k)\delta t\right) \\ &= \tilde{N}\int\mathcal{D}[x(t)]\exp\left(i\int_{t_i}^{t_f}\mathcal{L}(x(t),\dot{x}(t))\,\mathrm{d}t\right)\Big|_{x(t_i)=x_i,x(t_f)=x_f}\end{aligned}$$

The constant $\tilde{N}$ is formally infinite, but, as we will see, it doesn't matter since it drops out of any physical quantity. The procedure of path formulation did an effective Legendre transformation from the Hamiltonian to the Lagrangian formalism. The integral of the Lagrangian over time is called the *action*

$$S[x] = \int_{t_i}^{t_f}\mathrm{d}t\,\mathcal{L}(x,\dot{x}).$$

The operator $\mathcal{D}[x]$ sums over all paths $x(t)$ that connect the initial and final points $x(t_i) = x_i$ and $x(t_f) = x_f$.

This was the treatment of the path integral in non-relativistic quantum mechanics. The relativistic case is very similar, the main difference is that we can have different types of fields. For bosons, we use regular numbers, where, for example, the Gaussian integral is

$$\int\mathrm{d}x_1\,\ldots\,\mathrm{d}x_n\, e^{\frac{1}{2}x_iA_{ij}x_j} = \sqrt{\frac{(2\pi)^n}{\det A}}.$$

For fermions, we have to use the anti-commuting *Grassmann numbers* θ_i and $\overline{\theta}_i$. These numbers were extensively discussed in QFT1 course. Many results are very similar, some are exactly opposite. For example, the Gaussian integral is

$$\int \mathrm{d}\overline{\theta}_1 \, \mathrm{d}\theta_1 \, \ldots \, \mathrm{d}\overline{\theta}_n \, \mathrm{d}\theta_n \, e^{\frac{1}{2}\overline{\theta}_i A_{ij} \theta_j} = \det A.$$

To calculate the Green's functions, we start with the following integral:

$$I = N \int \mathcal{D}[\phi] \, \phi(x_1)\phi(x_2) \exp\left(i \int_{-T}^{T} \mathrm{d}^4 x \, \mathcal{L}(\phi, \dot{\phi}) \right)$$

with the boundary conditions $\phi(x, -T) = \phi_a(x)$ and $\phi(x, T) = \phi_b(x)$. To write these boundary conditions more explicitly, we can decompose measure into three components:

$$\int \mathcal{D}[\phi(x)] = \int \mathcal{D}[\phi_1(\mathbf{x})] \int \mathcal{D}[\phi_2(\mathbf{x})] \int_B \mathcal{D}[\phi(x)]$$

where the boundary B for the last integral is defined as

$$B = \begin{cases} \phi(x_1^0, \mathbf{x}) = \phi_1(x); \\ \phi(x_2^0, \mathbf{x}) = \phi_2(x). \end{cases}$$

In other words, with the integral over B we mean the integration over all field configurations which are constrained at the times x_1 and x_2 to be equal to $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$ respectively. The remaining two functional integrals account for the fact that we need to integrate over all possible configurations for these two fields at fixed times (being at fixed times, they only depend on the vector $\mathbf{x}$). In this integral, the fields $\phi_1(\mathbf{x}_1)$ and $\phi_2(\boldsymbol{\phi}_2)$ can be pulled out of the last measure:

$$I = \int \mathcal{D}[\phi_1] \int \mathcal{D}[\phi_2] \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2) \int_B \mathcal{D}[\phi] \exp\left(i \int_{-T}^{T} \mathrm{d}^4 x \, \mathcal{L}(\phi, \dot{\phi}) \right).$$

Assuming that the initial conditions are ordered such that $-T < x_1^0 < x_2^0 < T$, the last integral can be split into three parts:

$$\begin{aligned} \int_B \mathcal{D}[\phi] e^{i \int_{-T}^{T} \mathrm{d}^4 x \, \mathcal{L}} &= \left(\int_B \mathcal{D}[\phi] e^{i \int_{-T}^{x_1^0} \mathrm{d}^4 x \, \mathcal{L}} \right) \left(\int_B \mathcal{D}[\phi] e^{i \int_{x_1^0}^{x_2^0} \mathrm{d}^4 x \, \mathcal{L}} \right) \left(\int_B \mathcal{D}[\phi] e^{i \int_{x_2^0}^{T} \mathrm{d}^4 x \, \mathcal{L}} \right) \\ &= \langle \phi_b | e^{-iH(T - x_2^0)} | \phi_2 \rangle \, \langle \phi_2 | e^{-iH(x_2^0 - x_1^0)} | \phi_1 \rangle \, \langle \phi_1 | e^{-iH(x_1^0 + T)} | \phi_a \rangle \, . \end{aligned}$$

The full integral is

$$I_{x_1^0 < x_2^0} = \int \mathcal{D}[\phi_1] \, \mathcal{D}[\phi_2] \, \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2) \, \langle \phi_b | e^{-iH(T - x_2^0)} | \phi_2 \rangle \, \langle \phi_2 | e^{iH(x_1^0 - x_2^0)} | \phi_1 \rangle \, \langle \phi_1 | e^{-iH(x_1^0 + T)} | \phi_a \rangle \, .$$

At this point, the fields $\phi_1(\mathbf{x}_1)$ are just c-numbers. However, we know that $\phi_1(\mathbf{x}_1) \, |\phi_1\rangle = \hat{\phi}_{\mathrm{S}}(\mathbf{x}_1) \, |\phi_1\rangle$, where $\hat{\phi}_{\mathrm{S}}(\mathbf{x}_1)$ is the field *operator* in the Schrödinger picture (with eigenvalue ϕ_1). Furthermore, we can also use the completeness relation

$$\int \mathcal{D}[\phi_i(\mathbf{x})] \, |\phi_i\rangle \, \langle \phi_i| = \mathbb{1}.$$

We obtain

$$I_{x_1^0<x_2^0} = \langle\phi_b|e^{-iH(T-x_2^0)}\hat{\phi}_{\rm S}(\mathbf{x}_2)e^{-iH(x_2^0-x_1^0)}\hat{\phi}_{\rm S}(\mathbf{x}_1)e^{-iH(x_1^0+T)}|phi_a\rangle\,.$$

Next, we replace the Schrödinger operators with Hamilton ones

$$\phi_{\rm H}(\mathbf{x}_i) = e^{iHx_i^0}\hat{\phi}_{\rm S}(\mathbf{x}_i)e^{-iHx_i^0}.$$

The result is

$$I_{x_1^0<x_2^0} = \langle\phi_b|e^{-iHT}\phi_{\rm H}(\mathbf{x}_2)\phi_{\rm H}(\mathbf{x}_1)e^{-iHT}|\phi_a\rangle$$

and after time-ordering

$$I = \langle\phi_b|e^{-iHT}T\{\phi_{\rm H}(\mathbf{x}_2)\phi_{\rm H}(\mathbf{x}_1)\}e^{-iHT}|\phi_a\rangle\,.$$

This is almost the two point correlator. What is missing is the vacuum state at each end. We can project the vacuum state by sending $T\to\infty$, however we have to be careful that we only count physical states. The necessary limit is actually $T\to\infty(1-i\varepsilon)$. Expanding the Hamiltonian into the eigenstates $|n\rangle$, we get

$$e^{-iHT}\,|\phi_a\rangle = \sum_n e^{-iE_nT}\,|n\rangle\,\langle n|\phi_a\rangle\,.$$

The vacuum state is the one that has the smallest energy $E_\Omega < E_n$. In the limit $T\to\infty(1-i\varepsilon)$ one obtains

$$\lim_{T\to\infty(1-i\varepsilon)}\sum_n e^{-iE_nT}\,|n\rangle\,\langle n|\phi_a\rangle \propto \langle\Omega|\phi_a\rangle\,e^{-iE_\Omega T}\,|\Omega\rangle + \{\text{faster decreasing terms}\}.$$

The entire integral I is

$$\lim_{T\to\infty(1-i\varepsilon)}\left(N\int\mathcal{D}[\phi]\phi(x_1)\phi(x_2)e3i\int_{-T}^{T}\mathrm{d}^4x\,\mathcal{L}\right) = \mathcal{N}\,\langle\Omega|T\{\phi_{\rm H}(x_1)\phi_{\rm H}(x_2)\}|\Omega\rangle\,.$$

In order to fix the normalization we consider the zero-point correlator

$$\mathcal{N} = \mathcal{N}\,\langle\Omega|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)}\left(N\int\mathcal{D}[\phi]e^{i\int_{-T}^{T}\mathrm{d}^4x\,\mathcal{L}}\right).$$

Finally, we can write the two-point correlation function as

$$\begin{aligned}\langle\Omega|T\{\phi_{\rm H}(x_1)\phi_{\rm H}(x_2)\}|\Omega\rangle &= \lim_{T\to\infty(1-i\varepsilon)}\left(\frac{\int\mathcal{D}[\phi]\,\phi(x_1)\phi(x_2)e^{i\int_{-T}^{T}\mathrm{d}^4x\,\mathcal{L}}}{\int\mathcal{D}[\phi]\,e^{i\int_{-T}^{T}\mathrm{d}^4x\,\mathcal{L}}}\right)\\ &= \frac{\int\mathcal{D}[\phi]\,\phi(x_1)\phi(x_2)e^{iS[\phi]}}{\int\mathcal{D}[\phi]\,e^{iS[\phi]}}.\end{aligned}$$

This formula generalizes to any n-point function

$$\langle\Omega|T\{\phi_{\rm H}(x_1)\dots\phi_{\rm H}(x_n)\}|\Omega\rangle = \frac{\int\mathcal{D}[\phi]\,\phi(x_1)\dots\phi(x_n)e^{iS[\phi]}}{\int\mathcal{D}[\phi]\,e^{iS[\phi]}}.$$

There is another, sleeker way to compute the n-point functions, the *generating functional*. We start by adding a source term to the Lagrangian (or action)

$$S[\phi,\mathcal{J}] = \int\mathrm{d}^4x\,(\mathcal{L}(\phi)+\phi(x)\mathcal{J}(x)).$$

The generating functional is defined as

$$Z[\mathcal{J}] = \int \mathcal{D}[\phi] e^{iS[\phi,\mathcal{J}]}.$$

We also define the normalization $Z[0] = N\langle\Omega|\Omega\rangle$. The beauty of the generating functional is that it generates all n-point functions taking derivatives with respect to the source $\mathcal{J}$:

$$\langle\Omega|T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle = \frac{1}{Z[0]}\frac{\delta}{i\delta\mathcal{J}(x_1)}\cdots\frac{\delta}{i\delta\mathcal{J}(x_n)}Z[\mathcal{J}]\bigg|_{\mathcal{J}=0}.$$

With this we have introduced the main machinery we need in QFT. To get a better feeling of how it works, we will do an example. Let

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2$$

where $\phi \in \mathbb{R}$ is a real scalar field. The action can be written a bit differently using the integration by parts:

$$S[\phi] = \int \mathrm{d}^4x\ \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2\right) = -\frac{1}{2}\int \mathrm{d}^4x\ \phi(\Box + m^2)\phi + \text{surface term}.$$

However, when we take the limit $T \to \infty(1 - i\varepsilon)$, the action becomes

$$S[\phi] = -\frac{1}{2}\int \mathrm{d}^4x\ \phi(\Box + m^2 - i\varepsilon)\phi$$

and the surface term vanishes. This finally gives the generating functional

$$Z_0[\mathcal{J}] = \int \mathcal{D}[\phi] \exp\left(i\int \mathrm{d}^4x\ \left[-\frac{1}{2}\phi(\Box + m^2 - i\varepsilon)\phi + \mathcal{J}\phi\right]\right).$$

Formally, we can interpret this integral as a Gaussian integral

$$\int \mathrm{d}\mathbf{p} \exp\left(-\frac{1}{2}p_iA_{ij}p_j + J_ip_i\right) = \sqrt{\frac{(2\pi)^n}{\det A}}\exp\left(\frac{1}{2}J_i(A^{-1})_{ij}J_j\right).$$

In our case, the matrix A is the operator $A = i(\Box + m^2 - i\varepsilon)$. Since the solution requires the inverse, we'd better calculate it:

$$i(\Box + m^2 - i\varepsilon)\Pi_{\mathrm{F}}(x - y) = \delta(x - y).$$

The Fourier transform of this equation gives the explicit solution:

$$\Pi_{\mathrm{F}}(x - y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\varepsilon}e^{-ip(x-y)}.$$

The determinant of the operator A is independent of the position, so it can be pulled out of the integral, and it will be cancelled by the normalization factor $Z[0]$. The zero-point generating functional is

$$Z_0[\mathcal{J}] = N\exp\left(-\frac{1}{2}\int \mathrm{d}^4x\ \int \mathrm{d}^4y\ \mathcal{J}(x)\Pi_{\mathrm{F}}(x - y)\mathcal{J}(j)\right).$$

Moreover, we can show that the two-point function is exactly the Feynman propagator

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = (-i)^2\frac{1}{Z_0[\mathcal{J}]}\frac{\delta^2Z_0[\mathcal{J}]}{\delta\mathcal{J}(x)\delta\mathcal{J}(y)}\bigg|_{\mathcal{J}=0} = \Pi_{\mathrm{F}}(x - y).$$

4.2 Quantization of Abelian gauge theories

First, we will show how to quantize QED, the Yang-Mills theory with the gauge group U(1). Focusing just on the kinetic term of the gauge field, we have

$$S[A] = \int \mathrm{d}^4x \ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}.$$

The integral measure is now $\mathcal{D}[A] = \mathcal{D}[A_0]\dots\mathcal{D}[A_3]$. The integrand can be rewritten using the antisymmetry and integration by parts:

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\partial_\mu A_\nu F^{\mu\nu} = \frac{1}{2}A_\nu\partial_\mu F^{\mu\nu} + \text{surface terms}.$$

Ignoring the surface terms, we get

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}A_\nu\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{2}A_\mu(g^{\mu\nu}\Box\partial^\mu\partial^\nu)A_\nu.$$

The action is

$$S[A] = \frac{1}{2}\int \mathrm{d}^4x \ A_\mu(x)(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)A_\nu(x).$$

Denote $M^\mu\nu = (g^{\mu\nu}\Box - \partial^\mu\partial^\nu)$. This operator is not invertible! The solution for

$$-i(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)\Pi^{\mathrm{F}}_{\nu\sigma}(x-y) = \delta_\rho{}^\mu\delta^{(4)}(x-y)$$

does not exist. This is very related to the gauge invariance: there are many copies of physically equivalent fields and infinitely many of them diverge badly. This can be easily seen if we take a longitudinally polarized field, $A^{\mathrm{L}}_\mu = \alpha(x)k_\mu$. The propagator in Fourier space is

$$\widetilde{M}_{\mu\nu} = -k^2 g_{\mu\nu} + k_\mu k_\nu.$$

Then

$$\tilde{A}^{\mathrm{L}}_\mu(-k^2 g^{\mu\nu} + k^\mu k^\nu)\tilde{A}^{\mathrm{L}}_\nu = 0.$$

To mend this problem, we will use a special method to only count the physically relevant field configurations.

4.2.1 Faddeev-Popov procedure

Since the path integral might not be particularly well-defined, we will first demonstrate the *Faddeev-Popov method* on a finite dimensional space. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $F(x)$ a function that is invariant under some lie group H of dimension $m < n$:

$$F(h \cdot x) = F(x), \quad \forall h \in H.$$

The integral

$$Z = \int_{\mathbb{R}^n} F(x)\,\mathrm{d}x^n$$

is divergent as it double-counts all equivalent configurations on which F is constant. We would like to integrate only the physically distinct states, which can be identified with the quotient space $\mathbb{R}^n/H$. Imagine that we found a function $g(x)$ such that $g(x) = 0$ would give us exactly the physical states. This function should have m components, denoted by $g_i(x)$, $i = 1, \dots, m$.

We want to only integrate over the representatives of the cosets (denoted $x^h = h \cdot x$). As H is a Lie group it can be parametrized by m parameters α_i,

$$h(\alpha) = h(\alpha_1, \dots, \alpha_m) = e^{\alpha_i T_i}$$

where T_i are generators of the group. Now, we will use a particularly complicated expression for number 1:

$$1 = \int_H \mathrm{d}h \det\left(\frac{\delta g}{\delta h}\right) \delta(g(h \cdot x)) = \int \prod_{i=1}^{m} \mathrm{d}\alpha_i \, \delta(g(h(\alpha) \cdot x)) \det\left(\frac{\partial g(h(\alpha) \cdot x)}{\partial \alpha}\right).$$

Inserting this "1" into the not-so-well-defined partition function, we get

$$Z = \int \prod_{i=1}^{m} \mathrm{d}\alpha_i \int_{\mathbb{R}^n} \mathrm{d}^n x \, F(x) \delta(g(h(\alpha) \cdot x)) \det\left(\frac{\partial g_i(h(\alpha) \cdot x)}{\partial \alpha_j}\right).$$

Due to the delta function, the determinant must be evaluated at $g = 0$. We will denote $\Delta = \det(\dots)_{g=0}$. For a very general linear choice of function g we can write

$$g_i(h(\alpha) \cdot x) = M_{ij}(x)\alpha_j + f_j(x).$$

The determinant is only a function of x, $\Delta(x) = \det(M(x))$. This determinant is independent of α and hence invariant for the action of H. In the expression for the partition function, we would like to shift $x \to h^{-1}x$. The determinant is invariant for transformations of H and so is the function F (by assumption). The measure $\mathrm{d}^n x \to \det(h)\mathrm{d}^n x$ remains invariant for $\mathrm{SU}(N)$ (since $\det(h) = 1$). We are left with

$$\begin{aligned} Z &= \int \prod_{i=1}^{m} \int_{\mathbb{R}^n} \mathrm{d}^n x \, F(x) \delta(g(h(\alpha) \cdot h^{-1}x)) \Delta(x) \\ &= \int \prod_{i=1}^{m} \int_{\mathbb{R}^n} \mathrm{d}^n x \, F(x) \delta(g(x)) \Delta(x) \\ &= \mathcal{N} \int_{\mathbb{R}^n} \mathrm{d}^n x \, F(x) \delta(g(x)) \Delta(x). \end{aligned}$$

The integral over $\mathbb{R}^n$ is now finite as it is constrained to the physical states with the delta function. The whole generating functional is still divergent, but all divergences can be packed into a prefactor (which hopefully cancels for any physical observables).

Finally, we can consider gauge fields. The transformation of field A_μ by a lie group element parametrized by α (there is only one parameter in $\mathrm{U}(1)$) is

$$A_\mu^{[\alpha]}(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x).$$

This corresponds to $h(\alpha)x$ from before. The product of measures of $\mathrm{d}\alpha_i$ will be replaced by a functional measure over $\alpha(x)$, and we will choose $g(x_j)$ (in the new notation $G(A)$) so that it will fully specify α, i.e., *fix the gauge*. Most generally, we will require

$$G(A^{[\alpha]}) = 0.$$

There are several options we can choose for G. What we have already seen is the *Lorentz gauge* $G(A) = \partial_\mu A^\mu$. It holds

$$G(A^{[\alpha]}) = \partial_\mu A^\mu + \frac{1}{e}\Box\alpha = 0.$$

This constrains α to be harmonic, and the zero boundary conditions make it zero. For now, we will keep G as generic as possible. The generating functional is

$$Z = \mathcal{N} \int \mathcal{D}[A] e^{iS[A]} \xrightarrow{\text{F.-P.}} \mathcal{N} \int \mathcal{D}[\alpha]\, \mathcal{D}[A] \det\left(\frac{\delta G(A^{[\alpha]})}{\delta \alpha}\right).$$

We used that the measure $\mathcal{D}[A]$ and the action $S[A]$ are invariant under the gauge transformation. We will choose a slight generalization of the Lorentz gauge, a covariant gauge

$$G(A) = \partial^\mu A_\mu - f(x)$$

for some function $f(x)$. It holds

$$G(A^{[\alpha]}) = \partial^\mu A_\mu + \frac{1}{e}\Box\alpha(x) - f(x), \quad \frac{\delta G(A^{[\alpha]})}{\delta\alpha} = \frac{1}{e}\Box.$$

The derivative of the gauge fixing function is independent of $A^{[\alpha]}$ and α. Thus, the determinant can be pulled out of the integral. Since everything is gauge invariant, we can absorb the remaining $\alpha(x)$ in the delta function into the gauge field. This yields

$$Z = \mathcal{N} \int \mathcal{D}[\alpha] \int \mathcal{D}[A]\, \delta(\partial_\mu A^\mu - f(x)) e^{iS[A]}.$$

This formula is independent of the choice of the unspecified function $f(x)$. Consequently, the generating functional should stay the same if we average over all possible $f(x)$ with some weighting function $F[f(x)]$:

$$Z = \tilde{\mathcal{N}} \int \mathcal{D}[f] \int \mathcal{D}[A]\, \delta(\partial_\mu A^\mu - f(x)) e^{iS[A]} F[f(x)] = \tilde{\mathcal{N}} \int \mathcal{D}[A] e^{iS[A]} F[\partial_\mu A^\mu].$$

A typical choice for the weighting function is a Gaussian

$$F[f(x)] = \exp\left(-i \int \mathrm{d}^4x\, \frac{f^2(x)}{2\xi}\right),$$

where ξ is a parameter that controls the strength of the gauge fixing. Gauge fixed using this procedure is called the *R_ξ gauge*. This is equivalent adding an extra term to the Lagrangian

$$Z = \tilde{\mathcal{N}} \int \mathcal{D}[A] \exp\left[i \int \mathrm{d}^4x \left(\mathcal{L}[A] - \frac{1}{2}(\partial_\mu A^\mu)^2\right)\right]$$

This extra term is called the *gauge-fixing term*. Hence, any time we see a generating functional, we can fix the gauge by

$$\int \mathcal{D}[A]\, e^{iS[A]} \to \text{const.} \cdot \int \mathcal{D}[A] \exp(iS_\xi[A]).$$

We say that S_ξ is the gauge-fixed action. For any gauge-invariant operator $\mathcal{O}(A)$, we can write

$$\langle\Omega|T\{\mathcal{O}(A)\}|\Omega\rangle = \frac{\int \mathcal{D}[A]\, \mathcal{O}(A) e^{iS_\xi[A]}}{\int \mathcal{D}[A]\, e^{iS_\xi[A]}}.$$

If the operator is not gauge invariant, the result will depend on the parameter ξ. For example, the two point function is not gauge invariant, hence it depends on the parameter ξ. Its Fourier transform is

$$\Pi_{\mathrm{F}}^{\mu\nu}(k) = \frac{-i}{k^2 + i\varepsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right).$$

The specific choices of the parameter ξ are:

- $\xi = 0$: *Landau gauge* (equivalent to the Lorentz gauge);
- $\xi = 1$: *Feynman-t' Hooft gauge*.

4.3 Quantization of non-Abelian gauge theories

In the non-Abelian case, we will repeat all the steps from the Abelian case, but we will have to be careful about non-commutativity of the gauge fields. The Yang-Mills Lagrangian with only the gauge field reads

$$\mathcal{L}_{\mathrm{YM}} = \frac{1}{2g^2}\,\mathrm{tr}[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}], \qquad \mathbf{F}_{\mu\nu} = -igF^a_{\mu\nu}T^a$$

where T^a are the generators of the gauge group $\mathrm{SU}(N)$. The general form of the gauge transformation for a field is

$$\mathbf{A}'_\mu = \omega\mathbf{A}_\mu\omega^{-1} + \omega(\partial_\mu\omega^{-1}), \quad \omega \in \mathrm{SU}(N).$$

The gauge constraint must be generalized slightly:

$$\mathbf{G}(\mathbf{A}^{[\alpha]}) = (\partial^\mu\mathbf{A}^{[\alpha]}_\mu) - \mathbf{f}(x) = \partial_\mu[\omega_\alpha\mathbf{A}_\mu\omega_\alpha^{-1} + \omega_\alpha(\partial_\mu\omega_\alpha^{-1})] - \mathbf{f}(x).$$

The field transformation can be compactly written as

$$\mathbf{A}^{[\alpha]}_\mu = e^{i\alpha_b T^b}(-igA^a_\mu T^a + \partial_\mu)e^{-i\alpha_c T^c} = -ig(A^a_\mu + \partial_\mu\alpha^a/g + f^{abc}A^b_\mu\alpha^c)T^a = \mathbf{A}_\mu + \frac{1}{g}\mathbf{D}_\mu\alpha.$$

The gauge fixing condition becomes

$$G(\mathbf{A}^\alpha)^a = \partial^\mu(A^a_\mu + (\mathrm{D}_\mu\alpha)^a/g) - f^a(x).$$

The derivative of the gauge fixing function with respect to α is

$$M^{ab} = \frac{\delta G(\mathbf{A}^{[\alpha]}_\mu)^a}{\delta\alpha^b} = \frac{1}{g}(\partial^\mu\mathrm{D}_\mu)^{ab} = \frac{1}{g}\partial^\mu(\partial_\mu\delta^{ab} + f^{abc}A^c_\mu).$$

This transforms in the adjoint representation $M \to \omega M\omega^{-1}$ (hence not gauge invariant). However, the determinant is gauge invariant as it can be seen (after some work) from the expression

$$\det\left(\frac{\delta G(\mathbf{A}^{[\alpha]})^a}{\delta\alpha^b}\right)^{-1} = \int \mathcal{D}[\alpha^a]\delta(G(\mathbf{A}^{[\alpha]})^a),$$

which shows that the determinant is a gauge invariant number, since it can be written as an integration over all possible values of the gauge parameters on the right-hand side. However, this time the determinant depends on the field! Consequently, we cannot just absorb it into the normalization constant. Other than that, we can follow the same steps as before (introducing the Gaussian regulator) to get

$$Z = \mathcal{N}\int \mathcal{D}[A^a]\, e^{iS_\xi[A^a]}\det\left(\frac{(\partial^\mu\mathrm{D}_\mu)^{ab}}{g}\right)$$

where

$$S_\xi[A^a] = \int \mathrm{d}^4x\, \left(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right)$$

is the gauge-fixed action. Next, we'd better do something with the determinant. Remember the identity of Grassmann integrals

$$\int \mathrm{d}\overline{\theta}_1 \, \mathrm{d}\theta_1 \, \ldots \, \mathrm{d}\overline{\theta}_n \, \mathrm{d}\theta_n \, e^{-\overline{\theta}_i A_{ij} \theta_j} = \det A.$$

With this, we can rewrite the determinant as a functional integral over some new fields:

$$\det\left(\frac{(\partial^\mu \mathrm{D}_\mu)^{ab}}{g}\right) = \int \mathcal{D}[c]\,\mathcal{D}[\overline{c}]\,\exp\left(i\int \mathrm{d}^4x\,\overline{c}[-\partial^\mu \mathrm{D}_\mu]c\right).$$

The term can alternatively be written as

$$\overline{c}_a[-\partial^\mu \mathrm{D}_\mu]^{ab} c_b = \overline{c}_a[-\Box\delta^{ab} - g\partial_\mu f^{abc} A^c_\mu]c_b.$$

The fields are Grassmann numbers, so they anti-commute and at the same time they are scalars (spin zero). This means that they violate the spin-statistic theorem, which means they cannot be physical fields. They are called *Faddeev-Popov ghosts*. We can define the ghost action

$$S_{\mathrm{gh}} = \int \mathrm{d}^4x\,\overline{c}^a(-\Box\delta^{ab} - g\partial^\mu f^{abc} A^c_\mu)c^b$$

and from this the Yang-Mills generating functional

$$Z = \mathcal{N}\int \mathcal{D}[A^a]\,\mathcal{D}[c^b]\,\mathcal{D}[c^c]\,e^{i(S_\xi[A^\mu]+S_{\mathrm{gh}})}.$$

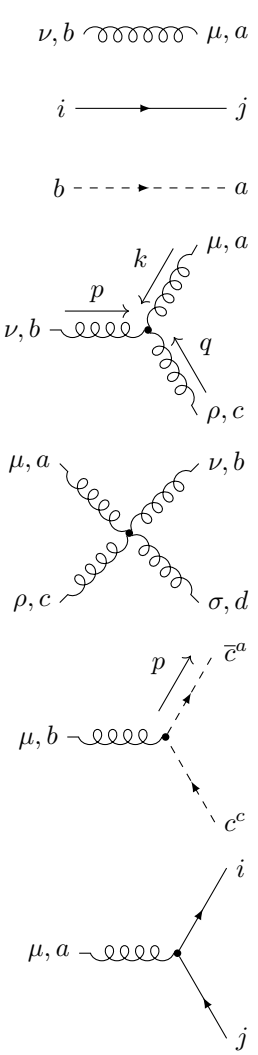
From this it is easy to calculate the propagator:

$$\langle\Omega|T\{c^a(x)\overline{c}^b(y)\}|\Omega\rangle = \int \frac{\mathrm{d}^4k}{(2\pi^4)}e^{-ik(x-y)}\frac{i\delta^{ab}}{k^2+i\varepsilon}.$$

Since ghosts are Grassmann variables, we have to keep the track of the momentum direction, so we denote them with a dashed arrow. If we also include the fermions, the full Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \overline{\Psi}(i\not{\mathrm{D}} - m)\Psi + \overline{c}^a(-\partial^\mu \mathrm{D}^{ab}_\mu)c^b.$$

Here is the list of all propagators and vertex factors in the Yang-Mills theory with fermions.



$$= i\frac{-g^{\mu\nu} + (1-\xi)p^\mu p^\nu/p^2}{p^2 + i\varepsilon}\delta^{ab},$$

$$= \frac{i\delta^{ij}}{\not{p} - m + i\varepsilon},$$

$$= \frac{i\delta^{ab}}{p^2 + i\varepsilon},$$

$$= gf^{abc}\left(g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right),$$

$$= \begin{aligned} &-ig^2[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})] \end{aligned}$$

$$= -gf^{abc}p^\mu,$$

$$= ig\gamma^\mu T^a_{ij}.$$

4.3.1 Gribov ambiguity and axial gauges

Faddeev-Popov procedure relies crucially on the fact that the gauge fixing function $G(A)$ has a unique solution for every gauge field configuration A_μ. However, this is not always the case. The solutions of the equation $G(A) = 0$ can be degenerate, i.e., there can be multiple α's that give the same gauge field configuration. This happens in particular with Coulomb and $\mathbb{R}_\xi$ gauges. The duplicate fields are called *Gribov copies*. However, there are two reassuring facts.

- We can get rid of Gribov copies if we require a positive definite determinant in the path integral.
- This ambiguity only affects large scale field configurations, i.e., non-perturbative calculations.

There is a set of gauges that don't suffer from the Gribov ambiguity. They are called the *axial gauges*:

$$[G(\mathbf{A})]^a = r^\mu A^a_\mu - f^a(x) = 0.$$

This is not covariant as it singles out a special direction r^μ. It is exactly the same acrobatics as we have already seen when calculating polarization vectors in spinor-helicity formalism. Following the same steps as before, the gauge-fixing and ghost terms in the Lagrangian are

$$\mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{gh}} = -\frac{1}{2\lambda}(r^\mu A^a_\mu)^2 + \overline{c}^a r^\mu(\delta^{ac}\partial_\mu + g f^{abc} A^b_\mu)c^c.$$

The parameter λ is a real number that has the same role as ξ in the R_ξ gauge. The propagator is

$$\Pi^{\mu\nu,ab}_{\text{axial}}(k) = \frac{i\delta^{ab}}{k^2 + i\varepsilon}\left(-g^{\mu\nu} + \frac{k^\mu r^\nu + k^\nu r^\mu}{k\cdot r} + \frac{r^2 + \lambda k^2}{(k\cdot r)^2}k^\mu k^\nu\right.$$

Choosing light-like vector r^μ and $\lambda = 0$ simplifies the propagator to

$$\Pi^{\mu\nu,ab}_{\text{axial}}(k) = \frac{i\delta^{ab}}{k^2 + i\varepsilon}\left(-g^{\mu\nu} + \frac{k^\mu r^\nu + k^\nu r^\mu}{k\cdot r}\right).$$

This propagator has two important properties.

1. The propagator is orthogonal to both k^μ and r^μ, which implies only two physical polarizations that can propagate.
2. The ghost Lagrangian includes a term that is proportional to $r^\mu A_\mu$. As the propagator is orthogonal to r^μ, the ghost field does not couple to the gauge field. The ghost fields completely decouple from the physical fields and don't need to be considered.

However, even though the axial gauge is free of Gribov copies, it is gauge invariant and hence the calculation of loop diagrams becomes extremely complicated. People usually choose the Feynman gauge as it is the lesser of two evils.

4.4 BRST symmetry

BRST symmetry is another symmetry of Yang-Mills Lagrangian that helps us prove the renormalizability of the theory. The Yang-Mills Lagrangian with one fermion reads

$$\mathcal{L}_{\text{YM}} = \underbrace{-\frac{1}{4}F^{\mu\nu,a}F^a_{\mu\nu} + \overline{\Psi}(i\not{D} - m)\Psi}_{\mathcal{L}_{\text{cl}}} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \overline{c}^a(-\partial^\mu D^{ab}_\mu)c^b.$$

Now we introduce an auxiliary field B^a and rewrite the Lagrangian as

$$\mathcal{L}_{\text{YM}} = \mathcal{L}_{\text{cl}} + \overline{c}^a(-\partial^\mu \mathrm{D}^{ab}_\mu)c^b + \frac{\xi}{2}B^a B^a + B^a\partial^\mu B^a_\mu.$$

The equation of motion for the auxiliary field B^a is

$$B^a + \frac{1}{\xi}\partial^\mu A^a_\mu = 0.$$

This field does not have a kinetic term, so it does not propagate (cannot be used in Feynman diagrams). Even more, it is equivalent to the gauge fixing term $(\partial^\mu A_\mu)^2/(2\xi)$, as we can see by writing

$$\frac{\xi}{2}\left(B^a B^a + \frac{2}{\xi} B^a \partial_\mu A^{\mu,a}\right) = \frac{\xi}{2}\left((B^a + \xi^{-1}\partial^\mu A^a_\mu)^2 - \frac{1}{\xi^2}(\partial^\mu A^a_\mu)^2\right).$$

In the path integral it becomes

$$\int \mathcal{D}[B_a] \exp\left(i\int \mathrm{d}^4x \left[\frac{\xi}{2}B^a B^a + B^a \partial^\mu A^a_\mu\right]\right) =$$
$$= \underbrace{\exp\left(i\int \mathrm{d}^4x \left[-\frac{(\partial^\mu \partial^a_\mu)^2}{2\xi}\right]\right)}_{\text{standard } R_\xi \text{ gauge term}} \underbrace{\int \mathcal{D}[B_a] \exp\left(i\int \mathrm{d}^4x \left[\frac{\xi}{2}B^a B^a\right]\right)}_{\text{trivial Gaussian integral}}.$$

The *BRST symmetry* is a symmetry of the Yang-Mills Lagrangian that is related to the gauge invariance. To construct it, we introduce a Grassmann parameter θ and the symmetry reads

$$\begin{aligned}
\delta A^a_\mu &= \theta D^{ab}_\mu c^b, \\
\delta \Psi &= ig\theta c^a T^a_{\mathrm{F}} \Psi, \\
\delta c^a &= -\frac{1}{2} g\theta f^{abc} c^b c^c, \\
\delta \overline{c}^a &= \theta B^a, \\
\delta B^a &= 0.
\end{aligned}$$

The first two lines are the gauge transformation of the gauge field and the fermion field, respectively, with the choice of the gauge transformation $\alpha(x) = g\theta c^b(x)$. We can also see that the number of ghost fields increases (or stays the same) under the BRST transformation (the number of anti-ghosts decrease). We say that BRST has *ghost number* of $+1$. It is also clear that the classical part of the Lagrangian is invariant under the BRST transformation. The same holds for the term $B^a B^a$ since $\delta B^a = 0$. For the other two terms, we will check term by term:

$$\begin{aligned}
B^a \partial^\mu A^a_\mu &\xrightarrow{\text{BRST}} B^a \partial^\mu A^a_\mu + \theta B^a \partial^\mu (D^{ab}_\mu c^b) \\
\overline{c}^a (-\partial^\mu D^{ab}_\mu) c^b &\xrightarrow{\text{BRST}} (\overline{c}^a + \theta B^a)(-\partial^\mu D^{ab}_\mu c^c - \partial^\mu \delta(\mathrm{D}^{ab}_\mu c^b)) \\
&= \overline{c}^a (-\partial^\mu \mathrm{D}^{ab}_\mu c^b - \partial^\mu \delta(\mathrm{D}^{ab}_\mu c^b)) + \theta B^a (-\partial^\mu \mathrm{D}^{ab}_\mu c^b).
\end{aligned}$$

Two terms cancel, but we are still left with the term $\delta(\mathrm{D}^{ab}_\mu c^b)$, which we will show equals zero:

$$\begin{aligned}
\delta(\mathrm{D}^{ab}_\mu c^b) &= \mathrm{D}^{ab}_\mu \delta c^b + g f^{abc} \delta A^b_\mu c^c \\
&= -\frac{1}{2} g\theta \partial_\mu (f^{abc} c^b c^c) - \frac{1}{2} g^2 \theta f^{abc} f^{cde} A^b_\mu c^d c^e + g f^{abc} (\partial_\mu c^b) c^c + g^2 \theta f^{abc} f^{cde} A^d_\mu c^e c^c
\end{aligned}$$

Some of the first term cancels with the third term, and we are left with

$$\begin{aligned}
\delta(\mathrm{D}^{ab}_\mu c^b) &= -\frac{1}{2} g^2 \theta f^{abc} f^{cde} [A^b_\mu c^d c^e + A^d_\mu c^e c^a + A^e_\mu c^b c^d] \\
&= -\frac{1}{2} g^2 \theta a^b_\mu c^d c^e (f^{abc} f^{cde} + f^{aec} f^{cbd} + f^{adc} f^{ced}) = 0.
\end{aligned}$$

The result is zero by the Jacobi identity. As we will show shortly, BRST symmetry is *nilpotent*, i.e. applying it twice gives zero. Using some differential geometry, we can then read this symmetry as a cohomology theory. Denote the set of all field $\phi = \{A^a_\mu, \Psi, c^a, B^a, \dots\}$ and the BRST operator as Q:

$$\delta^{\mathrm{BRST}}\phi = \theta Q\phi.$$

As an example, $QA^a_\mu = \mathrm{D}^{ab}_\mu c^b$. We want to show that $Q^2\phi = 0$, and we will do this field by field:

$$\begin{aligned}
Q^2 A^a_\mu &= \frac{1}{\theta}\delta^{\mathrm{BRST}}(D^{ab}_\mu c^b) = 0,\\
Q^2 c^a &= -\frac{1}{2}gf3abc\delta(c^b c^c)/\theta = \frac{1}{2}gf^{abc}\left(\frac{1}{2}g\theta f^{bde}c^d c^e c^c - \frac{1}{2}gc^b\theta f^{cde}c^d c^e\right) = \dots\\
&= \frac{1}{2}g^2 f^{abc}f^{bde}c^d c^e c^c = 0,\\
Q^2\Psi &= \delta(igc^a T^a\Psi)/\theta = ig\left(-\frac{1}{2}g\theta f^{abc}T^a c^b c^c\Psi + igc^a\theta T^a c^b T^b\Psi\right) = \dots\\
&= -\frac{1}{2}g^2\left(f^{abc}T^a c^b c^c - f^{abc}T^a c^b c^c\right) = 0.
\end{aligned}$$

In the last step for the ghost field, we used the Jacobi identity for the structure constants. Since this operator is nilpotent on any field, it is also nilpotent on any Lagrangian.

With the symmetry of the Lagrangian comes a conserved current J^μ_{B} and a conserved charge

$$Q^{\mathrm{B}} = \int \mathrm{d}^3x\; J^{\mathrm{B}}_0(x).$$

In the Hamiltonian picture, the transformation can be written in terms of a commutator

$$\delta\phi = \theta Q\phi = i[\theta Q^{\mathrm{B}}, \phi] = i\theta[Q^{\mathrm{B}}, \phi]_{\mp}.$$

The symmetry implies $[Q^{\mathrm{B}}, H] = 0$, where H is the Hamiltonian. The nilpotency of the operator Q implies that

$$0 = Q^2\phi = [Q^{\mathrm{B}}, [Q^{\mathrm{B}}, \phi]_{\mp}]_{\pm} = [Q^2_{\mathrm{B}}, \phi] = 0.$$

It implies that Q^{B} is either 0 or identity. It cannot be identity, as it should increase the ghost number by 2. Thus, we have shown that the conserved charge Q^{B} is also nilpotent $Q^2_{\mathrm{B}} = 0$.

Let $\mathcal{H}$ be the set of eigenstates of the Hamiltonian H and let $|\phi\rangle \in \mathcal{H}$. If it holds $Q^{\mathrm{B}}|\phi\rangle = 0$ then $|\phi\rangle \in \ker(Q^{\mathrm{B}})$. If for some $|\psi\rangle \in \mathcal{H}$ it holds $|\psi\rangle = Q^{\mathrm{B}}|phi\rangle$ for some $|\phi\rangle \in \mathcal{H}$, then $Q^{\mathrm{B}}|\psi\rangle = 0$ by nilpotency and $|\psi\rangle \in \ker(Q^{\mathrm{B}})$. But since $|\psi\rangle \in \mathrm{im}(Q^{\mathrm{B}})$ it must hold $\mathrm{im}(Q^{\mathrm{B}}) \subset \ker(Q^{\mathrm{B}})$. We can split the Hilbert space into three subspaces:

- $\mathcal{H}_1$ is the space of all states that are not annihilated by Q^{B}, i.e. $|\psi_1\rangle \in \mathcal{H}_1$ if $Q^{\mathrm{B}}|\psi_1\rangle \neq 0$.
- $\mathcal{H}_2$ is the image of Q^{B}, i.e. $|\psi_2\rangle \in \mathcal{H}_2$ if there exists $|\phi\rangle \in \mathcal{H}$ such that $|\psi_2\rangle = Q^{\mathrm{B}}|\phi\rangle$.
- $\mathcal{H}_0$ is the space of all states that are in the kernel but not in the image of Q^{B}, i.e. $|\psi_0\rangle \in \mathcal{H}_0$ if $Q^{\mathrm{B}}|\psi_0\rangle = 0$ and $|\psi_0\rangle \notin \mathcal{H}_2$.

The last subspace can be thought of as *cohomology* of the operator Q^{B}:

$$\mathcal{H}_0 = \frac{\ker(Q^{\mathrm{B}})}{\mathrm{im}(Q^{\mathrm{B}})}.$$

Let's make some observations:

1. If $|\psi_{2a}\rangle, |\psi_{2b}\rangle \in \mathcal{H}_2$, then $\langle\psi_{2a}|\psi_{2b}\rangle = 0$.
2. For all $|\psi_2\rangle \in \mathcal{H}_2$ and $|\psi_0\rangle \in \mathcal{H}_0$ it holds $\langle\psi_2|\psi_0\rangle = 0$.
3. If $|\psi_{0a}\rangle, |\psi_{0b}\rangle \in \mathcal{H}_2$, then $\langle\psi_{0a}|\psi_{0b}\rangle = \mathbb{1}\delta_{ab}$.

Using the Hodge decomposition, all states $\mathcal{H}_1$ can be chosen to be orthogonal to any state from the kernel $\ker\psi \in \ker(Q^{\mathrm{B}})$. To get a physical interpretation of the BRST symmetry, we can consider it for the asymptotic states where $g = 0$. The transformation is

$$\begin{aligned}\delta A^a_\mu &= \theta D^{ab}_\mu c^b,\\ \delta\Psi &= 0,\\ \delta c^a &= 0,\\ \delta\overline{c}^a &= \theta B^a,\\ \delta B^a &= 0.\end{aligned}$$

The equations of motion for B^a are $\xi B^a - \partial^\mu A^a_\mu$. The quanta of $B^a \sim k^\mu\varepsilon^a_\mu \neq 0$ cannot be transversely polarized, and not even forward polarized, hence they have to correspond to the backwards polarized gluons A_μ. The BRST transforms forward-polarized gluons to ghosts, it annihilates ghosts and turns anti-ghosts into backwards-polarized gluons. This means that

- forward polarized gluons and anti-ghosts belong to $\mathcal{H}_1$,
- backward polarized gluons and ghosts belong to $\mathcal{H}_2$,
- physical (transverse) gluons belong to $\mathcal{H}_0$.

This is a very hand-wavy proof that the physical states correspond exactly to $\mathcal{H}_0$, i.e., the cohomology of the BRST operator Q^{B}. Let's make this proof a bit more legitimate. The Lagrangian can be split in classical and new part, $\mathcal{L} = \mathcal{L}_{\mathrm{cl}} + \mathcal{L}_{\mathrm{new}}$. The classical part is trivially invariant under the BRST transformation, and hence the new part must be invariant on its own. We can write $\mathcal{L}_{\mathrm{new}} = Q\kappa$, where

$$\kappa = \overline{c}^a(\partial^\mu A^a_\mu + \xi B^a/2).$$

Reader can check that acting with Q on κ gives the new part of the Lagrangian. Using this, we can write the full action

$$S_{\mathrm{B}} = S_{\mathrm{cl}} + QK, \qquad K = \int \mathrm{d}^4x\ \kappa = \int \mathrm{d}^4x\ \overline{c}^a(\partial^\mu A^a_\mu + \xi B^a/2).$$

The consequence is that the S-matrix is gauge invariant between physical states. Even though K depends on ξ it does not affect the physical states. To see this, take two physical states $|\phi\rangle, |\psi\rangle$ and consider the path integral

$$Z = \int \mathrm{D}[\phi] e^{i(S_{\mathrm{cl}}+QK)}.$$

The variation of Z with respect to the gauge parameter ξ is $\delta_\xi Z = Z_\xi(iQ\delta_\xi K) \propto [Q^{\mathrm{B}}, \delta_\xi K]$. For two states $|\phi\rangle, |\psi\rangle$ we have

$$\delta_\xi\langle\phi|\psi\rangle \propto \langle\psi|[Q^{\mathrm{B}}, \delta_\xi K]|\phi\rangle = 0, \quad \forall\delta_\xi K.$$

This means that for physical states it must hold $Q^{\mathrm{B}}|\phi\rangle = 0$ and $\langle\psi|Q^{\mathrm{B}} = 0$, i.e., physical states must be annihilated by Q^{B}. This means that they must be in the kernel of Q^{B}. Nonetheless, this doesn't exclude the space $\mathcal{H}_2$ but for that we would need more formal treatment (which can be found in Weinberg). In the language of differential geometry, we can write

- $\mathcal{H}_1$: $Q_\mathrm{B}\,|\psi_1\rangle \neq 0$, not-closed states (anti-ghosts and forward polarized gluons),
- $\mathcal{H}_2 = \mathrm{im}(Q_\mathrm{B})$: exact states (ghosts and backward polarized gluons),
- $\mathcal{H}_0 = \ker(Q_\mathrm{B})/\,\mathrm{im}(Q_\mathrm{B})$: the cohomology of the BRST operator (physical states).

By the unitarity of the S-matrix, we can write for all $|A_0\rangle\,, |B_0\rangle \in \mathcal{H}_0$

$$\langle A_0|B_0\rangle\,\langle A_0|S^\dagger S|B_0\rangle = \sum_{\psi\in H}\langle A_0|S^\dagger|\psi\rangle\,\langle\psi|S|B_0\rangle = \sum_{C_0\in\mathcal{H}_0}\langle A_0|S^\dagger|C_0\rangle\,\langle C_0|S|B_0\rangle\,.$$

In the last step we restricted the S-matrix to $\mathcal{H}_0$. We can do this, because $[Q_\mathrm{B}, H] = 0$, and it holds

$$Q_\mathrm{B}S\,|A_0\rangle = SQ_\mathrm{B}\,|A_0\rangle = 0.$$

Denoting $S\,|A_0\rangle = |C_0\rangle \in \mathcal{H}_0$ and $Q_\mathrm{B}\,|A_0\rangle = |K_2\rangle \in \mathcal{H}_2$ we get $\langle A_0|K_2\rangle = 0$. And since $\langle J_2|K_2\rangle = 0$ for any two states in $\mathcal{H}_2$ we can entirely forget any states ψ in $\mathcal{H}_2$. In practice, the restriction of the S-matrix to H_0 is reflected in the fact that ghosts will always cancel the non-physical polarizations of the gluons in loop diagrams.

Next, we want to make some general statements about the n-point correlation functions (and in particular the Ward identity) using the BRST symmetry. The BRST equivalent of the Ward identity is called the *Slavnov-Taylor identity*:

$$\sum_{i=1}^{m}\langle\Omega|T\{\phi_{n_1}(x_1)\dots\phi_{n_{i-1}}(x_{i-1})\delta^\mathrm{BRST}(\phi_{n_i}(x_i))\phi_{n_{i+1}}(x_{i+1})\dots\phi_{n_n}(x_m)\}|\Omega\rangle = 0$$

where $\phi_n = \{\Psi, A^c_\mu, c^a, B^a, \dots\}$ and operator δ^BRST denotes the BRST action. To prove this, we start with the equations of motion for the Green's functions is (called the *Schwinger-Dyson equation*):

$$\langle\Omega|T\left\{\frac{\delta S[\phi_j]}{\delta\phi_n(x)}\phi_{n_1}(x_1)\dots\phi_{n_m}(x_m)\right\}|\Omega\rangle = \\ = i\sum_{i=1}^{m}\delta^{(4)}(x-x_i)\,\langle\Omega|T\{\phi_{n_1}(x_1)\dots\cancel{\phi_{n_i}(x_i)}\dots\phi_{n_m}(x_m)\}|\Omega\rangle\,.$$

The off-shell BRST current takes the form

$$\partial_\mu j^\mu(x) = -\sum_{n=1}^{m}\frac{\delta S[\phi_j]}{\delta\phi_n}\delta^\mathrm{BRST}(\phi_n).$$

Now we combine the two expressions from above

$$\partial_\mu\,\langle\Omega|T\{j^\mu(x)\phi_{n_1}(x_1)\dots\phi_{n_m}(x_m)\}|\Omega\rangle = \\ = -\sum_{n=1}^{m}\langle\Omega|T\left\{\frac{\delta S[\phi_j]}{\delta\phi_n}\delta^\mathrm{BRST}(\phi_n(x))\phi_{n_1}(x_1)\dots\phi_{n_m}(x_m)\right\}|\Omega\rangle \\ = -i\sum_{n=1}^{m}\delta^{(4)}(x-x_i)\,\langle\Omega|T\{\phi_{n_1}(x_1)\dots\delta^\mathrm{BRST}(\phi_{n_i}(x_i))\dots\phi_{n_m}(x_m)\}|\Omega\rangle\,.$$

When we integrate this equation over the whole space, the left side is zero (because the current is conserved) and the right side give exactly the Slavnov-Taylor identity:

$$\sum_{i=1}^{m} \langle\Omega|T\{\phi_{n_1}(x_1)\dots\phi_{n_{i-1}}(x_{i-1})\delta^{\mathrm{BRST}}(\phi_{n_i}(x_i))\phi_{n_{i+1}}(x_{i+1})\dots\phi_{n_n}(x_m)\}|\Omega\rangle = 0.$$

Using this property, we can prove the transversality of the gluon self-energy (using R_ξ gauge):

$$\langle\Omega|T\{A^a_\mu(x)A^b_\nu(y)\}|\Omega\rangle = \int \frac{\mathrm{d}^4k}{(2\pi^4)} e^{-ik(x-y)}\Pi^{ab}_{\mu\nu}(k).$$

It equals the geometric series of 1PI diagrams

$$\Pi^{ab}_{\mu\nu}(k) = \quad + \quad + \quad + \dots$$

$$= \frac{\delta^{ab}}{k^2+i\varepsilon}\left[F_1(k^2)g^{\mu\nu} + F_2(k^2)k^\mu k^\nu\right]$$

$$= \frac{i\delta^{ab}}{k^2+i\varepsilon}\left[\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}\right)A(k^2) + \frac{k_\mu k_\nu}{k^2}B(k^2)\right].$$

We have introduced the longitudinal and transversal projectors

$$P^{\mathrm{T}}_{\mu\nu} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2},$$
$$P^{\mathrm{L}}_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}.$$

They satisfy the following relations

$$P^{\mathrm{T}}_{\mu\nu}k^\mu = P^{\mathrm{T}}_{\nu\mu}k^\mu = 0, \qquad P^{\mathrm{L}}_{\mu\nu}k^\mu = k^\nu,$$
$$P^{\mathrm{T,L}}_{\mu\nu}g^{\nu\rho}P^{\mathrm{T,L}}_{\rho\sigma} = P^{\mathrm{T,L}}\mu\sigma, \qquad P^{\mathrm{L}}P^{\mathrm{T}} = 0.$$

To the leading order

$$\Pi^{ab}_{\mu\nu}(k) = \frac{i\delta^{ab}}{k^2+i\varepsilon}\left(-g_{\mu\nu}+(1-\xi)\frac{k_\mu k_\nu}{k^2}\right) = \frac{i\delta^{ab}}{k^2+i\varepsilon}\left(P^{\mathrm{T}}_{\mu\nu} - \xi P^{\mathrm{L}}_{\mu\nu}\right).$$

We would expect that $A(k^2) = 1 + \mathcal{O}(g^2)$ and $B(k^2) = \xi + \mathcal{O}(g^2)$, however, we will see that $B(k^2) = \xi$ exactly to all orders. The one-particle-irreducible self energy is

$$\Sigma_{\mu\nu} = diagram = k^2/P^{\mathrm{T}}_{\mu\nu}\Sigma^{\mathrm{T}}(k^2 - P^{\mathrm{L}}_{\mu\nu}\Sigma^{\mathrm{L}}(k^2)).$$

The expansion of the full propagator in terms of 1PI parts is

$$\Pi^{ab}_{\mu\nu} = \frac{i\delta^{ab}}{k^2+i\varepsilon}\left[P^{\mathrm{T}}_{\mu\nu} - \xi P^{\mathrm{L}}_{\mu\nu}\right] + \frac{i\delta^{ab}}{k^2+i\varepsilon}\left[P^{\mathrm{T}}_{\mu\rho} - \xi P^{\mathrm{L}}_{\mu\rho}\right](-\sigma_{\rho\sigma})\left[P^{\mathrm{T}}_{\sigma\nu} - \xi P^{\mathrm{L}}_{\sigma\nu}\right] + \dots$$

$$= \frac{i\delta^{ab}}{k^2+i\varepsilon}\left[P^{\mathrm{T}}_{\mu\nu} - \xi P^{\mathrm{L}}_{\mu\nu}\right] + \frac{i\delta^{ab}}{k^2+i\varepsilon}\left[P^{\mathrm{T}}_{\mu\nu}\Sigma^{\mathrm{T}} - P^{\mathrm{L}}_{\mu\nu}\xi\Sigma^{\mathrm{P}}\right] +$$
$$\frac{i\delta^{ab}}{k^2+i\varepsilon}\left[P^{\mathrm{T}}_{\mu\nu}(\sigma^{\mathrm{T}})^2 - P^{\mathrm{L}}_{\mu\nu}(\xi\Sigma^{\mathrm{P}})^2\right] + \dots$$

$$= \frac{i\delta^{ab}}{k^2+i\varepsilon}\left[P^{\mathrm{T}}_{\mu\nu}\left(\frac{1}{1-\Sigma^{\mathrm{T}}(k^2)}\right) - \xi P^{\mathrm{L}}_{\mu\nu}\left(\frac{1}{1-\xi\Sigma^{\mathrm{L}}(k^2)}\right)\right].$$

From this we see that

$$A(k^2) = \frac{1}{1-\Sigma^{\mathrm{T}}(k^2)}, \qquad B(k^2) = \frac{\xi}{1-\xi\Sigma^{\mathrm{L}}(k^2)}.$$

This means that we can only focus on $\Sigma^{\mathrm{L}}(k^2)$. Write

$$\langle\Omega|T\left\{\frac{\delta S[\phi]}{\delta B^a(x)}\partial_y^\mu A_\mu^b(y)\right\}|\Omega\rangle = 0$$

where $\partial_y^\mu = \partial/\partial y_\mu$. The derivative of action with respect to B^a is

$$\frac{\delta S}{\delta B^a(x)} = \partial_x^\nu A_\nu^a(x) + \xi B^a(x).$$

Note that this is not automatically zero, as we don't assume that the equations of motion are satisfied. We have

$$\langle\Omega|T\{(\partial_x^\mu A_\nu^a(x) + \xi B^a(x))\partial_y^\mu A_\mu^a(y)\}|\Omega\rangle = 0$$

which gives

$$\partial_x^\nu\partial_y^\mu \langle\Omega|T\{A_\nu^a(x)A_\mu^b(y)\}|\Omega\rangle = -\xi\langle\Omega|T\{B^a(x)\partial_y^\mu(y)\}|\Omega\rangle = -\xi\langle\Omega|T\{(Qc^a(x))\partial_y^\mu(y)\}|\Omega\rangle.$$

The Slavnov-Taylor identity gives us

$$0 = \langle\Omega|T\{\theta Q\bar{c}^a(x)\partial_y^\mu A_\mu^a(y)\}|\Omega\rangle + \langle\Omega|T\{\bar{c}^a(x)\theta Q(\partial_y^\mu A_\mu^a(y))\}|\Omega\rangle.$$

BRST acting on the derivative of the gauge field is

$$Q(\partial_y^\mu A_\mu^a(y)) = \partial_y^\mu(QA_\mu^a(y)) = \partial_y^\mu(\mathrm{D}_\mu^{ab}c^b(y)),$$

and we also have $\delta S/\delta\bar{c}^a(y) = -\partial_y^\mu(\mathrm{D}_\mu^{ab}c^b(y))$. The Slavnov-Taylor identity can be rewritten using these two relations as

$$0 = \langle\Omega|T\{Q\bar{c}^a(x)\partial_y^\mu A^b(y)\}|\Omega\rangle + \langle\Omega|T\{\bar{c}^a(\delta S/\delta\bar{c}^b(y))\}|\Omega\rangle.$$

Going back to the equation of motion, we can rewrite it as

$$\partial_x^\mu\partial_y^\nu \langle\Omega|T\{A_\nu^a(x)A_\mu^b(y)\}|\Omega\rangle = -\xi\langle\Omega|T\left\{\frac{\delta S}{\delta\bar{c}^b(y)}\bar{c}^a(x)\right\}|\Omega\rangle \overset{\text{EoM}}{=} -i\xi\delta^{ab}\delta^{(4)}(x-y)\underbrace{\langle\Omega|\Omega\rangle}_{1}.$$

We now preform a Fourier transformation of the propagator

$$\partial_x^\mu\partial_y^\nu\int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{-ik(x-y)}\Pi_{\mu\nu}^{ab}(k) = -i\xi\int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{-ik(x-y)}.$$

Hence, we get

$$\boxed{k^\mu k^\nu\Pi_{\mu\nu}^{ab}(k) = -i\xi\delta^{ab}.}$$

When we write the propagator written in terms of the projectors, we get

$$\frac{\delta^{ab}}{k^2+i\varepsilon}k^\mu k^\nu\left[A(k^2)P_{\mu\nu}^{\mathrm{T}} - B(k^2)P_{\mu\nu}^{\mathrm{L}}\right] = -i\delta^{ab}B(k^2) = -i\xi\delta^{ab}.$$

Hence,

$$\boxed{B(k^2) = \xi, \quad \Sigma^{\mathrm{L}}(k^2) = 1}$$

to *all orders*.

4.5 Renormalization of Yang-Mills theory

In order to renormalize any theory, we have to determine which diagrams are divergent and which are not. The Lagrangian can be split into the kinetic and interaction part $\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}}$, where

$$\mathcal{L}_{\text{int}} = -gf^{abc}(\partial_\mu A^a_\nu)A^{\mu,b}A^{\nu,c} - \frac{1}{4}g^2 f^{eab}f^{ecd}A^a_\mu A^b_\nu A^{\mu,c}A^{\nu,d} + gf^{abc}(\partial_\mu \overline{c}^a)A^{b,\mu}c^c + g\overline{\Psi}_i \not{A}^a \Psi_j T^a_{ij}.$$

For every interaction term i, we can calculate the *superficial degree of divergence* Δ_i by the following formula

$$\Delta_i = d - a_i - \sum_f n_{i,f}(d/2 - 1 + s_f)$$

where a_i is the number of derivatives in the interaction term, d is the space-time dimension, $n_{i,f}$ is the number of external legs of type f and s_f is a number that is non-zero only for fermion fields (where it equals $1/2$). Our Lagrangian has $\Delta_i = 0$ for all interaction terms, so it is power-counting renormalizable. The superficial degree of divergence of the diagram can be calculated with the formula

$$D = d - \frac{3}{2}E_\Psi - E_A - E_{c,\overline{c}}$$

where E_Ψ is the number of external fermion legs, E_A is the number of external gauge field legs and $E_{c,\overline{c}}$ is the number of external ghost and anti-ghost legs. Note, however, that the superficial degree of divergence gives us only the upper bound. The actual divergence might be less severe (see Table 4.1).

As we see, there are many divergent diagrams, however all of them are only logarithmically divergent. The discrepancy between the superficial degree of divergence and the actual divergence can be attributed to the gauge invariance and the BRSTT symmetry. For example, BRST constrains the divergence of gluon propagator by the Slavnov-Taylor identity, which gives us the transversality condition:

$$AA1PI = (k^\mu k^\nu - k^2 g^{\mu\nu})\Sigma_{\text{T}}(k^2).$$

Since the transverse projector is not divergent and Σ is dimensionless, it can only be logarithmically divergent.

There are nine terms in the Lagrangian and all of them need counter-terms

$$\begin{aligned}
\mathcal{L}_{\text{ren}} = &-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^{\mu,a})^2 + \overline{\Psi}(i\not{\partial})\Psi - m\overline{\Psi}\Psi + g\overline{\Psi}\gamma^\mu T^a A^a_\mu \Psi + \\
&- gf^{abc}(\partial_\mu A^a_\nu)A^{\mu,b}A^{\nu,c} - \frac{1}{4}g^2 f^{abc}f^{ade}A^a_\mu A^c_\nu A^{\mu,d}A^{\nu,e} + (\partial_\mu \overline{c}^a)(\partial^\mu c^a) + \\
&- gf^{abc}(\partial^\mu \overline{c}^a)c^b A^c_\mu - \overline{\delta}_3\frac{}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{\overline{\delta}_\xi}{2\xi}(\partial_\mu A^{\mu,a})^2 + \overline{\delta}_2\overline{\Psi}(i\not{\partial})\Psi + \\
&- \overline{\delta}_m m\overline{\Psi}\Psi + \overline{\delta}_1 g\overline{\Psi}\gamma^\mu T^a A^a_\mu \Psi - \overline{\delta}_1^{3g} gf^{abc}(\partial_\mu A^a_\nu)A^{\mu,b}A^{\nu,c} + \\
&- \overline{\delta}_1^{4g}\frac{1}{4}g^2 f^{abc}f^{ade}A^a_\mu A^c_\nu A^{\mu,d}A^{\nu,e} + \overline{\delta}_3^c(\partial_\mu \overline{c}^a)(\partial^\mu c^a) - \overline{\delta}_1^c gf^{abc}(\partial^\mu \overline{c}^a)c^b A^c_\mu.
\end{aligned}$$

There are 9 distinct counter-terms that we could add. However, we can instead renormalize the theory by just rescaling the bare fields and parameters. Let Ψ_0, A^μ_0, c^a_0 be the bare fields and g_0, m_0, ξ_0 the bare parameters. The scaling is

$$\Psi_0 = \sqrt{Z_2}\Psi, \quad A^{\mu,a}_0 = \sqrt{Z_3}A^{\mu,a}, \quad c^a_0 = \sqrt{Z^c_3}c^a,$$

$$m_0 = Z_m m, \quad g_0 = Z_g \tilde{\mu}^\varepsilon g, \quad \xi_0 = Z_\xi \xi, \quad \tilde{\mu} = \sqrt{\frac{e^{\gamma_E}}{4\pi}}\mu$$

where μ is the dimensional regularization scale. The Slavnov-Taylor identity is

$$Z_3 \left(\partial_x^\nu \partial_y^\mu \langle\Omega|T\{A_\nu^a(x)A_\mu^b(y)\}|\Omega\rangle\right) Z_\xi \left(-\xi \langle\Omega|T\{\frac{\delta S}{\delta \overline{c}^b(y)}\overline{c}^a(x)\}|\Omega\rangle\right).$$

This implies that the parameter ξ is preserved by the renormalization, $Z_3 = Z_\xi$. In other words,

$$1 + \overline{\delta}_\xi = \frac{Z_3}{Z_\xi} = 1 \quad \Longrightarrow \quad \overline{\delta}_\xi = 0.$$

So we can immediately forget about renormalization of ξ. Comparing $\mathcal{L}_{\text{ren}}$ with the parameter-rescaled Lagrangian gives

$$\overline{\delta}_2 = Z_2 - 1, \quad \overline{\delta}_m = Z_2 Z_m - 1, \quad \overline{\delta}_1^{3q} = Z_g Z_3^{2/3} - 1, \quad \overline{\delta}_3 = Z_3 - 1,$$

$$\overline{\delta}_1 = Z_g Z_2 \sqrt{Z_3} - 1, \quad \overline{\delta}_1^{4g} = (Z_g Z_3)^2 - 1, \quad \overline{\delta}_3^c = Z_3^c - 1, \quad \overline{\delta}_1^c = Z_g Z_3^c \sqrt{Z_3} - 1.$$

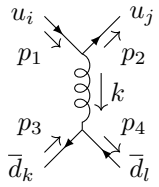
From renormalization of the propagators, we can determine three counter-terms. The other four divergent diagrams can be rendered finite by just one other counter-term (we won't prove this fact). So, we have in total 4 counter-terms that we need to calculate.

From now on, we will specialize to *quantum chromodynamics* (QCD). It is the Yang-Mills theory with $SU(3)$ gauge group and $N_f = 6$ different massless spin-1/2 particles, called *quarks*. They group neatly in three pairs, but from the QCD perspective they all have same properties.

Before calculating loop diagrams to obtain the counter-terms, we will first consider a tree level amplitude. Consider the following scattering process:

$$u(p_1) + \overline{d}(p_3) \to u(p_2)\overline{d}(p_4).$$

The Feynman rules give



$$= (ig_s^2) T_{ji}^a T_{kl}^b g_{\mu\nu} \frac{i(-g_{\mu\nu} + (1-\xi)k_\mu k_\nu/k^2)}{k^2} \delta^{ab} \overline{u}_j(p_2)\gamma^\mu u_i(p_1) \cdot \overline{v}_k(p_3)\gamma^\nu v_l(p_4)$$

$$= (ig_s)^2 T_{ij}^a T_{kl}^a \frac{1}{k^2} (\overline{u}_j \gamma_\mu u_i)(\overline{v}_k \gamma^\mu v_k).$$

The gauge dependent part of the amplitude identically equals zero, because $\overline{u}(p_2)\not{k}u(p_1) = 0$ (remember $k^\mu = p_1^\mu - p_2^\mu = p_4^\mu - p_3^\mu$). Matrices T_{ij}^a are the generators of SU(3) group in the fundamental representation, half of Gell-Mann matrices $T_{ij}^a = \lambda_{ij}^a/2$. Notice that the entire amplitude is exactly the same as in the QED case, except for the colour factor $T_{ij}^a T_{kl}^a$. Let's look at some examples.

1. Let u be red ($i = 1$) and $\overline{d}$ be anti-green ($k = 2$). Then the colour factor is

$$T_{j1}^a T_{2l}^a = \frac{1}{2}(\delta_{jl}\delta_{21} - \delta_{j1}\delta_{2l}/3) = -\frac{1}{6}\delta_{j1}\delta_{2l} = -\frac{1}{6}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   So the process is exactly the same as $e^-\mu^+ \to e^-\mu^+$ just with e (the coupling constant) replaced with $-g_s/6$. Since the sign is opposite, this interaction is actually *attractive*.

2. Let u be red ($i = 1$) and $\overline{d}$ be anti-red ($k = 1$). Then the colour factor is

$$T^a_{j1}T^a_{1l} = \frac{1}{2}\begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, the product state of the initial particles (fundamental times anti-fundamental) can be split into the direct sum of irreducible representations

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8},$$

a singlet and an octet. The colour singlet is expressed as

$$|\mathbf{1}\rangle = \frac{1}{\sqrt{3}}(|r\overline{r}\rangle + |g\overline{g}\rangle + |b\overline{b}\rangle)$$

(this corresponds to the second example) and the elements of the octet have the form $|r\overline{g}\rangle$ (first example). For the reaction in the second case $|\rangle$ missing

$$V^{\mathrm{QED}}(r) = -\frac{e^2}{4\pi r}, \qquad V^{\mathrm{QCD}}_{\mathrm{singlet}}(r) = -\frac{4}{3}\frac{g_\mathrm{s}^2}{4\pi r}, \qquad V^{\mathrm{QCD}}_{\mathrm{octet}}(r) = +\frac{1}{6}\frac{g_\mathrm{s}^2}{4\pi r}.$$

We can see that the octet interaction is repulsive, while the singlet interaction is attractive. This means that the quarks can form bound states only if they are in the singlet configuration.

4.5.1 Counter-terms at one loop

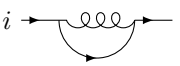
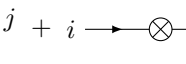
Generally, the counter-terms can be written as

$$Z_i = 1 - \frac{\alpha_\mathrm{s}(\mu^2)}{4\pi}\frac{1}{\varepsilon}C_i + \mathcal{O}(\alpha_\mathrm{s}^2)$$

where C_i are some constants and $\alpha_\mathrm{s}(\mu^2) = g_\mathrm{s}(\mu^2)/4\pi$.

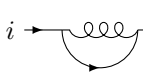
Quark self-energy

Let's start with quark self-energy:

$i\Sigma_\mathrm{R}(p) =$  $+$ 

$$= (ig)^2 T^a_{jk}T^a_{ki}\tilde{\mu}^{d-4}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}\gamma_\mu\frac{-i}{(k-p)^2+i\varepsilon} + \delta_{ij}(\overline{\delta}_2\not{p} - \overline{\delta}_m m).$$

The colour algebra gives the Casimir operator $T^a_{jk}T^a_{kl} = C_\mathrm{F}\delta_{ij} = (N^2-1)/(2N)\delta_{ij}$. The first part of the self-energy is

 $= i\delta_{ij}C_\mathrm{F}\left(-\frac{g_\mathrm{s}^2}{(4\pi)^2}\right)\int_0^1 \mathrm{d}x\,(2m - x\not{p})\left[\frac{2}{\varepsilon} + \ln\left(\frac{\tilde{\mu}^2}{(1-x)(m^2-xp^2)}\right)\right].$

Considering only the infinite terms we have

$$i\Sigma_\mathrm{R}(p) = i\delta_{ij}\left(\frac{g^2}{(4\pi)^2}C_\mathrm{F}(\not{p} - 4m)\frac{1}{\varepsilon} + (\overline{\delta}_2\not{p} - \overline{\delta}_m m) + \mathcal{O}(\varepsilon^0)\right).$$

From this, we get

$$\overline{\delta}_2 = -\frac{g^2}{16\pi^2} C_{\mathrm{F}} \frac{1}{\varepsilon} = -\frac{\alpha_{\mathrm{s}}}{4\pi} \frac{C_{\mathrm{F}}}{\varepsilon}, \quad \overline{\delta}_m = -4 \frac{\alpha_{\mathrm{s}}}{4\pi} \frac{C_{\mathrm{F}}}{\varepsilon}$$

and from $\overline{\delta}_2 = Z_2 - 1$ and $\overline{\delta}_m = Z_m Z_2 - 1$ we get

$$Z_2 = 1 - \frac{1}{\varepsilon} \frac{\alpha_{\mathrm{s}}}{4\pi} C_{\mathrm{F}}, \quad Z_m = 1 - \frac{3}{\varepsilon} \frac{\alpha_{\mathrm{s}}}{4\pi} C_{\mathrm{F}}.$$

There is a faster way to calculate only the divergent part of the self-energy. We write

$$-i\Sigma_{\mathrm{R}}(p) = (ig_{\mathrm{s}})^2 C_{\mathrm{F}} \delta_{ij} \tilde{\mu}^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \gamma^\mu \frac{i(\not{k} + m)}{k^2 - m^2 + i\varepsilon} \gamma_\mu \frac{-i}{(k-p)^2 + i\varepsilon}.$$

In $d = 4$ it holds that $\int \mathrm{d}^4 k \, k^{-n}$ diverges for $n \leq 4$. For $k \to \infty$ the propagators can be expanded as

$$\frac{1}{k^2 - m^2 + i\varepsilon} = \frac{1}{k^2} \left(1 + \frac{m^2}{k^2} + \mathcal{O}(k^{-4})\right), \qquad \frac{1}{(k-p)^2 + i\varepsilon} = \frac{1}{k^2} \left(1 + \frac{2kp}{k^2} + \mathcal{O}(k^{-3})\right).$$

Any terms of order k^{-2} in the parenthesis can be neglected. We insert the expansions in the first expression and obtain

$$\begin{aligned} i\Sigma(p) &= -g_{\mathrm{s}}^2 C_{\mathrm{F}} \delta_{ij} \tilde{\mu}^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\gamma^\mu}{k^2} (\not{k} + m) \frac{\gamma_\mu}{k^2} \left(1 + \frac{2kp}{k^2}\right) \\ &= -g_{\mathrm{s}}^2 C_{\mathrm{D}} \tilde{\mu}^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^k} \frac{1}{k^4} \left(\gamma^\mu \not{k} \gamma_\mu + m \gamma^\mu \gamma_\mu + \frac{2kp}{k^2} \gamma^\mu \not{k} \gamma_\mu + \dots\right). \end{aligned}$$

Now we use $\int \mathrm{d}^d k k_\mu = 0$ (due to oddness of the integrand), $\gamma^\mu \gamma_\mu = d$ and $\gamma^\mu \not{k} \gamma_\mu = (2-d)\not{k}$, and we get

$$\begin{aligned} -i\Sigma^{\mathrm{UV}} &= -g_{\mathrm{s}}^2 C_{\mathrm{F}} \delta_{ij} \tilde{\mu}^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^4} \left(dm^2 + (2-d) \frac{2kp}{k^2} \not{k}\right) \\ &= -g_{\mathrm{s}}^2 C_{\mathrm{F}} \delta_{ij} \tilde{\mu}^{2\epsilon} \left(dm^2 + \frac{2d}{d} 2\not{p} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^4}\right). \end{aligned}$$

Here, we used $k^\mu k^\nu = g^{\mu\nu} k^2 / d$ (only in dimensional regularization). To show this trick, write

$$\int \mathrm{d}^d k \, k^a k^\mu k^\nu = A(k^2) g^{\mu\nu},$$

which holds by Lorentz covariance. Contracting with $g_{\mu\nu}$ on both sides gives

$$\int \mathrm{d}^d k \, k^{a+2} = d \cdot A(k^2).$$

Clearly, $k^\mu k^\nu$ can be replaced with $g^{\mu\nu} k^2 / d$.

The final integral that remained is both UV and IR divergent. Those divergences can cancel each other, however, we are interested only in the UV divergence. Hence, we can add an IR regulator, without changing the UV result:

$$\begin{aligned} \tilde{\mu}^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta^2)^2} &= i\tilde{\mu}^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 + \Delta^2)^2} \\ &= i\tilde{\mu}^{2\epsilon} \frac{1}{(4\pi)^2} \frac{1}{\epsilon} \Gamma(1+\epsilon) (\Delta^2)^{-\epsilon} (4\pi)^\epsilon \\ &= \frac{i}{(4\pi)^2} \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0). \end{aligned}$$

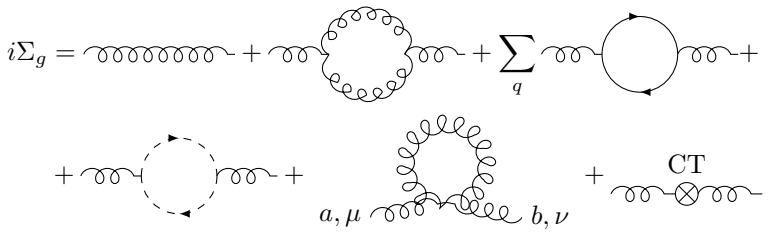
We see that UV-divergence is independent of the IR regulator Δ. The final result for the fermion propagator is

$$-i\Sigma^{\mathrm{UV}} = -i\delta_{ij}(4, -\not{p})C_{\mathrm{F}}\frac{g_{\mathrm{s}}^2}{(4\pi)^2}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0).$$

For the second approach, we had to calculate less but think more. (The law of conservation of misery applies here.)

Gluon self-energy

The gluon self energy is:



We will assume that all N_f fermions are massless. The counter-term has the form $\mathrm{CT} = -i\bar{\delta}_3(g_{\mu\nu}p^2 - p_\mu p_\nu)\delta^{ab}$. The colour algebra for the fermion loop gives

$$T^a_{ij}T^b_{ji} = \mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab} = T_{\mathrm{F}}\delta^{ab}$$

and for the gluon loop

$$f^{acd}f^{bcd} = N\delta^{ab} = C_{\mathrm{A}}\delta^{ab}.$$

The final result is (for 1-loop in UV limit)

$$\begin{aligned}-i\Sigma^{ab}_{\mu\nu}(p) =& -g_{\mathrm{s}}^2\delta^{ab}(p^2 g_{\mu\nu} - p_\mu p_\nu)\left(-\frac{5}{3}N + \frac{2}{3}N_f\right)\int\frac{\mathrm{d}^d k}{((2\pi)^d)}\frac{1}{k^4} + \\ &+ g_{\mathrm{s}}^2\delta^{ab}\frac{d-2}{d}g_{\mu\nu}\left((d-2)N - 2N_f\right)\int\frac{\mathrm{d}^d k}{((2\pi)^d)}\frac{1}{k^2} + \mathrm{CT}.\end{aligned}$$

This is not transverse (as we proved should be)! The reason is that the quadratically divergent terms in dimensional regularization cannot produce poles!! In fact, only logarithmically divergent parts create poles. This is very related to the fact that ζ function only produces pole at $z = 1$ and not at, say, $z = -1$ where it should naively be even more divergent. Quadratically divergent terms will produce a pole in two dimensions $1/(2-d)$, but this term is cancelled in our formula. To see this we add an IR regulator Δ and find

$$\frac{\mathrm{d}^d k}{k^2 + \Delta^2} = \Omega(d)\int_0^\infty \mathrm{d}k\, k^{d-3}\frac{k^2}{k^2+\Delta^2} = \Omega(d)\int_0^\infty \mathrm{d}k\, k^{d-3}\left(1 - \frac{\Delta^2}{k^2}\right).$$

The first term gives zero as it is scale-less (by Veltman's formula) and the second term is logarithmically divergent and vanishes when we send the regulator to zero, $\Delta \to 0$. The result for gluon propagator can thus be simplified

$$\begin{aligned}-i\Sigma^{ab}_{\mu\nu}(p) &= \mathrm{tr}(T^aT^b)g_{\mathrm{s}}^2\left(-2N_f\frac{d-2}{d-1} + N\frac{3d-2}{d-1}\right)(g_{\mu\nu}p^2 - p_\mu p_\nu)\,\mathrm{Bub}(p^2, d) \\ &= i\delta^{ab}\frac{\alpha_{\mathrm{s}}}{4\pi}(p^2 g_{\mu\nu} - p_\mu p_\nu)\frac{1}{\varepsilon}\left[\left(\frac{5}{3}N - \frac{2}{3}N_f\right) + \frac{31}{9}N - \frac{10}{9}N_f + \mathcal{O}(\varepsilon)\right],\end{aligned}$$

where

$$\mathrm{Bub}(p^2,d)=\tilde{\mu}^{2\varepsilon}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2(k-p)^2}=\frac{i}{16\pi^2}\left(\frac{\tilde{\mu}^2}{-p^2-i\delta}\right)^{\varepsilon}\left(\frac{1}{\epsilon}+2+\mathcal{O}(\varepsilon)\right).$$

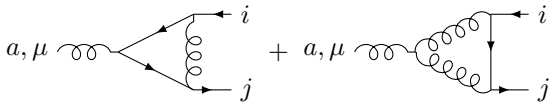
From this it follows

$$\overline{\delta}_3=-\frac{\alpha_\mathrm{s}}{4\pi}\frac{1}{\varepsilon}\left(-\frac{5}{3}N+\frac{2}{3}N_f\right)$$

and $Z_3=1+\overline{\delta}_3$.

Vertex one-loop correction

The last missing piece we need to calculate for the running of α_s is the vertex correction Z_g. We will be able to get it from $\overline{\delta}_1=Z_gZ_2\sqrt{Z_3}-1$ as we already know Z_2 and Z_3. The one-loop vertex correction is



$$=igT^a_{ij}\gamma^\mu\left(\frac{\alpha_\mathrm{s}}{4\pi}\frac{1}{\varepsilon}(C_\mathrm{A}+C_\mathrm{F})+\overline{\delta}_1+\mathcal{O}(\varepsilon^0)\right).$$

Consequently,

$$\overline{\delta}_1=\frac{\alpha_\mathrm{s}}{4\pi}\frac{1}{\varepsilon}\left(\frac{3}{2}N-\frac{1}{2N}\right)=-\frac{\alpha_\mathrm{s}}{4\pi}\frac{1}{\varepsilon}\left(C_\mathrm{S}+C_\mathrm{F}\right).$$

The vertex correction factor is

$$Z_g=1-\frac{\alpha_\mathrm{s}}{4\pi}\frac{1}{\varepsilon}\left(\frac{11}{6}N-\frac{1}{3}N_f\right)=1-\frac{\alpha_\mathrm{s}}{4\pi}\frac{1}{\varepsilon}\left(\frac{11}{6}C_\mathrm{A}-\frac{2}{3}T_\mathrm{F}N_f\right).$$

This factor relates the bare and physical coupling

$$g_0=Z_g\tilde{\mu}^{\varepsilon}g.$$

4.6 Running of the strong coupling

The expansion of α_s is

$$\alpha_\mathrm{s}^{(0)}=\alpha_\mathrm{s}\tilde{\mu}^{2\varepsilon}\left(1-\frac{\alpha_\mathrm{s}}{2\pi}\frac{\beta_0}{\varepsilon}+\dots\right)$$

where β_0 is the first coefficient of the beta function of QCD:

$$\beta_0=\frac{11}{6}C_\mathrm{A}-\frac{2}{3}T_\mathrm{F}N_f.$$

Since the bare coupling doesn't depend on the energy scale, it holds $\tilde{\mu}\partial\alpha_\mathrm{s}/\partial\tilde{\mu}=0$. It follows

$$\tilde{\mu}\frac{\partial\alpha_\mathrm{s}}{\partial\tilde{\mu}}\tilde{\mu}^{2\varepsilon}\left(1-\frac{\alpha_\mathrm{s}}{2\pi}\frac{\beta_0}{\varepsilon}\right)+\alpha_\mathrm{s}(2\varepsilon)\left(1-\frac{\alpha_\mathrm{s}}{2\pi}\frac{\beta_0}{\varepsilon}+\dots\right)-\frac{\alpha_\mathrm{s}}{2\pi}\frac{\tilde{\mu}^{2\varepsilon}}{\varepsilon}\beta_0\tilde{\mu}\frac{\partial\alpha_\mathrm{s}}{\partial\tilde{\mu}}=0$$

and

$$\tilde{\mu}\frac{\partial\alpha_\mathrm{s}}{\partial\tilde{\mu}} = -\frac{2\varepsilon\alpha_\mathrm{s}\left(1-\frac{\alpha_\mathrm{s}}{2\pi}\frac{\beta_0}{\varepsilon}\right)}{1-\frac{\alpha_\mathrm{s}}{2\pi}\frac{\beta_0}{\varepsilon}+\dots} = -2\varepsilon\alpha_\mathrm{s}\left(1+\frac{\alpha_\mathrm{s}}{2\pi}\beta_0+\mathcal{O}(\alpha_\mathrm{s}^2)\right).$$

We can now take the limit of $\varepsilon \to 0$ and rename $\tilde{\mu} \to \mu$, and we get the RGE equation for the strong coupling

$$\mu\frac{\partial\alpha_\mathrm{s}}{\partial\mu} = -\beta_0\frac{\alpha_\mathrm{s}}{\pi}.$$

To solve this equation, we integrate

$$\int_{\alpha_\mathrm{s}(\mu_i)}^{\alpha_\mathrm{s}(\mu_f)}\frac{\mathrm{d}\alpha_\mathrm{s}}{\alpha_\mathrm{s}^2} = -\frac{\beta_0}{\pi}\int_{\mu_i}^{\mu_f}\mathrm{d}\mu\,\mu^{-1}.$$

The solution is

$$\alpha_\mathrm{s}(\mu_f^2) = \frac{\alpha_\mathrm{s}(\mu_i^2)}{1+(\alpha_\mathrm{s}(\mu_i^2)/2\pi)\beta_0\ln(\mu_f^2/\mu_i^2)},$$

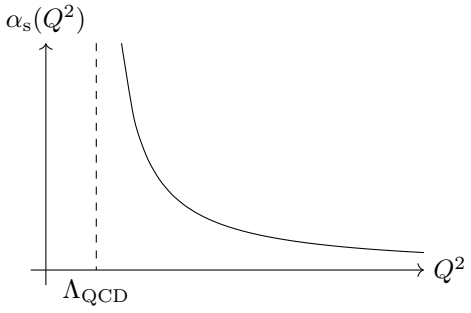
where, in the Yang-Mills theory, $\beta_0^{\mathrm{YM}} = (11/6)C_\mathrm{A} - (2/3)T_\mathrm{F}N_f$. As long as $\beta_0^{\mathrm{YM}} > 0$ the coupling runs ($\alpha_\mathrm{s}(\mu_f^2) \to 0$ as $\mu_f \to \infty$). Actually, from this expression we can also determine the beta function for QED using the process of abelianization. In the case $\mathrm{SU}(N) \to \mathrm{U}(1)$, we have $C_\mathrm{A} = 0$ and $T_\mathrm{F} = C_\mathrm{F} = 1$ and so $\beta_0^{\mathrm{QED}} = (-2/3)N_f$. In QCD, we have $N_f = 6$ flavours and $N = 3$ colours so the coefficient β_0 is negative as long as we have less than 17 flavours.

From the solution of the RGE, we see that coupling has a pole at Λ

$$\Lambda = M\exp\left(-\frac{\pi}{\beta_0\alpha_\mathrm{s}(M)}\right).$$

The value for Λ_{QCD} is between 200 MeV and 1 GeV. Using this energy as the reference scale for α_s, the running of α_s can be expressed more simply:

$$\alpha_\mathrm{s}(Q^2) = \frac{2\pi}{\beta_0\ln(Q^2/\Lambda_{\mathrm{QCD}}^2)}.$$



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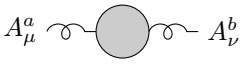
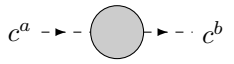
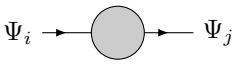
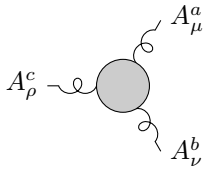
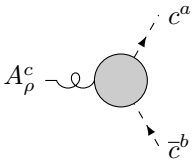
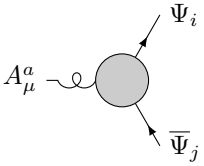
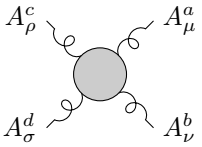
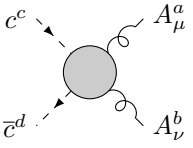
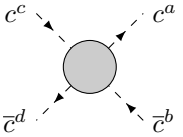
	2	0
	2	0
	1	0
	1	0
	1	0
	0	0
	0	0
	0	finite
	0	finite
anything else	finite	finite

Table 4.1: Divergent diagrams in the Yang-Mills theory.

Chapter 5

Divergences in QCD

Experimentally, we know the following facts. In quantum chromodynamics, quarks come in six flavours:

$$\begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} c \\ s \end{pmatrix}, \quad \begin{pmatrix} t \\ b \end{pmatrix},$$

up, down, charm, strange, top and bottom. They have fractional charge,

$$\text{up-type} : +\frac{2}{3}e_0, \qquad \text{down-type} : -\frac{1}{3}e_0.$$

Quarks are never observed as free particles – we only observe colourless *hadrons*. These are bound states of quarks: *mesons* are comprised of a quark and anti-quark, while *baryons* are comprised of three quarks. For example, proton is uud and neutron is udd.

An interesting particle is $\Delta^{++} = uuu$. It has electric charge of $+2$ and spin $-3/2$. That means that it has symmetric spin and quark wave function, however it is still a fermion. This means that there must be another quantum number, the *colour* in which it is antisymmetric. The symmetry group behind colour is SU(3), quarks are in the fundamental representation $\mathbf{3}$ and anti-quarks are in the anti-fundamental representation $\overline{\mathbf{3}}$. Observables are only colourless hadrons:

- mesons are $\overline{q}_i q_i$, $i \in \{r, g, b\}$, which is a colour singlet (it transforms trivially under SU(3)),
- baryons are $\varepsilon_{ijk} q_i q_j q_k$, which is also a colour singlet.

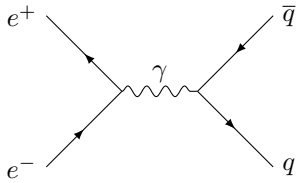
Now, we want to make sense of these observational facts using our knowledge of Yang-Mills theories. The easiest process to consider is the coupling of quarks to the photon.

5.1 $e^+e^- \to$ jets

First, we will study the annihilation of electron and positron into two gluons:

$$e^-(p_1) + e^+(p_2) \to q(p_3) + \overline{q}(p_4).$$

We will denote the cross-section of electron-positron scattering to muons with $\sigma_0 = \sigma(e^+e^- \to \mu^+\mu^-)$. The Feynman diagram is



The right vertex factor is $-ieQ_f\gamma^\mu$, where Q_f is the charge of the fermion in units of elementary charge e_0. Hence, the scattering cross-section should be

$$\sigma(e^+e^- \to q_f\overline{q}_f) = 3\sigma_0 \sum_f Q_f^2.$$

The factor of 3 comes from the fact that there are three colours of quarks. We define the *R-factor*

$$R = \frac{\sigma(e^+e^- \to q\overline{q})}{\sigma(e^-e^+ \to e^-e^+)} + \mathcal{O}(\alpha_\mathrm{s}, \alpha_\mathrm{em}) \approx 3{,}67.$$

However, the experimentally measured value is $R^\mathrm{exp} = 20{,}79 \pm 0{,}04$. However, we forgot to correct for the weak force. The same process can occur with a Z boson instead of the photon. We will know how to correct for this in few weeks, the theoretical prediction is

$$R^\mathrm{th}_{\gamma+Z} = \frac{\left| \begin{array}{l} e^+ \\ e^- \end{array} \!\!\!\

and (in Feynman gauge)

$$\Gamma^\mu = (ig)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \gamma^\rho \frac{i}{\not{k} + \not{p}_3} \gamma^\mu \frac{i}{\not{k} - \not{k}_4} \gamma^\nu \frac{-ig_{\rho\nu}}{k^2}.$$

However, due to Lorentz structure, we can write B^μ_{ij} to all orders as

$$B^\mu_{ij} = -ieQ_q \delta_{ij} F(s) \overline{u}(p_3) \gamma^\mu u(p_4).$$

There is nothing else that would have the same Lorentz structure and couldn't be expressed in this way. There could be terms such as $\overline{u}(p_3) p_3^\mu u(p_4)$, but this would be zero due to the Dirac equation. However, note that this only holds in the massless case. At tree level, it holds that $F(s) = 1$. The left side of the amplitude will be considered only at tree level (the loop level would be suppressed by α_{em} which is much smaller than α_{s}):

$$A_\mu = -ie\overline{u}(p_2) \gamma_\mu u(p_1).$$

At ℓ loops in QCD, the expression for the amplitude is

$$-i\mathcal{M}^{(\ell)} = ie^2 Q_q \delta_{ij} \frac{F^{(\ell)}(s)}{s} \overline{u}(p_2) \gamma^\mu u(p_1) \overline{u}(p_3) \gamma_\mu u(p_4).$$

To get the cross-section, we need to square the amplitude and average over the initial states and sum over the final states. The squared amplitude is

$$\left| \mathcal{M}^{(\ell)} \right|^2 = |\mathcal{M}^{(0)}|^2 + 2\mathrm{Re}\, (\mathcal{M}^{(0)*} \mathcal{M}^{(1)}) + \dots.$$

We will introduce a new symbol

$$X^{(\ell,m)} = \left(\frac{e^2 Q_q}{s} \right)^2 N M^{(\ell,m)} \sum_{\text{spins}} \overline{u}(p_2) \gamma^\mu u(p_1) \overline{u}(p_1) \gamma^\nu u(p_2) \sum_{\text{spins}} \overline{u}(p_3) \gamma_\mu u(p_4) \overline{u}(p_4) \gamma_\nu u(p_3),$$

where

$$M^{(\ell,m)}(s) = \begin{cases} \left(\frac{\alpha_{\mathrm{s}}}{2\pi}\right)^{\ell+m} 2\mathrm{Re}\, \left[F^{(\ell)}(s) F^{(m)}(s) \right], & \ell \neq m; \\ \left(\frac{\alpha_{\mathrm{s}}}{2\pi}\right)^{2\ell} \left| F^{(\ell)}(s) \right|^2 & \ell = m, \end{cases}$$

and N is the number of colours. After performing the spin-sums, we get

$$X^{(\ell,m)} = \left(\frac{e^2 Q_q}{s} \right)^2 N M^{(\ell,m)}(s) 8 \left[s_{13}^2 + s_{23}^2 - \varepsilon (s_{13} + s_{23})^2 \right]$$

where $s_{13} = (p_1 - p_3)^2 = -p_1 p_3 = t$ and $s_{23} = (p_2 - p_3)^2 = -p_2 p_3 = u$. The differential cross-section is

$$\mathrm{d}\sigma^{(\ell,m)}_{e^+e^- \to q\overline{q}} = \frac{1}{4J} \cdot \frac{1}{4} \cdot X^{(\ell,m)} \cdot \mathrm{d}\Phi_{q\overline{q}}.$$

The first term is the flux factor, the second is from the spin average and the last is the phase space:

$$\mathrm{d}\Phi_{q\overline{q}} = \frac{\mathrm{d}^{d-1}\mathbf{p}_3}{(2\pi^d) 2E_3} \frac{\mathrm{d}^{d-1}\mathbf{p}_4}{(2\pi)^d 2E_4} (2\pi)^d \delta^{(d)}(p_1 + p_2 - p_3 - p_4).$$

A very helpful relation in calculation of the phase space is

$$1 = \frac{\mathrm{d}^d Q}{(2\pi)^d} (2\pi)^d \delta^{(4)}(Q - p_1 - p_2) \mathrm{d}M^2 \delta(Q^2 - M^2).$$

With this, we can write the phase space as

$$\mathrm{d}\Phi_{q\bar{q}} = \underbrace{\mathrm{d}M^2\delta(M^2 - s_{12})}_{e^+e^-\to\gamma^*, M^2=s_{12}} \underbrace{\frac{\mathrm{d}^{d-1}\mathbf{p}_3}{(2\pi)^d 2E_3}\frac{\mathrm{d}^{d-1}\mathbf{p}_4}{(2\pi)^4}(2\pi)^d\delta^{(d)}(Q-p_3-p_4)}_{\gamma^*\to q\bar{q}}.$$

The spin sum can be rewritten to

$$\left[s_{13}^2 + s_{23}^2 - \varepsilon(s_{13}+s_{23})^2\right] = 4\underbrace{(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu})}_{e^+e^-\to\gamma^*}\underbrace{((1-\varepsilon)p_3^\mu p_4^\nu - \varepsilon p_3^\mu p_3^\nu)}_{\gamma^*\to q\bar{q}}.$$

Denoting

$$\mathrm{d}\Pi = \frac{\mathrm{d}^{d-1}\mathbf{p}_3}{(2\pi)^d 2E_3}\frac{\mathrm{d}^{d-1}\mathbf{p}_4}{(2\pi)^4}(2\pi)^d\delta^{(d)}(Q-p_3-p_4)$$

the differential cross-section is

$$\frac{16\pi^2\alpha_{\mathrm{em}}^2 Q_q^2 N}{8s}\frac{32}{s}\left[p_{1\mu}p_{2\nu}+p_{2\mu}p_{1\nu}\right]\mathrm{d}M^2\delta(M^2-s_{12})M^{(\ell,m)}(s_{12})\mathrm{d}\Pi((1-\varepsilon)p_3^\mu p_4^\nu - \varepsilon p_3^\mu p_3^\nu).$$

The calculation is rather long, but after the dust settles, we get the final result

$$\begin{aligned} I^{\mu\nu} &= \int \mathrm{d}\Pi\ ((1-\varepsilon)p_3^\mu p_4^\nu - \varepsilon p_3^\mu p_3^\nu) \\ &= I_1 Q^2 g^{\mu\nu} + I_2 Q^\mu Q^\nu \\ &= \frac{1}{4(3-2\varepsilon)}\left[Q^2 g^{\mu\nu} + 2(1-4\varepsilon+2\varepsilon^2)Q^\mu Q^\nu\right]\int \mathrm{d}\Pi \end{aligned}$$

and

$$\tilde{\mu}^{2\varepsilon}\int \mathrm{d}\Pi = \frac{1}{8\pi}\left(\frac{s}{\mu^2}\right)^{-\varepsilon}\left[\frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)e^{\varepsilon\gamma_{\mathrm{E}}}}\right]$$

where γ_{E} is the Euler-Mascheroni constant and $d = 4-2\varepsilon$. The cross-section is

$$\mathrm{d}\sigma^{(\ell,m)}_{e^+e^-\to q\bar{q}} = \underbrace{(1-\varepsilon)^2\left[\frac{\Gamma(1-\varepsilon)e^{\varepsilon\gamma_{\mathrm{E}}}}{\Gamma(2-2\varepsilon)(1-2\varepsilon/3)}\left(\frac{s}{\mu^2}\right)^{-\varepsilon}\right]}_{C(\varepsilon)}\left(\frac{4\pi\alpha_{\mathrm{em}}Q_q^2N}{3s}\right)\mathrm{d}M^2\delta(M^2-s)M^{(\ell,m)}(s).$$

When $\varepsilon\to 0$, $C(\varepsilon)\to 1$. For the leading-order cross-section we have $M^{(0,0)}(s) = |F^{(0)}(s)|^2 = 1$ and so

$$\sigma^{(0)}_{q\bar{q}} = \int \mathrm{d}\sigma^{(0)}_{e^+e^-\to q\bar{q}} = \frac{4\pi\alpha_{\mathrm{em}}^2 Q_q^2 N}{3s}$$

and

$$\sigma^{(0)}_{\mathrm{tot}} = 3\underbrace{\frac{4\pi\alpha_{\mathrm{em}}^2}{3s}}\sum_f Q_f^2.$$

Next, we want to calculate the form factor $F(s)$ at one loop. The full calculation will be done on the homework, but the result is

$$\left(\frac{\alpha_{\mathrm{s}}}{2\pi}\right)F^{(1)}(s) = ig_{\mathrm{s}}^2 C_{\mathrm{F}}\frac{d^2-7d+16}{4-d}\tilde{\mu}^{2\varepsilon}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k+p_3)^2(k-p_1)^2}.$$

The integral is actually the bubble integral (which means that the triangle integral can be reduced). In this case, we need the bubble calculated to one ε higher precision than usual, since the reduction from triangle to bubble brings one $1/(4-d)$ factor in front:

$$\mathrm{Bub}(s,d) = \tilde{\mu}^{2\varepsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2(k-p_3-p_1)^2} = \frac{i}{16\pi^2} \left(-\frac{s}{\tilde{\mu}^2}\right)^{\varepsilon} \left(\frac{1}{\varepsilon} + 2 + \left(4 - \frac{\pi^2}{12}\right)\varepsilon + \mathcal{O}(\varepsilon^2)\right).$$

The reason why we get another factor of $1/(4-d)$ is that the bubble () is UV divergent (for $k^\mu \to \infty$ we get $1/\varepsilon_{\mathrm{UV}}$). However, the triangle diagram () is quadratically IR divergent (once softly $k^\mu \to 0$ and once collinearly $k^\mu \to p_1^\mu, p_3^\mu$), $1/\varepsilon_{\mathrm{IR}}^2$. When we transition from one to the other, we need to get another factor of ε. The form factor is

$$\left(\frac{\alpha_{\mathrm{s}}}{2\pi}\right) F^{(1)}(s) = -C_{\mathrm{F}} \frac{\alpha_{\mathrm{s}}}{2\pi} \left(-\frac{s}{\tilde{\mu}^2}\right)^{-\varepsilon} \left[\frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} + 4 - \frac{\pi^2}{12} + \mathcal{O}(\varepsilon)\right].$$

For the one-loop cross-section we get

$$\sigma^{(1)}_{e^+e^-\to q\bar{q}} = -C_{\mathrm{F}} \frac{\alpha_{\mathrm{s}}}{2\pi} \sigma^{(0)}_{q\bar{q}} \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 8 - \frac{\pi^2}{6} + \mathcal{O}(\varepsilon)\right] \underbrace{\left(\frac{3(1-\varepsilon)^2\Gamma(1-\varepsilon)e^{\varepsilon\gamma_{\mathrm{E}}}}{\Gamma(2-2\varepsilon)(3-2\varepsilon)}\right) \mathrm{Re}\left(-\frac{s}{\tilde{\mu}^2} - i\delta\right)^{-\varepsilon} \left(\frac{s}{\tilde{\mu}^2}\right)^{-\varepsilon}}_{H(\varepsilon)}.$$

We used the Feynman prescription $(-1-i\delta)^{-\varepsilon} = 1+i\pi\varepsilon+\pi^2\varepsilon^2$. We can also write $e^{\varepsilon\gamma_{\mathrm{E}}}/\Gamma(1-\varepsilon) = 1-\pi^2\varepsilon^2/12+\mathcal{O}(\varepsilon)$, so we can absorb the π-dependent terms in the definition of H. The virtual cross-section is

$$\sigma^{(1,\mathrm{v})}_{e^+e^-\to q\bar{q}} = H(\varepsilon) C_{\mathrm{F}} \frac{\alpha_{\mathrm{s}}}{2\pi} \sigma^{(0)}_{q\bar{q}} \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \mathcal{O}(\varepsilon)\right).$$

This amplitude is doubly IR divergent and once UV divergent. To remove the divergences, we employ two methods. The first method is adding the counter-terms. There are three possibilities: Z_g, Z_3 and Z_2. Z_3 is not present at one loop order and the correction to the vertex Z_g only affects the higher order corrections. Hence, the only counter-term we need is

$$Z_2 = 1 - \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{C_{\mathrm{F}}}{2\varepsilon}.$$

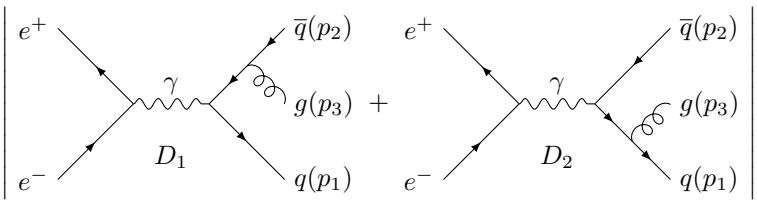
This one will regulate only the UV divergence, as we only calculated it for high momenta. We have to be careful, this counter-term only works in the massive case (in massless case, all bubbles are zero and Z_2 and Z_3 are exactly equal to 1 at all orders). The IR divergences will be regulated by making the gluon massive and sending its mass λ to 0. When we do the calculation for massive gluon, we get

$$F^{(1)}_{\lambda}(s) = C_{\mathrm{F}} \left[\frac{1}{2\varepsilon_{\mathrm{UV}}} - 2 + \frac{\pi^2}{6} - \frac{1}{2}\ln^2\left(\frac{s}{\lambda^2}\right) + 2\ln\left(\frac{s}{\lambda^2}\right) + i\pi\left(\ln\left(\frac{s}{\lambda^2}\right) - \frac{3}{2}\right)\right].$$

When we renormalize the quark field strength, we get

$$F^{\mathrm{ren}} = Z_2 \left(F^{(0)} + \frac{\alpha_{\mathrm{s}}}{2\pi} F^{(1)} + \mathcal{O}(\alpha_{\mathrm{s}}^2)\right) = 1 + \frac{\alpha_{\mathrm{s}}}{2\pi}(F^{(1)} - C_{\mathrm{F}}/(2\varepsilon)) + \mathcal{O}(\alpha_{\mathrm{s}}^2).$$

However, there are some other processes that are of the same order as the one we calculated and physically indistinguishable from the one above.

$\sigma^{\text{real}} =$ 

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For example, the first part of the amplitude (just the QCD portion) is

$$D_1 = ieQ_g T^a_{ij} \overline{u}(p_1)\gamma^\nu \frac{\not{p}_1 + \not{p}_3}{(p_1+p_3)^2}\gamma^\mu u(p_2)\varepsilon(p_3).$$

The denominator of the propagator is $(p_1+p_3)^2 = 2E_1E_2(1-\cos\theta)$. The diagrams diverge if the gluon is soft ($E_3 \to 0$) and if the gluon is collinear with one of the quark momenta. We hope that the sum of this cross-section and the previous one gives something finite.

$$\sigma^{(0,\text{r})}_{e^+e^-\to q\overline{q}} = H(\varepsilon)C_\text{F}\frac{\alpha_\text{s}}{2\pi}\sigma^{(0)}_{\text{q}\overline{\text{q}}}\left(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{19}{2}+\mathcal{O}(\varepsilon)\right).$$

Finally, we arrived to the actual physical amplitude in the NLO:

$$\sigma^{\text{NLO}}_{e^+e^-\to q\overline{q}} = \sigma^{(1,\text{v})}_{e^+e^-\to q\overline{q}} + \sigma^{(1,\text{r})}_{e^+e^-\to q\overline{q}} = N\sum_q Q_q^2\sigma_0\left(\frac{3}{2}\frac{\alpha_\text{s}}{2\pi}C_\text{F}+\mathcal{O}(\alpha_\text{s}^2)\right).$$

The corrected R-ratio for γ propagator is

$$R^{\text{NLO}}_\gamma = 3\sum_q Q_q^2\left(1+\frac{\alpha_\text{s}}{\pi}+\mathcal{O}(\alpha_\text{s}^2)\right).$$

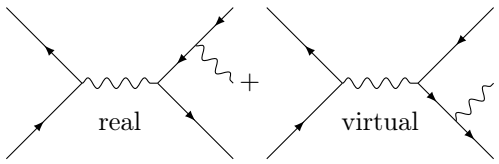
If we also include the Z-boson exchange, we get

$$R^{\text{NLO}}_{\gamma+Z} = R^{\text{LO}}_{\gamma+Z}\left(1+\frac{\alpha_\text{s}}{\pi}+\mathcal{O}(\alpha_\text{s}^2)\right).$$

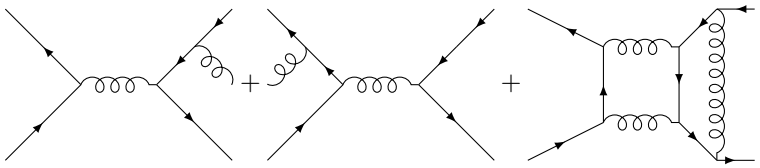
Using this, we can measure the value of α_s. The experimentally measured value for R-ratio at the mass of the Z boson is $R^{\text{exp}}_{\gamma+Z} = 20{,}79 \pm 0{,}04$. The leading order prediction is $R^{\text{LO}}_{\gamma+Z} = 20{,}09$. This means that the correction should be $\alpha_\text{s}/(2\pi) \approx 0{,}035$ which gives $\alpha_\text{s} \approx 1/10$. A more careful analysis gives the value $\alpha_\text{s}(M_Z^2) = 0{,}118$.

5.2 Infrared divergences

There are two important theorems in quantum Yang-Mills theories. The *Bloch-Nordsieck (NB) theorem* applies to massive Abelian gauge theories (QED). It states that all (soft) IR divergences cancel if we sum over virtual and real processes – summing over degenerate final states.

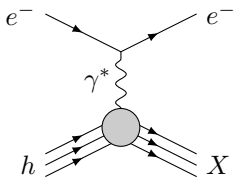


The *KLN theorem* (named after Kinoshita, Lee and Nauenberg) applies to massless non-Abelian theories. It states that all IR (soft and collinear) if we sum over all degenerate *initial and final* states.

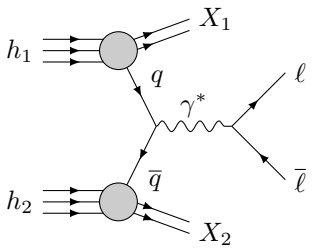


This is very useless, as in practice we only deal with 2 particle initial states, and we cannot add soft gluons to the initial states. In practice, quantum field theorists typically just ignore the degenerate initial states and only sum over the final states. In our example it worked, because the initial state only contained electrons, which don't couple to gluons (in the first order). Particularly, there are two important processes where the KLN theorem is a bit stronger.

The first is the *deep inelastic scattering* (DIS). In this process, we scatter a high-energy electron off a hadron h (typically a proton). The electron scatters off a quark inside the hadron.



Another option is to scatter two hadrons (h_1, h_2) off each other. In practice, only one quark from each hadron will interact. If they are antiparticles, they annihilate to a photon and if the photon produces a lepton–anti-lepton pair $(\ell\overline{\ell})$, the process is called the *Drell-Yan process* (DY).



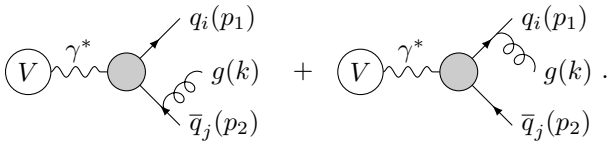
For DIS and DY we can prove the *collinear factorization*. For DIS, we can write the differential cross-section as

$$\mathrm{d}\sigma_{he^-\to e^-X} = \sum_q \int \mathrm{d}z\, f_q^h(z) \mathrm{d}\hat{\sigma}_{qe^-\to e^-X}(z).$$

The function $f_q^h(z)$ is called the *parton distribution function* (PDF) and z is the energy fraction of the quark q in the hadron h. Initially, the PDF is bare, and after renormalization the IR divergences are packed into the PDF, which we can measure at a certain energy $f_q^h(z;\mu)$.

5.2.1 Soft divergences

We want to study infrared divergences in general. Consider a real gluon emission process



There is also a virtual gluon emission process (where the gluon flies directly from the blob), but we are not interested in that since a gluon that is created off-shell ($k^2 \neq 0$) will never produce infrared divergences. The differential cross-section for the real emission is

$$\mathrm{d}\sigma_{\mathrm{R}} \propto \int \mathrm{d}g_k F(1,2,k;V)$$

where $F(1,2,k;V)$ is the scattering amplitude for initial QED state V with quarks 1 and 2 and the gluon k in the final state. The integration measure is

$$\mathrm{d}g_k = \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2E_k}\theta(E_{\max} - E_k)$$

and the integrand is

$$F(1,2,k;V) = N_A \mathrm{d}\Pi_{q\bar{q}} \|\mathcal{M}(1,2,k;V)\|^2.$$

The number N_A is the averaging factor and $\mathrm{d}\Pi_{q\bar{q}}$ is the phase space measure for the quark-antiquark pair

$$\mathrm{d}\Pi_{q\bar{q}} = \frac{\mathrm{d}^{d-1}p_1}{(2\pi)^{d-1}2E_1}\,\frac{\mathrm{d}^{d-1}p_2}{(2\pi)^{d-1}2E_2}\,(2\pi)^d\delta^{(2)}(p_V - p_1 - p_2 - k).$$

The *soft divergences* arise from the region where the gluon momentum k is small, $k^\mu \to 0$. For example, the first diagram is proportional to

$$\lim_{k\to 0}\frac{\not{p}+\not{k}+m_q}{(p+k)^2-m_q^2} \sim \lim_{k\to 0}\frac{\not{p}_1+m_q}{2pk_1+k^2+(p^2-m_q^2)} = \lim_{k\to 0}\frac{\not{p}_1+m_q}{2kp_1} \to \infty.$$

For simplicity, we will assume that the quark is massless, $m_q = 0$, but this doesn't change any physics. The scattering amplitude is

$$\mathcal{M}^\rho(1,2,k;V) = ig_{\mathrm{s}}T^a_{ij}\overline{u}(p_1)\left[\gamma^\mu\frac{i(\not{p}_1+\not{k})}{(k+p_1)^2}\tilde{\mathcal{M}}^\rho_{q\bar{q}\gamma} - \tilde{\mathcal{M}}^\rho_{q\bar{q}\gamma}\frac{i(\not{p}_2+\not{k})}{(k+p_2)^2}\gamma^\mu\right]u(p_2)\varepsilon_\mu(k).$$

In the soft limit, we replace

$$\frac{\not{k}+\not{p}}{(k+p)^2} \to \frac{\not{p}}{2kp}$$

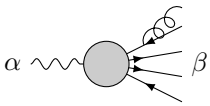
and obtain

$$\begin{aligned}\mathcal{M}^\rho(1,2,k;V) &= -g_{\mathrm{s}}T^a_{ij}\overline{u}(p_1)\left[\gamma^\mu\frac{\not{\varepsilon}(k)\not{p}_1}{2kp_1}\tilde{\mathcal{M}}^\rho_{q\bar{q}\gamma} - \tilde{\mathcal{M}}^\rho_{q\bar{q}\gamma}\frac{\not{p}_2\not{\varepsilon}(k)}{2kp_2}\gamma^\mu\right]u(p_2)\\ &= -gT^a_{ij}\left[\frac{p_1\varepsilon(k)}{p_1k} - \frac{p_2\varepsilon(k)}{p_2k}\right]\overline{u}_i(p_1)\tilde{\mathcal{M}}^\rho_{q\bar{q}\gamma}u_j(p_2).\end{aligned}$$

In the last term, we used $\not{p}\not{k} = 2pk - \not{k}\not{p}$. The term with the swapped slashed momenta is cancelled by the Dirac equation. Note that the term after the square brackets exactly corresponds to the amplitude $\gamma^* \to q\bar{q}$ and the factor in the brackets is very universal, independent of the process. It is called the *soft universal factor* or Eikonal factor. We can write the Eikonal factor as an *Eikonal current*:

$$\frac{p_1\varepsilon(k)}{p_1 k} - \frac{p_2\varepsilon(k)}{p_2 k} = \varepsilon^\mu(k) J_\mu(k).$$

It fulfils the Ward identity, $J^\mu(k)k_\mu = 0$. It generalizes to any number of outgoing quark-antiquark pairs and gluons (an initial state α and final state β):



$$= \mathcal{M}^\mu_{\beta\alpha}\varepsilon_\mu(k) \xrightarrow{k^\mu \to 0} g\sum_n \frac{\varepsilon(k)p_n}{kp_n}\mathbf{T}_n\mathcal{M}_{\beta\alpha}$$

where

$$\mathbf{T}_n = \begin{cases} +T^a_{ij} & \text{if } n \text{ is a quark,} \\ -T^a_{ij} & \text{if } n \text{ is an antiquark,} \\ +f^{abc} & \text{if } n \text{ is a gluon.} \end{cases}$$

In QED, this equation is modified slightly:

$$\sum_n \left(e_n \frac{\varepsilon(k)p_n}{kp_n}\lambda_n\right)\mathcal{M}_{\beta\alpha}$$

where $\lambda_n = \pm 1$ ($+$ corresponds to a particle and $-$ to an antiparticle). Consider a physically (transverse) polarized polarization vector $\varepsilon^\mu_\pm(k)$. Under Lorentz transformations it changes to $C_+\varepsilon^\mu_+(k) + C_-\varepsilon^\mu_-(k) + Ck^\mu$. The Ward identity says $\mathcal{M}^\mu_{\alpha\beta}k_\mu = 0$. If we use the soft factorization ($\varepsilon^\mu(k) \to k^\mu$), in the soft limit we get

$$\mathcal{M}_{\beta\alpha\sum_n e_n\lambda_n} = 0$$

in QED (this corresponds to charge conservation) and

$$\mathcal{M}_{\beta\alpha}\sum_n \mathbf{T}_n = 0$$

in QCD. This is the colour conservation,

$$\sum_n \mathbf{T}_n = 0.$$

We can generalize this discussion to particles with spin 2, for example the gravitons. The Ward identity says $k_\mu\mathcal{M}^{\mu\nu}_{\beta\alpha} = k_\nu\mathcal{M}^{\mu\nu}_{\beta\alpha} = 0$. In the soft limit, we have

$$\mathcal{M}^{\mu\nu}_{\beta\alpha}\varepsilon_{\mu\nu}(k) \xrightarrow{k^\mu \to 0} \sum_n \xi_n \left(\frac{p^\mu_n p^\nu_n \varepsilon_{\mu\nu}(k)\lambda_n}{p_n k}\right)\mathcal{M}_{\beta\alpha}$$

where ξ_n is the "particle charge" and $\lambda_n = \pm 1$ (1 for incoming and -1 for outgoing). The Ward identity implies

$$\sum_n \xi_n p^\mu_n \lambda_n = 0.$$

If ξ_n are generic, this only has a trivial solution, $p_n^\mu = 0$. This means that the spin-2 particle does not interact with other particles. If, however, ξ_n is universal for all particles, the equation implies the momentum conservation.

To get the cross-section in the soft limit, we write

$$\sum_{\substack{\text{spins}\\ colors}} |\mathcal{M}(1,2,k,V)|^2 = g^2 T^a_{ij} T^a_{ji} \sum_\lambda \left(\frac{p_1 \varepsilon^\lambda(k)}{p_1 k} - \frac{p_2 \varepsilon^\lambda(k)}{p_2 k} \right) \sum_{\text{sums}} |\tilde{\mathcal{M}}_{q\bar{q}\gamma}|.$$

The sum over the polarizations of ε is

$$\sum_\lambda \varepsilon^\lambda_\mu(k)(\varepsilon^\lambda_\nu(k))^* = -g_{\mu\nu} + \frac{k_\mu n_\nu - k_\nu n_\mu}{kn}$$

in the axial gauge. However, the second term gives zero when contracted with the Eikonal current (thanks to Ward, again). The colour factor is $T^a_{ij} T^a_{ji} = NC_\mathrm{F}$. The factor N will be absorbed into $\overline{|\mathcal{M}_{q\bar{q}\gamma}|^2}$. We contract the polarization sum with the Eikonal current and obtain

$$\sum_{\substack{\text{spins}\\ colors}} |\mathcal{M}(1,2,k,V)|^2 = g^2 C_\mathrm{F} \frac{2p_1p_2}{(p_1k)(p_2k)} \overline{|\mathcal{M}_{q\bar{q}\gamma}|^2}.$$

5.2.2 Collinear divergences

The first diagram of the two we are considering is proportional to

$$D_1 = \frac{i(\not{p}_1 - \not{k})}{2kp_1} = \frac{i(\not{p}_1 + \not{k})}{2E_1E_2(1-\cos\theta)} \sim \frac{1}{\theta^2}.$$

The angle θ is the angle between the quark and the gluon. This divergence is called the *collinear divergence* and only exists in the massless case (hence, it is sometimes called the *mass divergence*). In the massive case, the denominator is

$$2E_1E_2(1 - |\mathbf{p}_1|/E_1\cos\theta) \xrightarrow{\theta\to 0} 2E_2(E_1 - |\mathbf{p}_1|) \neq 0.$$

In the collinear case, we can perform the *Sudakov decomposition*, where the two quarks fly in exactly the opposite directions. We denote the direction of p_1 as *collinear* and the direction of p_2 as *anti-collinear*. The momenta can be chosen in a convenient way

$$p_1 = (E_1, 0, 0, E_1), \quad p_2 = (E_2, 0, 0, -E_2), \quad k_\perp = (0, k_x, k_y, 0).$$

As it clearly holds $p_1k_\perp = p_2k_\perp = 0$, we can write

$$k^\mu = xp_1^\mu + \beta p_2^\mu + k_\perp^\mu.$$

Since the gluon is on shell, it holds

$$0 = k^2 = x\beta s + k_\perp^2.$$

Hence, $k_\perp^2 = -x\beta s < 0$ and $|\mathbf{k}_\perp|^2 = -k_\perp^2 > 0$. The denominator of the scattering amplitude is $2kp_1 = \beta p_1p_2 \sim \theta^2$. Therefore, $\beta \propto \theta^2$ and $|\mathbf{k}_\perp| \propto \theta$. Both terms go to 0 as θ goes to zero, so in

this limit only x-contribution remains. The amplitude is

$$\mathcal{M}^{\rho}(1,2,k;V) = igT^a_{ij}\overline{u}(p_1)\left[\gamma^{\mu}\frac{i((1+x)\not{p}_1+\beta\not{p}_2+\not{k}_\perp)}{\beta s}\tilde{\mathcal{M}}^{\rho}_{q\bar{q}\gamma}+\right.$$
$$\left.-\tilde{\mathcal{M}}^{\rho}_{q\bar{q}\gamma}\frac{i(x\not{p}_1+(1+\beta)\not{p}_2+\not{k}_\perp)}{\beta s}\gamma^{\mu}\right]u(p_2)\varepsilon_{\mu}(k).$$

Superficially, the differential cross-section is

$$\mathrm{d}\sigma_{\mathrm{R}} \propto \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2E_{\mathbf{k}}}|\mathcal{M}|^2 \propto \frac{E_{\mathbf{k}}\mathrm{d}E_{\mathbf{k}}}{E_b\mathbf{k}}\mathrm{d}\theta\,\sin\theta|\mathcal{M}|^2 \xrightarrow{\theta\to 0} \theta\mathrm{d}\theta\,\frac{1}{\theta^4}.$$

This is a severe divergence, however some divergences actually cancel, and it only remains a logarithmic divergence. To see this, denote

$$\begin{aligned}\mathrm{Div} &= igT^a_{ij}\overline{u}(p_1)\gamma^{\mu}\frac{i((1+x)\not{p}_1+\not{k}_\perp)}{\beta s}\tilde{\mathcal{M}}^{\rho}_{q\bar{q}\gamma}u(p_2)\varepsilon_{\mu}(k)\\ &= igT^a_{ij}\overline{u}(p_1)\frac{(1+x)2\varepsilon p_1+\not{\varepsilon}\not{k}_\perp}{\beta s}\tilde{\mathcal{M}}^{\rho}_{q\bar{q}\gamma}u(p_2)\\ &= -gT^a_{ij}\overline{u}(p_1)\left(-\frac{2(1+x)}{x}\frac{\varepsilon k_\perp}{\beta s}+\frac{\not{\varepsilon}\not{k}_\perp}{\beta s}\right)\tilde{\mathcal{M}}^{\rho}_{q\bar{q}\gamma}.\end{aligned}$$

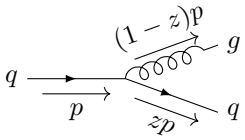
In the last step, we used that the gluon is physical, so $k\varepsilon = 0$ and consequently

$$2\varepsilon p_1 = -\frac{2}{x}(\beta\varepsilon p_2+\varepsilon k_\perp) \approx -\frac{2\varepsilon k_\perp}{x}.$$

When we sum over polarizations and colours, we get

$$\sum_{\substack{\mathrm{spins}\\ colors}}|\mathcal{M}(1,2,k,V)|^2 = 2g^2\underbrace{\left[C_{\mathrm{F}}\left(\frac{1+(1+x)^2}{x(1+x)}-\varepsilon\frac{x}{1+x}\right)\right]}_{P_{qq(x)}}\frac{1}{(k+p_1)^2}|\mathcal{M}(k+1,2;V)|^2$$

where $P_{qq}(x)$ is the *quark splitting function* and $\varepsilon = (4-d)/2$.



We can reparametrize the splitting function in terms of the variable $z = x/(1+x)$, which is the energy fraction of the daughter quark in the gluon direction. The splitting function is then

$$P_{qq}(z) = C_{\mathrm{F}}\left(\frac{1+z^2}{1-z}-\varepsilon(1-z)\right).$$

There are other splitting functions, for example the gluon and quark-gluon splitting functions, P_{gg} and P_{gq}.

$$
\begin{aligned}
P_{qg}(z) &= C_{\mathrm{F}} \left(\frac{1 + (1+z)^2}{z} - \varepsilon z \right), \\
\overline{P}_{qq}(z) &= 2C_{\mathrm{A}} \left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right), \\
\overline{P}_{q\overline{q}}(z) &= T_{\mathrm{F}} \left(1 - \frac{2z(1-z)}{1-\varepsilon} \right).
\end{aligned}
$$

The last two can actually change the colours, so the given expression refer to colour-averaged splitting functions.

Chapter 6

Spontaneous symmetry breaking

There are two main ways to break a symmetry:

- *Explicit symmetry breaking* (ESB) happens when a non-symmetric term is added to the Lagrangian $\mathcal{L} = \mathcal{L}_{\text{sym}} + \mathcal{L}_{\text{asym}}$. If the term $\mathcal{L}_{\text{asym}}$ is small, we can do perturbation theory. Here, we are not interested in this case.
- *Spontaneous symmetry breaking* (SSB) happens when the vacuum state is not invariant under the symmetry of the Lagrangian.

6.1 Discrete symmetry breaking

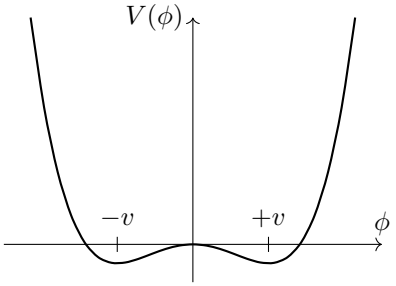
Assume, we are dealing with Lagrangian with a single real scalar field $\phi(x)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

This Lagrangian is invariant under the $\mathbb{Z}_2$ symmetry $\phi \to -\phi$. The potential term $V(\phi) = \lambda\phi^4/4!$ has a minimum at $\phi = 0$. This minimum still satisfies the $\mathbb{Z}_2$ symmetry. We will modify the Lagrangian slightly by replacing the mass with $m^2 \to -\mu^2$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \underbrace{\frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4}_{V(\phi)}.$$

This Lagrangian represents a massless particle in a double-well potential $V(\phi)$.



The minima of the potential are found by taking the derivative of the potential and setting it to zero:

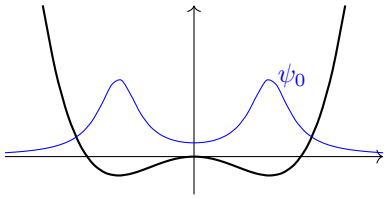
$$\frac{\mathrm{d}V}{\mathrm{d}\phi} = -\mu^2\phi + \frac{\lambda}{6}\phi^3 = 0, \quad \Longrightarrow \quad \phi = \pm\sqrt{\frac{6\mu^2}{\lambda}} \equiv \pm v.$$

Since there are multiple vacuum solutions, we choose (randomly) one of them, for example $\phi = v$. The expansion of the field around the minimum value is $\phi(x) = v + \sigma(x)$ and the vacuum state corresponds to $\langle\sigma\rangle = 0$. Expanding the Lagrangian in terms of the new field σ, we obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \underbrace{\frac{1}{2}\mu^2v^2 - \frac{\lambda}{4!}v^4}_{\text{constant}} + \underbrace{\left(\mu^2v - \frac{\lambda}{6}v^3\right)}_{0}\sigma + \underbrace{\left(\frac{1}{2}\mu^2 - \frac{\lambda}{4}v^2\right)}_{-\mu^2}\sigma^2 - \underbrace{\frac{\lambda}{3!}v\sigma^3 - \frac{\lambda^4}{4!}\sigma^4}_{\text{interaction}}.$$

The new field σ has acquired a mass $m_\sigma^2 = 2\mu^2$ and 3- and 4-point interaction.

In 1-dimensional quantum mechanics, the corresponding problem would be a potential with a double-well potential $V(x) = -\mu^2x^2/2 + \lambda x^4/4!$. The ground states are the two minima of the potential, $x = \pm v$. The wave function is a superposition of the two states, $\psi_0(x) = \frac{1}{\sqrt{2}}(\psi_+(x) + \psi_-(x))$, where $\psi_\pm(x)$ are the wave functions in the minima. If we put a particle in one of the minima, it will tunnel to the other minimum.



This means that spontaneous symmetry breaking doesn't happen in quantum mechanics. However, in QFT there are infinitely many degrees of freedom, so the tunnelling probability goes to zero and SSB still occurs. There is a *Mermin-Wagner-Coleman theorem* that says

- A discrete symmetry can only be spontaneously broken in $D > 1$ dimensions.
- A continuous symmetry can only be spontaneously broken in $D > 2$ dimensions (this part was proven by Coleman).

6.2 Continuous symmetry breaking

The most studied example of continuous symmetry breaking is the *linear sigma model*. Let $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$ be N real scalar fields and let the Lagrangian be

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\boldsymbol{\phi})(\partial^\mu\boldsymbol{\phi}) + \frac{\mu^2}{2}\boldsymbol{\phi}\cdot\boldsymbol{\phi} - \frac{\lambda}{4}(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2.$$

This Lagrangian is symmetric under O(N) group, which rotates the fields into each other $\phi_i = R_{ij}\phi_j$. The potential is

$$V(\boldsymbol{\phi}) = -\frac{\mu^2}{2}\boldsymbol{\phi}\cdot\boldsymbol{\phi} + \frac{\lambda}{4}(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2$$

with a vacuum value

$$|\boldsymbol{\phi}_{\text{vac}}| = v = \sqrt{\frac{\mu^2}{\lambda}}.$$

For the vacuum state, we make a choice $\boldsymbol{\phi}_{\text{vac}} = (0, \dots, 0, v)$ and parametrize

$$\boldsymbol{\phi}(x) = (\pi_1(x), \dots, \pi_{N-1}(x), v + \sigma(x)) = (\boldsymbol{\pi}(x), v + \sigma(x)).$$

The Lagrangian can be expanded in terms of the new field σ:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\pi})(\partial^\mu \boldsymbol{\pi}) + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\mu^2 v(\boldsymbol{\pi}^2 + (v+\sigma)^2) - \frac{\lambda}{4}(\boldsymbol{\pi}^2 + (v+\sigma)^2)^2.$$

We can ignore the constant terms. We find that there are no linear terms in σ or $\boldsymbol{\pi}$ and there is a quadratic term only for σ. This means that the fields $\boldsymbol{\pi}$ are *massless*. The full expansion of the Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\pi})(\partial^\mu \boldsymbol{\pi}) + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu\boldsymbol{\pi}^2\sigma - \frac{\lambda}{4}(\sigma^4 + (\boldsymbol{\pi}^2)^2) - \frac{\lambda}{2}\boldsymbol{\pi}^2\sigma^2.$$

This Lagrangian is still symmetric under the $\mathrm{O}(N-1)$. The original symmetry had $\dim(\mathrm{O}(N)) = N(N-1)/2$ dimensions and the final symmetry has $\dim(\mathrm{O}(N-1)) = (N-1)(N-2)/2$ dimensions. The difference is $N-1$, exactly the number of massless fields $\boldsymbol{\pi}$. The newly created massless fields $\boldsymbol{\pi}$ are called *Goldstone bosons*. The relationship between the number of Goldstone bosons and the number of broken symmetries is not coincidental.

To make a more general statement, let's consider more general setting. Let $\mathcal{L} = (\partial_\mu \boldsymbol{\phi})^2/2 - V(\boldsymbol{\phi})$ where $\boldsymbol{\phi} = (\phi_1, \dots, \phi_N)$ are real scalar fields.

1. Assume that $V(\boldsymbol{\phi})$ is invariant under some group G transformation, where $\dim G = n$. This means $V(g\boldsymbol{\phi}) = V(\boldsymbol{\phi})$ for all $g \in G$.
2. The ground state of the potential is $\boldsymbol{\phi}_0$ such that $\partial V/\partial\phi_i|_{\boldsymbol{\phi}_0} = 0$.
3. The field $\boldsymbol{\phi}$ is expanded around the vacuum state $\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \delta\boldsymbol{\phi}$. We can write

$$V(\boldsymbol{\phi}_0 + \delta\boldsymbol{\phi}) = V(\boldsymbol{\phi}_0) + \frac{1}{2}\frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\bigg|_{\boldsymbol{\phi}_0} \delta\phi_i\, \delta\phi_j + \dots = V(\boldsymbol{\phi}_0) + M_{ij}^2 \delta\phi_i \delta\phi_j + \dots$$

   We introduced the mass matrix M_{ij} (a Hessian) which has to be semi-positive definite (non-negative eigenvalues) since we are in the minimum of the potential.
4. The field $\boldsymbol{\phi}_0$ is invariant under a subgroup $H \leq G$ with dimension $\dim H = n - m$. This means that there were m broken symmetries.

How many zero eigenvalues does the mass matrix M_{ij} have? To determine this, we expand the group element $g \in G$ infinitesimally using the group generators Δ^a

$$g = \mathbb{1} + \alpha^a \Delta^a$$

where α^a are (small) parameters. Writing the invariance of the potential under the group transformation, we have

$$V(\boldsymbol{\phi}) = V(g\boldsymbol{\phi}) = V(\boldsymbol{\phi}) + \alpha^a \frac{\partial V}{\partial\phi_i}\Delta^a_{ij}\phi_j + \mathcal{O}(\alpha^2).$$

It follows immediately that

$$\frac{\partial V}{\partial\phi_i}\Delta^a_{ij}\phi_j = 0.$$

Differentiating this equation with respect to ϕ_k gives

$$0 = \frac{\partial}{\partial \phi_k}\left(\frac{\partial V}{\partial \phi_i}\Delta^a_{ij}\phi_j\right) = \frac{\partial^2 V}{\partial \phi_k \partial \phi_k}\Delta^a_{ij}\phi_j + \frac{\partial V}{\partial \phi_i}\Delta^a_{ik} = 0.$$

Next, we evaluate the last expression at $\boldsymbol{\phi}_0$. The right term vanishes (because $\boldsymbol{\phi}_0$ is the minimum of the potential) and the left term becomes proportional to the mass matrix:

$$M^2_{ki}\Delta^a_{ij}\phi_{0,j} = 0.$$

We have two options.

1. $\boldsymbol{\phi}_0$ is invariant under group G. There is no symmetry breaking.
2. $\boldsymbol{\phi}_0$ is invariant under a subgroup $H < G$ (of the dimension $n - m$). This means that

   $$\Delta^b_{ij}\phi_j = \xi^b_i \neq 0, \qquad b \in \{1, \ldots, m\}$$

   and consequently $M_{ki}\xi^b_i = 0$. Hence, there are m zero eigenvalues of the mass matrix which implies m massless particles.

According to Noether's theorem, every conserved symmetry corresponds to a conserved current J^μ. Classically, $\partial_\mu J^\mu = 0$ and this also holds in QFT (if we exclude anomalies). With every conserved current comes a conserved charge

$$Q = \int \mathrm{d}^3\mathbf{x}\, J^0(x) = \int \mathrm{d}^3\mathbf{x} \sum_i \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}\frac{\partial \phi_i}{\partial \alpha}.$$

In Hamiltonian picture

$$[Q, \phi_j(\mathbf{y})] = \sum_i \int \mathrm{d}^3\mathbf{x}\, \frac{\delta \phi}{\delta \alpha}[\pi_i(\mathbf{x}), \phi_j(\mathbf{y})] = -i\frac{\delta \phi_j(\mathbf{y})}{\delta \alpha}$$

where Q is the generator of the symmetry transformation $[Q, H] = 0$, $\partial_t Q = 0$ and $\pi_i(\mathbf{x}) = \delta\mathcal{L}/\delta\dot{\phi}_1(\mathbf{x})$ is the conjugate momentum. The original vacuum $|\Omega\rangle_{\mathrm{sym}}$ is not stable, since $Q\,|\Omega\rangle_{\mathrm{sym}} = 0$. The real (true) vacuum state $|\Omega\rangle_{\mathrm{true}}$ is created after SSB, and it is "charged" in a sense that

$$H\,|\Omega\rangle_{\mathrm{true}} = E_0\,|\Omega\rangle_{\mathrm{true}}\,, \quad HQ\,|\Omega\rangle_{\mathrm{true}} = E_0 Q\,|\Omega\rangle_{\mathrm{true}}\,.$$

The state of the particle π is

$$|\pi(\mathbf{p})\rangle = -\frac{2i}{F}\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}} J_0(\mathbf{x})\,|\Omega\rangle_{\mathrm{true}}$$

where F is a normalization factor with a unit of energy. If $\mathbf{p} = 0$, then $|\pi(0)\rangle = -(2i/F)Q\,|\Omega\rangle_{\mathrm{true}}$ which is a state with energy E_0. For $\mathbf{p} \neq 0$, we have

$$\hat{P}^i\,|\pi(\mathbf{p})\rangle = -\frac{2i}{F}\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{P}}[\hat{P}^i, J_0(\mathbf{x})]\,|\Omega\rangle_{\mathrm{true}} = p^i\,|\pi(\mathbf{p})\rangle\,.$$

This state has energy E_π, where $E_\pi = E_0 + E(\mathbf{p}) > 0$. In the limit $\mathbf{p} \to 0$, the energy of the particle $E(\mathbf{p})$ needs to go to zero. This means that this describes a massless particle, i.e., a Goldstone boson.

$$\langle\pi(\mathbf{q})|\pi(\mathbf{p})\rangle = 2\omega_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}).$$

We get

$$\omega_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) = -\frac{i}{F}\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\,\langle\pi(\mathbf{q})|J_0(\mathbf{x})|\Omega\rangle_{\text{true}}\,.$$

The Fourier transformation of this equation gives

$$\omega_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{y}} = -\frac{i}{F}\int \mathrm{d}^3\mathbf{x}\,\delta^{(3)}(\mathbf{x}-\mathbf{y})\,\langle\pi(\mathbf{q})|J_0(\mathbf{x})|\Omega\rangle_{\text{true}} = -\frac{i}{F}\,\langle\pi(\mathbf{q})|J_0(\mathbf{y})|\Omega\rangle_{\text{true}}\,.$$

Finally, we can write the amplitude of the current creating a Goldstone boson as (it can be done also for μ)

$$\langle\pi(q)|J_\mu(y)|\Omega\rangle_{\text{true}} = iq_\mu F e^{iqy}.$$

6.3 Higgs mechanism

We will discuss the Higgs mechanism which describes the spontaneous symmetry breaking of a gauge symmetry. We will treat it in two steps. First, we will consider an Abelian gauge theory, and then we will generalize it to a non-Abelian gauge theory.

6.3.1 Abelian Higgs mechanism

We are working in the only Abelian gauge theory, the U(1), with one complex scalar field ϕ. The Lagrangian is

$$\mathcal{L} = (\mathrm{D}_\mu\phi)^*(\mathrm{D}^\mu\phi) - V(\phi^*\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

The complex scalar field is conveniently split into two real fields $\phi = (\phi_1+i\phi_2)/\sqrt{2}$. The covariant derivative and potential are

$$\mathrm{D}_\mu = \partial_\mu + ieA_\mu, \quad V(\phi^*\phi) = -\mu^2|\phi|^2 + \frac{\lambda}{4}|\phi|^4.$$

To get the vacuum state, it is most convenient to reparametrize the field with polar coordinates $\phi(x) = \rho(x)e^{i\theta(x)}$. The minimum of the potential is given by

$$\frac{\partial V}{\partial\rho} = 0, \quad \Longrightarrow \quad \rho = \pm\sqrt{\frac{2\mu^2}{\lambda}} \equiv \pm v.$$

The covariant derivative in the polar coordinates is

$$\mathrm{D}_\mu\phi = \mathrm{D}_\mu(\rho e^{i\theta}) = e^{i\theta}\left(\partial_\mu\rho + i(\partial_\mu\theta + eA_\mu)\rho\right), \qquad (\mathrm{D}_\mu\phi)^*(\mathrm{D}_\mu\phi) = \partial^\rho_\mu\partial^\mu\rho + \rho^2(\partial_\mu\theta + eA_\mu)^2.$$

The symmetry U(1) is most clearly expressed in the polar parametrization

$$\phi(x) \to \phi'(x) = \rho e^{i\theta+\alpha}.$$

The Lagrangian is currently

$$\mathcal{L} = (\partial_\mu\rho)^2 + \rho^2(\partial_\mu\theta + eA_\mu)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\rho).$$

We don't like that the kinetic term for ρ doesn't have a factor of $1/2$, and we also don't like that θ field has no dimension. To fix this, we make another reparametrization which is essentially an expansion around the vacuum state:

$$\phi(x) = \left(v + \frac{\sigma(x)}{\sqrt{2}}\right) e^{i\pi(x)/F_\pi}.$$

The Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{v^2}{F_\pi^2}(\partial_\mu\pi)^2 + \frac{1}{2F_\pi^2}\sigma^2(\partial_\mu\pi)^2 + \frac{\sqrt{2}v}{F_\pi^2}\sigma(\partial_\mu\pi)^2 - \frac{1}{2}(\sqrt{2}\mu)^2\sigma^2 - \frac{1}{2}\sqrt{\lambda\mu^2}\sigma^3 - \frac{\lambda}{16}\sigma^4 \\ &= \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 - \frac{1}{2}(2\mu^2)\sigma^2 + \text{interactions}.\end{aligned}$$

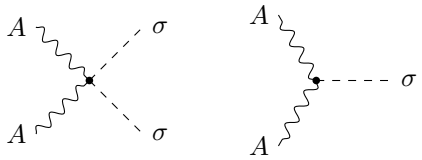
In order for the kinetic term of the π field to have a factor of $1/2$, we need to set $F_\pi = \sqrt{2}v = 2\mu/\sqrt{\lambda}$. The π field is a massless Goldstone boson, and the σ field has a mass $m_\sigma^2 = 2\mu^2$. The quadratic term in A_μ is $(\partial_\mu\theta + eA_\mu)^2 = (\partial_\mu\pi/F_\pi + eA_\mu)^2$. We can perform a gauge transformation of the field A_μ to get rid of the mixing $\partial_\mu\pi A_\mu$ term:

$$A_\mu \to A_\mu - \frac{1}{eF_\pi}\partial_\mu\pi, \qquad \mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + (v + \sigma/\sqrt{2})^2 e^2 A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(v + \sigma/\sqrt{2}).$$

The second term gives

$$(v + \sigma/\sqrt{2})^2 e^2 A_\mu A^\mu = \frac{2\mu^2 e^2}{\lambda} A_\mu A^\mu + \frac{\sigma^2}{2} e^2 A_\mu A^\mu + e^2\sqrt{2}v\sigma A_\mu A^\mu.$$

This expression represents the mass of the field A_μ and the two couplings σAA and $\sigma\sigma AA$



Focusing only on the field A on its own, we see that it has acquired a mass

$$m_A^2 = \frac{4\mu^2}{\lambda}e^2.$$

This is called the *Higgs mechanism* (or Englert-Brout-Higgs-GuralnikHagen-Kibble mechanism) The equation of motion for the gauge field is $\partial_\mu F^{\mu\nu} + m_A^2 A^\nu = 0$. Taking another derivative of this equation, we get

$$0 = \partial_\mu(\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) - m_A^2 A^\nu) = \Box(\partial_\mu A^\nu - \partial_\mu A^\mu) + m_A^2\partial_\mu A^\mu = m_A^2\partial_\mu A^\mu = 0.$$

An interesting exercise is to count the degrees of freedom before and after the symmetry breaking. Before the breaking, we had one massless gauge boson (2 degrees) and a complex scalar (2 degrees). After the breaking we were left with a real scalar (1 degree) and a massive gauge boson (3 degrees). The total degrees of freedom before and after the symmetry breaking are the same, 4.

6.3.2 Non-Abelian Higgs mechanism

Instead of a complex scalar field, we will now use a couple of complex scalar fields $\phi = (\phi_1, \phi_2)$ in fundamental representation and a SU(2) gauge group. The Lagrangian is only modified slightly

$$\mathcal{L} = (\mathrm{D}_\mu \phi)^\dagger (\mathrm{D}^\mu \phi) - V(\phi^\dagger \phi) - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a}.$$

The potential is $V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda/4 (\phi^\dagger \phi)^\dagger$. For the vacuum state ϕ_v it holds

$$\phi_v^\dagger \phi_v = \frac{2\mu^2}{\lambda} = v^2.$$

We can choose the specific vacuum state and expand the field around it in terms of four real scalar fields ξ_i

$$\phi_v = v \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \phi(x) = \begin{pmatrix} \xi_1(x) + i\xi_2(x) \\ v + \xi_3(x) + i\xi_4(x) \end{pmatrix}. \tag{6.1}$$

The gauge transformation of the field ϕ is

$$\phi(x) \to \phi'(x) = \omega(x)\phi(x) = (1 + i\varepsilon_a(x)\tau_a)\phi(x)$$

$\tau^a = \sigma^a / a$ are the Pauli matrices, $\varepsilon_a(x)$ are real infinitesimal parameters and $\omega(x)$ is an element of SU(2):

$$\omega(x) = \begin{pmatrix} 1 + i\varepsilon_3/2 & \varepsilon_2 + i\varepsilon_1/2 \\ -\varepsilon_2 + i\varepsilon_1/2 & 1 - i\varepsilon_3/2 \end{pmatrix}.$$

Now we choose a specific field configuration for some real scalar field $h(x)$

$$\varphi(x) = \begin{pmatrix} 0 \\ v + h(x)/\sqrt{2} \end{pmatrix}$$

and apply the gauge transformation

$$\varphi'(x) = \omega(x)\varphi(x) = \begin{pmatrix} v\varepsilon_2/2 + iv\varepsilon_2/2 \\ v + h/\sqrt{2} + \end{pmatrix}. \tag{6.2}$$

We dropped all the terms of order εh as we assume that h is also infinitesimal. As we can see by comparing Eq. (6.1) and Eq. (6.2), the gauge transformation of the field φ can give as any possible other expansion of the field ϕ around the vacuum state. The equations are much simpler if we choose φ, so we will do that. This choice is called the *unitary gauge* and is actually quite a delicate matter since it is not renormalizable.

The covariant derivative on the field φ is

$$\mathrm{D}_\mu \varphi = ig\tau^a A^a_\mu \begin{pmatrix} 0 \\ v \end{pmatrix} + (\partial_\mu ig\tau^a A^a_\mu) \begin{pmatrix} 0 \\ h(x)/\sqrt{2} \end{pmatrix}.$$

The full derivative term on the vacuum state becomes

$$(\mathrm{D}_\mu \phi_v)^\dagger (\mathrm{D}_\mu \phi_v) = \begin{pmatrix} 0 & v \end{pmatrix} (-ig\tau^a A^a_\mu)(ig\tau^b A^{\mu,b}) \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{4} g^2 v^2 A^a_\mu A^{\mu,a}.$$

This gives the mass of the gauge boson as $m_A^2 = g^2 v^2 / 2$. Let's count the degrees of freedom again. Before the symmetry breaking we had 2 complex scalar with two degrees of freedom each and three massless gauge bosons with two degrees of freedom each. After the symmetry breaking, we have one real scalar with one degree of freedom and three massive gauge bosons with three degrees of freedom each. The total number of degrees of freedom is the same, 10.

6.4 Georgi-Glashow model

We will now repeat the steps from the last section, but we will have a real scalar field $\boldsymbol{\phi}$ in the adjoint representation. The Lagrangian is

$$\mathcal{L} = \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} - \mathrm{Tr}[(\mathrm{D}_\mu\boldsymbol{\phi})^2] - V(\mathrm{Tr}[\boldsymbol{\phi}^2])$$

where the potential is $V(\mathrm{Tr}[\boldsymbol{\phi}^2]) = -\mu^2\,\mathrm{Tr}[\boldsymbol{\phi}^2] + \lambda/4\,\mathrm{Tr}[\boldsymbol{\phi}^2]^2$ and $\boldsymbol{\phi}(x) = \phi^a(x)\tau^a$. The covariant derivative becomes

$$\mathrm{D}_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} + g[\boldsymbol{\phi}, \mathbf{A}_\mu].$$

The trace formalism is very easy in our case $\mathrm{Tr}(\boldsymbol{\phi}^2) = \phi_a\phi_a/2$, so the potential and the vacuum expectation value are

$$V(\phi)/2 = -\mu^2(\phi_1^2+\phi_2^2+\phi_3^2) + \frac{\lambda}{4}(\phi_1^2+\phi_2^2+\phi_3^2)^2,$$

$$(\phi_v)^2 = (\phi_1^v)^2 + (\phi_2^v)^2 + (\phi_3^v)^2 = v^2 = \frac{2\mu^2}{\lambda}.$$

We can choose that only ϕ_3^v is non-zero, so the derivative on the vacuum state is

$$\mathrm{D}_\mu\boldsymbol{\phi}_v = -gv(A^1_\mu\tau^2 - A^2_\mu\tau^1), \quad (\mathrm{D}_\mu\boldsymbol{\phi}_v)^\dagger(\mathrm{D}^\mu\boldsymbol{\phi}_v) = \frac{(gv)^2}{4}\left[(A^1_\mu)^2 + (A^2_\mu)^2\right]\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

We see that only two of the three gauge fields acquired mass. The degrees of freedom before the breaking were 9 (3 for three real scalar fields and 6 for three massless gauge bosons) and 9 after the breaking (6 for two massive gauge fields, 2 for one massless gauge field and 1 for the remaining real scalar).

6.5 The Glashow-Weinberg-Salam model

Georgi-Glashow model was already quite good at describing different masses of gauge bosons. The failure was only in the coupling to fermions. The next attempt, that actually turns out to work, it is the spontaneous breaking of symmetry

$$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{\mathrm{em}}.$$

The gauge part of the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}W^a_{\mu\nu}W^{\mu\nu,a} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (\mathrm{D}_\mu H)^\dagger(\mathrm{D}^\mu H) + \mu^2 H^\dagger H - \lambda(H^\dagger H)^2.$$

Here, W^a_μ are SU(2) gauge fields, B_μ is $\mathrm{U}(1)_Y$ gauge field and H is a pair of complex scalar fields (*Higgs field*) that transforms in fundamental representation of SU(2) and has *hypercharge* $Y = 1/2$. The expanded covariant derivative for the Higgs field reads

$$\mathrm{D}_\mu H = \partial_\mu H - igW^a_\mu\tau^a H - ig'YB_\mu H.$$

The vacuum expectation value for the Higgs field is $|H_v| = v/\sqrt{2}$, where $v^2 = \mu^2/\lambda$. We choose

$$H_v = \begin{pmatrix}0\\ v/\sqrt{2}\end{pmatrix} \quad H = e^{2i\pi^a\tau^a/v}\begin{pmatrix}0\\ \frac{v+h(x)}{\sqrt{2}}\end{pmatrix} \xrightarrow{\text{unitary gauge}} \begin{pmatrix}0\\ \frac{v+h(x)}{\sqrt{2}}\end{pmatrix}.$$

The covariant derivative is

$$\mathrm{D}_\mu = \partial_\mu - ig\frac{\tau^a}{2}W^a_\mu - i\frac{g'}{2}B_\mu \mathbb{1} = \partial_\mu - \frac{i}{2}\begin{pmatrix} gW^3_\mu + g'B_\mu & g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^3_\mu) & -gW^3_\mu + g'B_\mu \end{pmatrix}.$$

The kinetic term of the Higgs field, evaluated at the vacuum expectation value, gives

$$(\mathrm{D}_\mu \langle H\rangle)^\dagger (\mathrm{D}^\mu \langle H\rangle) = \frac{g^2 v^2}{8}\left((W^1_\mu)^2 + (W^2_\mu)^2 + (-W^3_\mu + (g'/g)B_\mu)^2\right).$$

Fields W^3 and B mix, so we need to diagonalize the mass matrix. The mass eigenvalues A_μ and Z_μ are

$$\begin{aligned} B_\mu &= A_\mu \cos\theta_\mathrm{W} - Z_\mu \sin\theta_\mathrm{W} \\ W^3_\mu &= A_\mu \sin\theta_\mathrm{W} + Z_\mu \cos\theta_\mathrm{W} \end{aligned}$$

To make this transformation work, we choose $\tan\theta_\mathrm{W} = g'/g$. The angle θ_W is called the *Weinberg angle* or the *weak mixing angle*. After the transformation, the mass matrix becomes diagonal

$$(\mathrm{D}_\mu H_v)^\dagger (\mathrm{D}^\mu H_v) = \frac{1}{2}\left(\frac{gv}{2}\right)^2 \left((W^1_\mu)^2 + (W^2_\mu)^2\right) + \frac{1}{2}\left(\frac{gv}{2\cos\theta_\mathrm{W}}\right)^2 Z^2_\mu.$$

From this, we can read the masses of the gauge bosons

$$M_W = \frac{gv}{2}, \qquad M_Z = \frac{gv}{2\cos\theta_\mathrm{W}} = \frac{M_W}{\cos\theta_\mathrm{W}}.$$

The kinetic term for the gauge fields is

$$\mathcal{L}_\mathrm{kin} = -\frac{1}{4}W^1_{\mu\nu}W^{\mu\nu,1} - \frac{1}{4}W^2_{\mu\nu}W^{\mu\nu,2}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\frac{1}{4}Z_{\mu\nu}Z^{\mu\nu}.$$

The mass of the Higgs boson can be read from the potential, expanded around the vacuum state

$$\mathcal{L}_H = \lambda(H^\dagger H - v^2/2)^2 = \lambda(h^2v^2 + h^4/4 + vh^3) = -\frac{1}{2}h(\Box + m_h^2)h - g\frac{m_h^2}{4m_W}h^3 - \frac{g^2}{32}\frac{m_h^2}{m_W^2}h^4,$$

$$\implies \qquad m_h^2 = 2\lambda v^2.$$

Measuring the values of M_W, M_Z and G_F gives us the values of θ_W, v and g. Measuring the value of m_h gives us the value of λ. Before we go to interaction of the gauge bosons with other particles, we have to make another modification. Experimentally, the W bosons come with charge, so they have to be complex fields (and also conjugate to each other). We can rotate the fields W^1 and W^2 within the mass eigenspace to achieve this

$$W^\pm_\mu = \frac{W^1_\mu \mp W^2_\mu}{\sqrt{2}}.$$

The derivative of the Higgs field is thus

$$\mathrm{D}_\mu H = \left(\partial_\mu - \frac{ig}{\sqrt{2}}(W^+_\mu\tau^+ + W^-_\mu\tau^-) + \frac{ig}{2\cos\theta_\mathrm{W}}\right)\begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix}$$

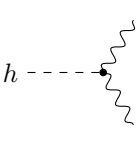
where

$$\tau^+ = \tau^1 + i\tau^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau^- = \tau^1 - i\tau^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The full interaction part of the Higgs boson with the gauge bosons is

$$\mathcal{L}_h = \mathcal{L}_h^{\text{self}} + 2\frac{h}{v}\left(M_W^2 W_\mu^+ W_\mu^- + \frac{1}{2}M_Z^2 Z_\mu Z^\mu\right) + \left(\frac{h}{v}\right)^2\left(M_W^2 W_\mu^+ W_\mu^- + \frac{1}{2}M_Z^2 Z_\mu Z^\mu\right).$$

From this, we can extract all the vertex factors. For example,



$$= 2i\frac{M_W^2}{v}g_{\mu\nu} = \frac{ie}{\sin\theta_{\text{W}}}M_W g_{\mu\nu}.$$

The coupling of fermions to the gauge fields is a bit more tricky. For now, assume we have a fermion field Ψ in the fundamental representation of SU(2) and with hypercharge Y. The covariant derivative is

$$\text{D}_\mu = \partial_\mu - \frac{ig}{\sqrt{2}}(W_\mu^+\tau^+ + W_\mu^-\tau^-) - \frac{i}{\sqrt{g^2+g'^2}}Z_\mu(g^2\tau^3 - g'^2 Y) - \frac{igg'}{\sqrt{g^2+g'^2}}A_\mu(\tau^3 + Y).$$

Using $Q = \tau^3 + Y$, we can (with a bit of algebra) rewrite the covariant derivative as

$$\text{D}_\mu = \partial_\mu - \frac{ig}{\sqrt{2}}(W_\mu^+\tau^+ + W_\mu^-\tau^-) - \frac{ig}{\cos\theta_{\text{W}}}Z_\mu(\tau^3 - Q\sin^2\theta_{\text{W}}) - ieQA_\mu,$$

where we have identified $e = g\sin\theta_{\text{W}}$. Assuming that the fermion fields are massless, we can split them in the right- and left-handed components

$$\Psi = P_{\text{L}}\Psi_{\text{L}} + P_{\text{R}}\Psi_{\text{R}}$$

where the projection operators are $P_{\text{R,L}} = (1\pm\gamma^5)/2$. Next, we assign different representations to the left- and right-handed components of the fermion field. In particular, left-handed components are in the fundamental representation of SU(2) and the right-handed components are in the trivial representation. This means that hypercharge of the right-handed components has to be the same as their electromagnetic charge $Y_{\text{R}} = Q_{\text{R}}$:

$$Y_{u_{\text{R}}} = 2/3, \quad Y_{d_{\text{R}}} = -1/3, \quad Y_{e_{\text{R}}} = -1.$$

The left-handed components have the weak isospin $T_{\text{L}} = 1/2$, so the third component of it is either $+1/2$ or $-1/2$. The hypercharge must be the same for both components of the doublet, so the difference in electromagnetic charge must be equal to one. This is exactly what we need for the leptonic doublet of electron and neutrino:

$$\tau^3 L_{\text{L}} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} \nu_e \\ e_{\text{L}} \end{pmatrix} = \begin{pmatrix} \nu_e/2 \\ -e_{\text{L}}/2 \end{pmatrix}.$$

We need $Q_e = -1/2 + Y = -1$ and $Q_\nu = 1/2 + Y = 0$. The solution is $Y_L = -1/2$. We can follow the same steps for the quark doublet $Q_{\text{L}} = (u_{\text{L}}, d_{\text{L}})^T$ and obtain $Y_Q = 1/6$. This all works great. But we still have a huge problem – we have assumed massless fermions. How do we include the

mass? A regular mass term for fermions $\mathcal{L}_m = -m_e(\overline{e}_\mathrm{L} e_\mathrm{R} + \overline{e}_\mathrm{R} e_\mathrm{L})$ is not gauge invariant. What we can write down, though, is a Yukawa term (this time for a general lepton ℓ and neutrino ν_ℓ)

$$\mathcal{L}_\mathrm{Yuk} = -\lambda_\ell \overline{L}_\mathrm{L} H \ell_\mathrm{R} + \mathrm{h.c.}.$$

This is invariant under SU(2) because $\overline{L}_\mathrm{L}$ and H are in the conjugate representations (and ℓ_R in trivial). The charge under $\mathrm{U}(1)_Y$ is $1/2 + 1/2 - 1 = 0$. After the spontaneous symmetry breaking, the Higgs field acquires a vacuum expectation value $H_v = v/\sqrt{2}$, so the Yukawa term becomes

$$\mathcal{L}_\mathrm{Yuk} = -\frac{\lambda_\ell v}{\sqrt{2}} \overline{\ell}_\mathrm{L} \ell_\mathrm{R} + \mathrm{h.c.}$$

This is exactly what we need for the mass term! We can read $m_\ell = \lambda_\ell v/\sqrt{2}$. However, if we repeat the same procedure for quarks, the up-type quarks will be massless as they play the same role as neutrino. To fix this, we construct the conjugate Higgs field $\tilde{H} = i\sigma_2 H^*$. This one has a vacuum expectation value in the other component

$$\tilde{H}_v = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix}.$$

Thus, we can write

$$\mathcal{L}_u = -\lambda_d \overline{Q}_\mathrm{L} H d_\mathrm{R} + \mathrm{h.c.} \xrightarrow{\mathrm{SSB}} -\frac{\lambda_d v}{\sqrt{2}} \overline{d}_\mathrm{L} d_\mathrm{R} + \mathrm{h.c.}$$

$$\mathcal{L}_d = -\lambda_d \overline{Q}_\mathrm{L} \tilde{H} u_\mathrm{R} + \mathrm{h.c.} \xrightarrow{\mathrm{SSB}} -\frac{\lambda_u v}{\sqrt{2}} \overline{u}_\mathrm{L} u_\mathrm{R} + \mathrm{h.c.}$$

This can work also for other generations, but with quarks it happens that weak eigenstates of quarks do not coincide with the mass eigenstates. This is corrected by a rotation matrix, called the *Cabibbo-Kobayashi-Maskawa matrix* (CKM matrix). The CKM matrix is a 3×3 unitary matrix that rotates the left-handed quark doublets into the mass eigenstates. For more details, see the notes from theoretical particle physics course. We could also add the right-handed neutrinos (using $\tilde{H}$), and it is an open problem of the Standard Model if they actually exist.

6.6 Effective potential

Consider a single real scalar field ϕ and some external source $J(x)$. The generating functional is

$$Z[J] = \langle \Omega | e^{iHt} | \Omega \rangle = \int \mathcal{D}[\phi] \, \exp\left(i \int \mathrm{d}^4 x \, (\mathcal{L}[\phi] + J\phi) \right).$$

This function generates all the correlation functions of the field ϕ and the source J. Moreover, we can define the generating functional of the connected correlation functions $E[J]$ (denoted by $W[J]$ in QFT1) with

$$Z[J] = e^{-iE[J]}.$$

The function Z is equivalent to polarization density in the statistical physics and function E is equivalent to the Helmholtz free energy. Taking a functional derivative of E with respect to the current gives

$$\frac{\delta E[J]}{\delta J(x)} = i \frac{\delta}{\delta J(x)} \ln Z[J] = -\frac{\int \mathcal{D}[\phi] \phi(x) \exp\left(i \int \mathrm{d}^4 x \, (\mathcal{L} - J\phi)\right)}{\int \mathcal{D}[\phi] \exp\left(i \int \mathrm{d}^4 x \, (\mathcal{L} - J\phi)\right)} = -\langle \Omega | \phi(x) | \Omega \rangle_J \equiv -\phi_\mathrm{cl}(x).$$

We have defined the classical field $\phi_{\mathrm{cl}}(x) = \langle\Omega|\phi(x)|\Omega\rangle_J$ as the vacuum expectation value of the field ϕ in the presence of the current J. Since we know that the derivative of E with respect to J gives us $-\phi_{\mathrm{cl}}$, we can use this equation and integrate it (adding an integration constant $\Gamma[\phi]$ which is independent of J)

$$E[J] = -\int \mathrm{d}^4x\; J(x)\phi_{\mathrm{cl}}(x) + \Gamma[\phi_{\mathrm{cl}}].$$

We can interpret the integration constant as a Legendre transformation of the generating functional E

$$\Gamma[\phi_{\mathrm{cl}}] = -E[J] - \int \mathrm{d}^4x\; J(x)\phi_{\mathrm{cl}}(x).$$

This is equivalent to the Gibbs free enthalpy in statistical physics, in QFT it is called the *effective action*. The derivative of the effective action with respect to the classical field gives us the current

$$\begin{aligned}
\frac{\delta\Gamma[\phi_{\mathrm{cl}}]}{\delta\phi_{\mathrm{cl}}(x)} &= -\frac{\delta E[J]}{\delta\phi_{\mathrm{cl}}(x)} - \int \mathrm{d}^4y\; \frac{\delta J(y)}{\delta\phi_{\mathrm{cl}}(x)}\phi_{\mathrm{cl}}(y) + J(X) \\
&= -\int \mathrm{d}^4y\; \frac{\delta E[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\phi_{\mathrm{cl}}(x)} - \int \mathrm{d}^4y\; \frac{\delta J(y)}{\delta\phi_{\mathrm{cl}}(x)}\phi_{\mathrm{cl}}(y) + J(X) \\
&= \int \mathrm{d}^4y\; \frac{\delta J(y)}{\delta\phi_{\mathrm{cl}}(x)}\phi_{\mathrm{cl}}(y) - \int \mathrm{d}^4y\; \frac{\delta J(y)}{\delta\phi_{\mathrm{cl}}(x)}\phi_{\mathrm{cl}}(y) + J(X) \\
&= J(x).
\end{aligned}$$

In short,

$$\boxed{\frac{\delta\Gamma[\phi_{\mathrm{cl}}]}{\delta\phi_{\mathrm{cl}}(x)} = -J(x).}$$

In the case of zero current, the derivative of the effective action with respect to the classical field gives zero

$$\left.\frac{\delta\Gamma[\phi_{\mathrm{cl}}]}{\delta\phi_{\mathrm{cl}}(x)}\right|_{J=0} = 0$$

which classically represents the vacuum. We can perform the gradient expansion of the effective action

$$\Gamma[\phi_{\mathrm{cl}}] = \int \mathrm{d}^dx \left[-V_{\mathrm{eff}}(\phi_{\mathrm{cl}}) - \frac{1}{2}Z(\phi_{\mathrm{cl}})(\partial_\mu\phi_{\mathrm{cl}})(\partial^\mu\phi_{\mathrm{cl}})\right] = -(VT)V_{\mathrm{eff}}(\phi_{\mathrm{cl}}).$$

If we are working in the Lorentz invariant theory, the vacuum must be Lorentz invariant as well. This means that the vacuum expectation value of the field cannot depend on the space-time coordinates

$$\phi_{\mathrm{cl}}(x) = \langle\Omega|\phi(x)|\Omega\rangle = \phi_{\mathrm{cl}} \neq f(x).$$

Under this assumption, the effective action becomes exactly proportional to the *effective potential*

$$\boxed{V_{\mathrm{eff}}(\phi_{\mathrm{cl}}) = -\frac{\Gamma[\phi_{\mathrm{cl}}]}{VT}}.$$

The condition for the minimum of the effective action becomes a normal derivative on the effective potential

$$\frac{\partial V_{\mathrm{eff}}(\phi_{\mathrm{cl}})}{\partial\phi_{\mathrm{cl}}} = 0.$$

If the current equals zero, then the effective action describes the energy $\Gamma[\phi_{\rm cl}] = -E[0]$ and the effective potential becomes the energy density $V_{\rm eff}(\phi_{\rm cl}) = E[0]/(VT)$.

How do we compute the effective potential (or action)? If we could solve the theory exactly, that wouldn't be a problem (we would just expand the generating functional and perform the Legendre transformation). However, we almost always lack exact solutions to useful theories, so we have to take the perturbative approach. We write the Lagrangian in two parts $\mathcal{L} = \mathcal{L}_1 + \delta\mathcal{L}$ where the second part includes the counter-terms, and we will also split the current $J = J_1 + \delta J$. We want this splitting to be such that the following equation is satisfied exactly (to all orders)

$$\left.\frac{\partial\mathcal{L}_1}{\partial\phi}\right|_{\phi_{\rm cl}} + J_1(x) = 0.$$

The generating functional becomes

$$Z[J] = e^{-iE[J]} = \int \mathcal{D}[\phi] \exp\left\{ i\int \mathrm{d}^4x\, (\mathcal{L}_1[\phi] + J_1\phi)\right\} \exp\left\{ i\int \mathrm{d}^4x\, (\delta\mathcal{L}[\phi] - \delta J\phi)\right\}.$$

We expand the field around its classical value $\phi(x) = \phi_{\rm cl} + \eta(x)$. The first term in the exponent becomes

$$\begin{aligned}\int \mathrm{d}^4x\, (\mathcal{L}_1[\phi] + J_1\phi) = \int \mathrm{d}^4x\, (\mathcal{L}_1(\phi_{\rm cl}) + J_1\phi_{\rm cl}) + \int \mathrm{d}^4x\, \eta(x) \left(\left.\frac{\partial\mathcal{L}_1}{\partial\phi}\right|_{\phi_{\rm cl}} + J_1\right) \\ + \frac{1}{2!}\int \mathrm{d}^4x\, \mathrm{d}^4y\, \eta(x)\eta(y) \left.\frac{\delta^2\mathcal{L}_1}{\delta\phi(x)\delta\phi(y)}\right|_{\phi_{\rm cl}} + \dots\end{aligned}$$

The second term is identically equal to zero. When we write it in the generating functional, we have

$$\begin{aligned}\int \mathcal{D}[\eta] \exp\left[i\int \mathrm{d}^4x\, (\mathcal{L}_1[\phi_{\rm cl} + J_1\phi_{\rm cl}]) + \frac{i}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, \eta(x)\eta(y) \left.\frac{\delta^2\mathcal{L}_1}{\delta\phi(x)\delta\phi(y)}\right|_{\phi_{\rm cl}}\right] = \\ = \exp\left[i\int \mathrm{d}^4x\, (\mathcal{L}_1 + J_1\phi_{\rm cl})\right] \det\left(-\frac{\delta^2\mathcal{L}_1}{\delta\phi\delta\phi}\right)^{-1/2}.\end{aligned}$$

Including also the counter-terms, we get

$$\delta\mathcal{L} + \delta J\phi = \delta\mathcal{L}[\phi_{\rm cl}] + \delta J\phi_{\rm cl} + (\delta\mathcal{L}[\phi_{\rm cl} + \eta] - \delta\mathcal{L}[\phi_{\rm cl}] + \delta J\eta).$$

Going back to the generating functional

$$e^{-E[J]} = \exp\left(i\int \mathrm{d}^4x\, (\mathcal{L}_1 + J_1\phi_{\rm cl} + \delta\mathcal{L}[\phi_{\rm cl}] + \delta J\phi_{\rm cl})\right) \det\left(-\frac{\delta^2\mathcal{L}_1}{\delta\phi\delta\phi}\right)^{-1/2} e^{\text{higher order terms}}.$$

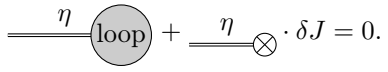
Taking the logarithm of this equation gives

$$-E[J] = \int \mathrm{d}^4x\, (\mathcal{L}_1 + J_1\phi_{\rm cl}) + \int \mathrm{d}^4x\, (\delta\mathcal{L}[\phi_{\rm cl}] + \delta J\phi_{\rm cl}) + \frac{i}{2}\log\det\left(-\frac{\delta^2\mathcal{L}_1}{\delta\phi\delta\phi}\right) + \text{higher order terms.}$$

Lastly, we perform the Legendre transformation to get the effective action

$$\Gamma[\phi_{\rm cl}] = \int \mathrm{d}^4x\, \mathcal{L}_1[\phi_{\rm cl}] + \frac{i}{2}\log\det\left(-\frac{\delta^2\mathcal{L}_1}{\delta\phi\delta\phi}\right) + \int \mathrm{d}^4x\, \delta\mathcal{L}_1[\phi_{\rm cl}] + \text{higher terms.}$$

In Lorentz invariant theories, can replace the integrals with (VT). This equation has no J dependence, and not even the δJ. However, at higher orders, the δJ will appear, and we can use it as a counter-term for the tadpole diagrams (which will keep the field at classical value at all orders). To do this, we require



Let's do an example of the effective action construction with the linear sigma model. We have the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 + \frac{1}{2}\mu^2 \phi_i^2 - \frac{\lambda}{4}(\phi_i^2)^2.$$

The field is $\phi_i = (\phi_1, \dots, \phi_N)$ which we expand around the classical value $\phi_i(x) = \phi^i_{\text{cl}} + \eta^i(x)$. Up to quadratic terms, the Lagrangian becomes

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}(\partial_\mu \eta_i)^2 + \frac{1}{2}\mu^2(\phi^i_{\text{cl}})^2 + \frac{1}{2}\mu^2\eta_i^2 - \frac{\lambda}{4}[(\phi^i_{\text{cl}})^2]^2 - (\mu^2 - \lambda(\phi^i_{\text{cl}})^2)\phi^i_{\text{cl}}\eta^i + \\ &+ \frac{\lambda}{2}\left[(\phi^i_{\text{cl}})^2(\eta^i)^2 + 2(\phi^i_{\text{cl}}\eta^i)^2\right] + \text{higher terms}.\end{aligned}$$

We split the Lagrangian into finite part and the counter-terms $\mathcal{L} = \mathcal{L}_1 + \delta\mathcal{L}$, and we do the same with the current $J = J_1 + \delta J$. The equation

$$\left.\frac{\partial \mathcal{L}_1}{\partial \phi_i}\right|_{\phi_{\text{cl}}} + J_1^i = 0$$

gives the solution

$$\mu^2\phi^i_{\text{cl}} - \lambda\phi^i_{\text{cl}}(\phi^j_{\text{cl}})^2 + J_1^i(x) = 0 \quad \Longrightarrow \quad J_1^i = -\phi^i_{\text{cl}}(\mu^2 - \lambda(\phi^j_{\text{cl}})^2).$$

We have

$$\begin{aligned}\mathcal{L}_1 + J_1^i\phi^i =& \frac{1}{2}(\partial_\mu \eta_i)^2 + \frac{1}{2}\mu^2(\phi^i_{\text{cl}})^2 + \frac{1}{2}\mu^2\eta_i^2 - \frac{\lambda}{4}[(\phi^i_{\text{cl}})^2]^2 - (\mu^2 - \lambda(\phi^i_{\text{cl}})^2)\phi^i_{\text{cl}}\eta^i + \\ &- \frac{\lambda}{2}\left[(\phi^i_{\text{cl}})^2(\eta^i)^2 + 2(\phi^i_{\text{cl}}\eta^i)^2\right] - (\phi^i_{\text{cl}})^2(\mu^2 - \lambda(\phi^i_{\text{cl}})^2) - (\eta^i\phi^i_{\text{cl}})(\mu^2 - \lambda(\phi^i_{\text{cl}})^2) + \\ &+ \text{higher terms}\end{aligned}$$

When we collect all the quadratic terms in η, we get

$$\begin{aligned}\mathcal{L}_{\eta\eta} &= \frac{1}{2}(\partial_\mu \eta_i)^2 + \frac{1}{2}(\mu^2 - \lambda(\phi^i_{\text{cl}})^2)\eta_i^2 - \lambda(\phi^i_{\text{cl}}\eta^i)(\phi^j_{\text{cl}}\eta^j) \\ &\overset{\text{IBP}}{=} \frac{1}{2}\eta^i(-\partial^2 + \mu^2 - \lambda(\phi^k_{\text{cl}})^2)\eta^j\delta_{ij} - \lambda(\phi^i_{\text{cl}}\phi^j_{\text{cl}})\eta^i\eta^j.\end{aligned}$$

We have done all this work to get to the value of the second derivative of the Lagrangian

$$\left.\frac{\delta\mathcal{L}_1}{\delta\phi_i\delta\phi_j}\right|_{\phi_{\text{cl}}} = (-\partial^2 + \mu^2 - \lambda(\phi^i_{\text{cl}})^2) - 2\lambda\phi^i_{\text{cl}}\phi^j_{\text{cl}}.$$

To compute the functional determinant, we can exploit the $\mathrm{O}(N)$ symmetry so that we align the classical field in the last direction $\phi^i_{\text{cl}} = (0, \dots, 0, \phi_{\text{cl}})$. With this simplification, we get

$$\left.\frac{\delta\mathcal{L}_1}{\delta\phi_i\delta\phi_j}\right|_{\phi_{\text{cl}}} = -(\partial^2 + M^2)\delta_{ij}, \qquad M^2 = \begin{cases}\lambda\phi^2_{\text{cl}} - \mu^2, & i = j < N; \\ 3\lambda\phi^2_{\text{cl}} - \mu^2, & i = j = N.\end{cases}$$

Continuing the computation yields

$$\det\left(\left.\frac{\delta\mathcal{L}_1}{\delta\phi_i\delta\phi_j}\right|_{\phi_{\rm cl}}\right) = \det\left(\partial^2 + (\lambda\phi_{\rm cl}^2 - \mu^2)\right)^{N-1} \det\left(\partial^2 + (3\lambda\phi_{\rm cl}^2 - \mu^2)\right).$$

To perform the determinants, we use

$$\log\det(\partial^2 + M^2) = \operatorname{tr}\log(\partial^2 + M^2) = \sum_k \log(-k^2 + M^2) = (VT)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\log(-k^2 + M^2).$$

The integral has to be evaluated using the dimensional regularization:

$$\begin{aligned}\tilde{\mu}_0^{2\varepsilon}\int\frac{\mathrm{d}^dk}{(2\pi)^d} &= i\tilde{\mu}_0^{2\varepsilon}\int\frac{\mathrm{d}^dk_{\rm E}}{(2\pi)^d}\ln(k_{\rm E}^2 + M^2) = -i\tilde{\mu}^{2\varepsilon}\left[\frac{\partial}{\partial\alpha}\int\frac{\mathrm{d}^dk_{\rm E}}{(2\pi)^d}\frac{1}{(k_{\rm E}^2 + M^2)^\alpha}\right]_{\alpha=0}\\ &= -i\tilde{\mu}_0^{2\varepsilon}\left[\frac{\partial}{\partial\alpha}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)}\frac{1}{(M^2)^{\alpha-d/2}}\right]_{\alpha=0} = \frac{i\tilde{\mu}_0^{2\varepsilon}}{(4\pi)^{d/2}}(M^2)^{d/2}\Gamma(-d/2).\end{aligned}$$

Hence,

$$\ln\det(\partial^2 + M^2) = VT\left[-\frac{i\Gamma(d/2)}{(4\pi)^{d/2}}(M^2)^{d/2}\right]\tilde{\mu}_0^{2\varepsilon}.$$

Plugging everything back into the effective potential (including the counter-term) gives

$$\begin{aligned}V_{\rm eff}(\phi_{\rm cl}) = &-\frac{1}{2}\mu^2\phi_{\rm cl}^2 + \frac{\lambda}{4}\phi_{\rm cl}^4 - \frac{1}{2}\frac{\Gamma(-d/2)}{(4\pi)^{d/2}}\left[(N-1)(\lambda\phi_{\rm cl}^2 - \mu^2)^{d/2} + (3\lambda\phi_{\rm cl}^2 - \mu^2)^{d/2}\right]\tilde{\mu}_0^{2\varepsilon} +\\ &+\frac{1}{2}\delta_\mu\phi_{\rm cl}^2 + \frac{1}{4}\delta_\lambda\phi_{\rm cl}^4 + \text{higher order terms}.\end{aligned}$$

We will set $d = 4-2\varepsilon$ and expand around $\varepsilon = 0$. For now, we are only concerned by the divergent terms

$$V_{\rm eff}^{\rm poles} = -\frac{1}{4(4\pi)^2}\frac{1}{\varepsilon}\left[N\mu^4 - 2(N+2)\lambda\mu^2\phi_{\rm cl}^2 + (N+8)\lambda^2\phi_{\rm cl}^4\right].$$

The first divergent term is just a constant, so we ignore it. The other two terms are exactly of the form $\phi_{\rm cl}^2$ and $\phi_{\rm cl}^4$ which can both be absorbed by the counter-terms. They are

$$\delta_\mu = -\frac{1}{(4\pi)^2}\frac{1}{\varepsilon}(N+2)\mu^2, \qquad \delta_\lambda = \frac{\lambda^2}{(4\pi)^2}\frac{1}{\varepsilon}(N+8)$$

which are exactly the same as the counter-terms we have used in the perturbation theory of the linear sigma model. The renormalized effective potential is thus

$$\begin{aligned}V_{\rm eff}^{\rm ren}(\phi_{\rm cl}) = &-\frac{1}{2}\mu^2\phi_{\rm cl}^2 + \frac{\lambda}{4}\phi_{\rm cl}^4 + \frac{1}{4(4\pi)^2}\left[-\frac{3}{2}(N-1)(\mu^2 - \lambda\phi_{\rm cl}^2)^2 - \frac{3}{2}(\mu^2 - 3\lambda\phi_{\rm cl}^2)^2 +\right.\\ &\left.+(N-1)(\mu^2 - \lambda\phi_{\rm cl}^2)^2\ln\left(\frac{\lambda\phi_{\rm cl}^2 - \mu^2}{\mu_0^2}\right) + (\mu^2 - 3\lambda\phi_{\rm cl}^2)\ln\left(\frac{3\lambda\phi_{\rm cl}^2 - \mu^2}{\mu_0^2}\right)\right].\end{aligned}$$

The potential is no longer real all values of $\phi_{\rm cl}$, however the minima are still in the region where the potential is real. The potential doesn't change much, the well is a bit steeper, the minima are a bit deeper and a bit further apart. What happens if we set $\mu = 0$. This would imply that

classically, there is no symmetry breaking, but after quantum corrections, some new minima might appear. The effective potential becomes a bit simpler

$$V_{\text{eff}}^{\text{ren}}(\phi_{\text{cl}}) = \frac{\lambda}{4}\left[1 + \frac{\lambda^2}{(4\pi)^2}\left(-\frac{3}{2}(N+8) + 9\ln 3 + (N+8)\ln\left(\frac{\lambda\phi_{\text{cl}}^2}{\mu_0^2}\right)\right)\right]\phi_{\text{cl}}^4.$$

This potential retains the minimum and zero at ϕ_{cl}=0, but there is an extra zero at

$$\phi_{\text{cl}}^2 = \frac{\mu_0^2}{\lambda}\exp\left(\frac{3}{2} - \frac{9\ln 3}{N+8} - \frac{(4\pi)^2}{\lambda(N+8)}\right) \xrightarrow{\lambda\ll 1} \frac{\mu_0}{\lambda}\exp\left(-\frac{(4\pi)^2}{\lambda(N+8)}\right)$$

and an extra minimum at

$$\phi_{\text{cl}}^2 = \frac{\mu_0^2}{\lambda}\exp\left(1 - \frac{9\ln 3}{N+8} - \frac{(4\pi)^2}{\lambda(N+8)}\right) \xrightarrow{\lambda\ll 1} \frac{\mu_0}{\lambda}\exp\left(-\frac{(4\pi)^2}{\lambda(N+8)}\right).$$

The result seems appealing, but for small λ each logarithm $\ln(\lambda\phi_{\text{cl}}^2/\mu_0^2)$ brings approximately λ^{-1}. This then cancels with the λ in front of the expression and the contribution is of order 1. At higher loop orders, we would get terms of form $\lambda^n \ln^n$ which all are of order one. This means that the perturbation theory breaks down, and we need to resum the logarithms using some other method. Fortunately, we can do this (at least in this case) using the running of the coupling λ:

$$\overline{\lambda} = \frac{\lambda}{1 + \frac{\lambda}{(4\pi)^2}(N+8)\ln(\phi_{\text{cl}}^2/\mu_0^2)}$$

and the resummed effective potential becomes

$$V_{\text{eff}}^{\text{ren}}(\phi_{\text{cl}}, \overline{\lambda}) = \frac{\overline{\lambda}}{4}\left[1 + \frac{\overline{\lambda}}{(4\pi)^2}\left((N+8)(\ln\overline{\lambda} - 3/2 + 9\ln 3)\right)\right]\phi_{\text{cl}}^4.$$

If we do the same calculation for the entire Standard Model, we get the *Coleman-Weinberg potential*. It has been calculated and resummed up to three loops, and it happens that there is another minimum, which would be smaller than the one we are living in now. It has been calculated the Standard model is stable if the mass of the Higgs boson is bigger than 129 GeV. Now, the actual mass is $m_H = 126\,\text{GeV}$, which might be close enough (the value might be corrected by higher orders) or the universe might be in a metastable state. This means that the expected time for the universe to tunnel to the other minimum is much larger than the age of the universe.

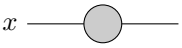
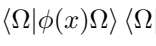
Let's return to the effective action and the functional E. Remember that

$$\frac{\delta E[J]}{\delta J(x)} = -\langle\Omega|\phi(x)|\Omega\rangle = -\phi_{\text{cl}}(x).$$

The second derivative with respect to J is

$$\begin{aligned}
\frac{\delta^2 E[J]}{\delta J(x)\delta J(y)} &= -\frac{\delta}{\delta J(x)}\left[\frac{1}{Z[J]}\int \mathcal{D}[\phi]\,\phi(y)e^{i\int \mathrm{d}^4x\,(\mathcal{L}+\phi J(y))}\right] \\
&= \frac{i}{Z[J]}\int \mathcal{D}[\phi]\phi(x)\phi(y)e^{i\int \mathrm{d}^4x\,(\mathcal{L}+\phi J)} + \frac{i}{Z[J]^2}[-]\ldots \\
&= -i\left(\langle\Omega|\phi(x)\phi(y)|\Omega\rangle - \langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle\right) \\
&= i\langle\Omega|\phi(x)\phi(y)|\Omega\rangle_{\text{conn}} = iD(x,y).
\end{aligned}$$

Pictorially, we have the following.

$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle =$  , $\langle\Omega|\phi(x)\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle =$  .

Since the derivative of the effective action with respect to the classical field is the current, we have

$$\frac{\delta}{\delta J(y)}\frac{\delta\Gamma[\phi_{\mathrm{cl}}]}{\delta\phi_{\mathrm{cl}}} = -\delta^{(4)}(x-y).$$

We will use a fancy functional chain rule for the derivative with respect to J:

$$\frac{\delta}{\delta J(y)} = \int \mathrm{d}^4 z\, \frac{\delta\phi_{\mathrm{cl}}(z)}{\delta J(y)}\frac{\delta}{\delta\phi_{\mathrm{cl}}}.$$

Applying this rule to the effective action gives

$$\int \mathrm{d}^4 z\, \frac{\delta\phi_{\mathrm{cl}}(x)}{\delta J(y)}\frac{\delta^2\Gamma[\phi_{\mathrm{cl}}]}{\delta\phi_{\mathrm{cl}}(z)\delta\phi_{\mathrm{cl}}(x)} = -\delta(x,y).$$

Furthermore, we use

$$\frac{\delta}{\delta J(y)}\phi_{\mathrm{cl}}(z) = -\frac{\delta}{\delta J(y)}\frac{\delta E[J]}{\delta J(z)} = -\frac{\delta^2 E[J]}{\delta J(y)\delta J(z)}$$

and plug it in the equation above

$$\int \mathrm{d}^4 z\, \frac{\delta^2 E[J]}{\delta J(y)\delta J(z)}\frac{\delta^2\Gamma[\phi_{\mathrm{cl}}]}{\delta\phi_{\mathrm{cl}}(z)\delta\phi_{\mathrm{cl}}(x)} = \delta(x-y).$$

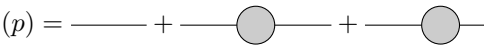
This means that the two terms in the integral are inverse to each other:

$$\frac{\delta^2\Gamma}{\delta\phi_{\mathrm{cl}}(x)\delta\phi_{\mathrm{cl}}(y)} = iD^{-1}(x,y) = iD^{-1}(x-y).$$

With the Lorentz invariance of the vacuum, we require $\phi_{\mathrm{cl}}(x) = \phi_{\mathrm{cl}}$. Moving to momentum space, we have

$$D(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{ip(x-y)}\tilde{D}(p),$$

where

$\tilde{D}(p) =$  $= \dfrac{i}{p^2 - m^2 - \Sigma(p^2)}.$

Hence, it follows

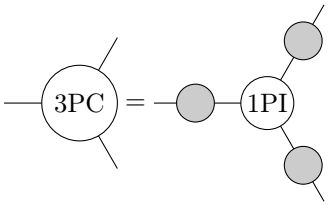
$$\left(\frac{\delta^2\Gamma}{\delta\phi_{\mathrm{cl}}(x)\delta\phi_{\mathrm{cl}}(y)}\right)_{\mathrm{F.T.}} = \tilde{D}^{-1}(p) = -i(p^2 - m^2 - \Sigma(p^2)).$$

To get the third derivative, we compactify the notation a bit: $J(x) = J_x$, $\phi_{\mathrm{cl}}(x) = \phi^x_{\mathrm{cl}}$ and

$D(x-y) = D_{xy}$. We have

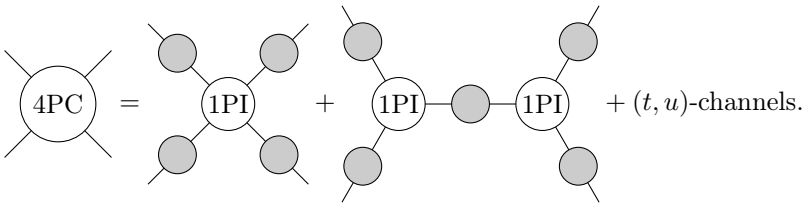
$$\begin{aligned}
\frac{\delta}{\delta J_z}\frac{\delta^2 E}{\delta J_y \delta J_x} &= \frac{\delta}{\delta J_z}\left(\frac{\delta^2\Gamma}{\delta\phi^x_{\rm cl}\delta\phi^y_{\rm cl}}\right)^{-1} \\
&= \int \mathrm{d}^4 w\, \frac{\delta\phi^w_{\rm cl}}{\delta J_z}\frac{\delta}{\delta\phi^w_{\rm cl}}\left(\frac{\delta\Gamma}{\delta\phi_{\rm cl}3^x\delta\phi^y_{\rm cl}}\right)^{-1} \\
&= -\int \mathrm{d}^4 w\, \left(\frac{\delta^2 E[J]}{\delta J_z \delta J_w}\right)\frac{\delta}{\delta\phi^w_{\rm cl}}\left(\frac{\delta\Gamma}{\delta\phi_{\rm cl}3^x\delta\phi^y_{\rm cl}}\right)^{-1} \\
&= -\int \mathrm{d}^4 w\, D_{zw}\frac{\delta}{\delta\phi^w_{\rm cl}}\left(\frac{\delta\Gamma}{\delta\phi_{\rm cl}3^x\delta\phi^y_{\rm cl}}\right)^{-1} \\
&= \int \mathrm{d}^4 w\, \mathrm{d}^4 u\, \mathrm{d}^4 v\, D_{xu}D_{yv}D_{zw}\frac{i\delta^3\Gamma}{\delta\phi^w_{\rm cl}\delta\phi^u_{\rm cl}\delta\phi^v_{\rm cl}}.
\end{aligned}$$

Pictorially, we have shown the following:



The grey blobs represent any connected diagrams. When we amputate the connected diagrams from the three-point function on each leg, we get a three-point one-particle irreducible. Hence, the effective action generates the one-particle irreducible correlation functions. For example, for four-point function, we get

$$\begin{aligned}
\frac{\delta^4 E}{\delta J_w \delta J_x \delta J_y \delta J_z} =& D_{ws}D_{xt}D_{yu}D_{zw}\left(\frac{i\delta^4\Gamma}{\delta\phi^s_{\rm cl}\delta\phi^t_{\rm cl}\delta\phi^u_{\rm cl}\delta\phi^v_{\rm cl}} + \frac{i\delta^3\Gamma}{\delta\phi^s_{\rm cl}\delta\phi^t_{\rm cl}\delta\phi^r_{\rm cl}}D_{qr}\frac{i\delta^3\Gamma}{\delta\phi^q_{\rm cl}\delta\phi^u_{\rm cl}\delta\phi^v_{\rm cl}} + \right. \\
&\left. + (t\leftrightarrow u) + (t\leftrightarrow u)\right).
\end{aligned}$$



We can algorithmically show to every order that

$$\frac{i\delta^4\Gamma}{\delta\phi^{x_1}_{\rm cl}\delta\phi^{x_2}_{\rm cl}\ldots\delta\phi^{x_n}_{\rm cl}} = -i\,\langle\Omega|\phi(x_1)\phi(x_2)\ldots\phi(x_n)|\Omega\rangle_{\rm 1PI}\,.$$

Using the effective potential to repeat the Goldstone's theorem almost verbatim from the classical case, we just need to replace the classical potential with the effective potential. Namely, every broken generator corresponds to a zero eigenvalue of the matrix $\frac{\delta^2 V_{\rm eff}}{\delta\phi^i_{\rm cl}\delta\phi^j_{\rm cl}}$, and, at zero momentum, the second derivative of the effective potential equals the second derivative of the effective action, which equals the inverse of the propagator which equals the mass of the particle.

Chapter 7

Anomalies in QFT

7.1 The Chiral anomaly

An anomaly means that a classical symmetry is broken by quantum corrections. Consider the following Lagrangian

$$\mathcal{L} = \overline{\Psi} i \not{\partial} \Psi.$$

This is the theory of a massless free fermion. This theory has two symmetries:

$$\begin{aligned} &\Psi \to \Psi' = e^{i\alpha} \Psi && \text{(usual U(1))}, \\ &\Psi \to \Psi' = e^{i\alpha\gamma_5} \Psi, && \text{(axial U(1))}. \end{aligned}$$

To see why the second is a symmetry, we write

$$\overline{\Psi}' = (\Psi^\dagger \gamma^0)' = \Psi^\dagger e^{-i\alpha\gamma_5} \gamma^0 \Psi^\dagger \gamma^0 e^{i\alpha\gamma_5}.$$

Equivalently, we could split the Lagrangian into left- and right-handed part

$$\mathcal{L} = \Psi_\mathrm{L} i \not{\partial} \Psi_\mathrm{L} + \Psi_\mathrm{R} i \not{\partial} \Psi_\mathrm{R},$$

where

$$\Psi_{\mathrm{L,R}} = P_{\mathrm{L,R}} \Psi = \frac{1 \mp \gamma^5}{2} \Psi.$$

We can see the full symmetry of the Lagrangian as a direct product of the two symmetries

$$\mathrm{U}(1)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{R} = \mathrm{U}(1)_V \times \mathrm{U}(1)_A.$$

The first one is the *vector symmetry* and the second one is the *axial symmetry*. Each symmetry is associated with a conserved current

$$J^V_\mu = \overline{\Psi} \gamma_\mu \Psi, \qquad J^A_\mu = \overline{\Psi} \gamma_\mu \gamma^5 \Psi.$$

It is straightforward to show that the divergence of the vector current on the equations of motion ($i \not{\partial} \Psi = 0$) is zero

$$\partial^\mu J^V_\mu = (\partial^\mu \overline{\Psi}) \gamma_\mu \Psi + \overline{\Psi} \gamma_\mu \partial^\mu \Psi = (\not{\partial} \overline{\Psi}) \Psi + \overline{\Psi} (\not{\partial} \Psi) = 0.$$

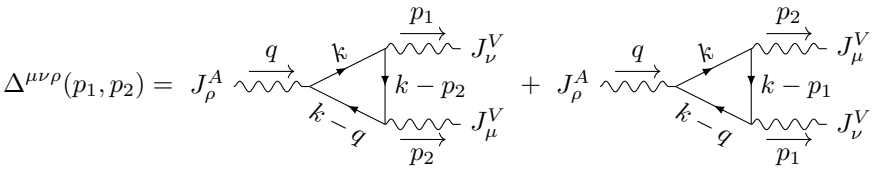
For the axial current, we have to be a bit more careful. We have

$$\partial^\mu J^A_\mu = (\partial^\mu \overline{\Psi})\gamma_\mu\gamma^5\Psi + \overline{\Psi}\gamma_\mu\partial^\mu\gamma^5\Psi = (\not{\partial}\overline{\Psi})\gamma^5\Psi - \overline{\Psi}\gamma^5(\not{\partial}\Psi) = 0$$

where we used that γ^5 and γ_μ anti-commute. The big question of this chapter is whether these symmetries still hold at the quantum level. The answer is no, the axial symmetry is broken by quantum corrections. As an example, consider a correlator of two vectors and one axial current

$$\tilde{\Delta}^{\mu\nu\rho}(x_1, x_2, x_3) = \langle\Omega|T\{J^V_\mu(x_1)J^V_\nu(x_2)J^A_\rho(x_3)\}|\Omega\rangle\,.$$

The Fourier transform of this correlator is

$\Delta^{\mu\nu\rho}(p_1,p_2) =$ 

Clearly, it holds $q = p_1 + p_2$. Evaluating the Feynman diagrams gives

$$\Delta^{\mu\nu\rho}(p_1,p_2) = C\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left[\mathrm{Tr}\left(\gamma^\rho\gamma^5\frac{1}{\not{k}-\not{q}}\gamma^\mu\frac{1}{\not{k}-\not{p}_2}\gamma^\nu\frac{1}{\not{k}}\right) + \mathrm{Tr}\left(\gamma^\rho\gamma^5\frac{1}{\not{k}-\not{q}}\gamma^\nu\frac{1}{\not{k}-\not{p}_1}\gamma^\mu\frac{1}{\not{k}}\right)\right].$$

We have collected all (unimportant) constants into one constant C. To calculate this, we would need to use dimensional regularization. This is very tricky to do, as it is not straight-forward to generalize γ^5 to d dimensions. Thus, we will first look at this naively in 4 dimensions. If the Ward identities are satisfied, then we should find

$$(p_1)_\mu\Delta^{\mu\nu\rho} = (p_2)_\nu\Delta^{\mu\nu\rho} = q_\rho\Delta^{\mu\nu\rho} = (p_1+p_2)_\rho\Delta^{\mu\nu\rho} = 0.$$

The first two equations are gauge invariant and correspond to the conservation of charge, so let's hope they hold. The last one is a bit different:

$$q_\rho\Delta^{\mu\nu\rho}(p_1,p_2) = C\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left[\mathrm{Tr}\left(\not{q}\gamma^5\frac{1}{\not{k}-\not{q}}\gamma^\mu\frac{1}{\not{k}-\not{p}_2}\gamma^\nu\frac{1}{\not{k}}\right) + \mathrm{Tr}\left(\not{q}\gamma^5\frac{1}{\not{k}-\not{q}}\gamma^\nu\frac{1}{\not{k}-\not{p}_1}\gamma^\mu\frac{1}{\not{k}}\right)\right].$$

Now we use $\not{q}\gamma^5 = \not{k}\gamma^5 + \gamma^5(\not{k}-\not{q})$. The expression becomes

$$\begin{aligned}q_\rho\Delta^{\mu\nu\rho}(p_1,p_2) =& C\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left[\mathrm{Tr}\left(\gamma^5\frac{1}{\not{k}-\not{q}}\gamma^\mu\frac{1}{\not{k}-\not{p}_2}\gamma^\nu\right) + \mathrm{Tr}\left(\gamma^5\gamma^\mu\frac{1}{\not{k}-\not{p}_2}\gamma^\nu\frac{1}{\not{k}}\right) + \right.\\ &\left. + \mathrm{Tr}\left(\gamma^5\frac{1}{\not{k}-\not{q}}\gamma^\nu\frac{1}{\not{k}-\not{p}_1}\gamma^\mu\right) + \mathrm{Tr}\left(\gamma^5\gamma^\nu\frac{1}{\not{k}-\not{p}_1}\gamma^\mu\frac{1}{\not{k}}\right)\right].\end{aligned}$$

We denote the four traces by A, B, C and D. Using the shift $k \to k + p_1$ we can show that $C = -B$ and $D = -A$. The amplitude hence vanishes and the Ward identity is seemingly satisfied. We can take similar steps to show that the first two Ward identities are satisfied as well. However, our manipulations were ill-defined. The Ward identity for axial symmetry is not satisfied, which can be experimentally confirmed by observing the decay of neutral pions into two photons. If axial symmetry was not anomalous, this decay should be forbidden. To get the correct result, have to treat our loop integral with dimensional regularization. The problem is

that γ^5 is not well-defined in d dimensions. Let's try to make a sensible definition of γ^5. In 4 dimensions, we have the following relations

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \{\gamma^5, \gamma^\mu\} = 0, \quad \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\varepsilon^{\mu\nu\rho\sigma}.$$

In d dimensions, we have $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and hence $\gamma^\mu\gamma_\mu = d$. If we keep the last relation, we get

$$\begin{aligned} d\,\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\tau\gamma_\tau) \\ &= -2g^{\mu\tau}\,\mathrm{Tr}(\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\tau) + \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\tau\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\tau) \\ &= \cdots = (8-d)\,\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma). \end{aligned}$$

We skipped some steps, where we also used $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$. The final result is

$$(d-4)\,\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 0$$

which implies that, after analytical continuation, we have

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 0.$$

This is quite troublesome. One way to deal with this is due to 't Hooft and Veltman, who defined $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ as an explicitly 4-dimensional object such that

$$\begin{cases} \{\gamma^\mu, \gamma^5\} = 0, & \text{if } \mu = 0, 1, 2, 3, \\ [\gamma^\mu, \gamma^5] = 0, & \text{otherwise.} \end{cases}$$

This means that we will have to extract a 2ε component out of every d dimensional object. This is called the *dimensional splitting*:

$$k^\mu\gamma_\mu = k^{[4]}\gamma^{[4]} + k^{[2\varepsilon]}\gamma^{[2\varepsilon]} = \tilde{k}_\mu\tilde{\gamma}^\mu + k^\perp_\mu\gamma^\mu_\perp.$$

With this splitting, we get

$$\{\tilde{\gamma}^\mu, \gamma^\nu_\perp\} = 0, \quad \{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\tilde{g}^{\mu\nu}, \quad \{\gamma^\mu_\perp, \gamma^\nu_\perp\} = 2g^{\mu\nu}_\perp, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

The other identities are also preserved:

$$\mathrm{Tr}(\gamma^5\gamma^\mu) = 0, \quad \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho) = 0, \quad \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\varepsilon^{\mu\nu\rho\sigma}.$$

We start again with the Feynman diagram for the loop diagram, this time in d dimensions.

$$\Delta^{\mu\nu\rho}(p_1, p_2) = C\int \frac{\mathrm{d}^d k}{(2\pi)^d}\left[\mathrm{Tr}\left(\gamma^\rho\gamma^5\frac{1}{\not{k}-\not{q}}\gamma^\mu\frac{1}{\not{k}-\not{p}_2}\gamma^\nu\frac{1}{\not{k}}\right) + \mathrm{Tr}\left(\gamma^\rho\gamma^5\frac{1}{\not{k}-\not{q}}\gamma^\nu\frac{1}{\not{k}-\not{p}_1}\gamma^\mu\frac{1}{\not{k}}\right)\right].$$

Call the first trace A and the second trace B. We write $\not{q}\gamma^5$ as

$$\not{q}\gamma^5 = \not{k}\gamma^5 - (\not{k}-\not{q})\gamma^5 = \not{k}\gamma^5 + \gamma^5(\tilde{\not{k}}-\not{q}) - \gamma^5\not{k}_\perp = \not{k}\gamma^5\gamma^5(\not{k}-\not{q}) - 2\gamma^5\not{k}_\perp.$$

For the trace A, we now get three terms:

$$A = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[\mathrm{Tr}\left(\gamma^5 \frac{1}{\not{k} - \not{q}} \gamma^\mu \frac{1}{\not{k} - \not{p}_2} \gamma^\nu \right) + \mathrm{Tr}\left(\gamma^5 \gamma^\mu \frac{1}{\not{k} - \not{p}_2} \gamma^\nu \frac{1}{\not{k}} \right) + \right.$$
$$\left. -2\,\mathrm{Tr}\left(\gamma^5 \not{k}_\perp \frac{1}{\not{k} - \not{q}} \gamma^\mu \frac{1}{\not{k} - \not{p}_2} \gamma^\nu \frac{1}{\not{k}} \right) \right].$$

We get a very similar expression for the trace B. We can take the same steps as before to cancel the first two terms of A with the first two terms of B because μ and ν are contracted by external 4-dimensional polarization vectors:

$$A + B = I + I(\mu \leftrightarrow \nu, p_1 \leftrightarrow p_2)$$

where

$$I = -2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}(\gamma^5 \not{k}_\perp (\not{k} - \not{q}) \gamma^\mu (\not{k} - \not{p}_1) \gamma^\nu \not{k})}{k^2 (k - p_1)^2 (k - q)^2}.$$

Then we split $\not{k} = \tilde{\not{k}} + \not{k}_\perp$ and use the same identities as before. There is also one additional useful identity (that we won't prove here):

$$\mathrm{Tr}(\Gamma_\perp \tilde{\Gamma}) = \frac{\mathrm{Tr}(\Gamma_\perp)\,\mathrm{Tr}(\tilde{\Gamma})}{\mathrm{Tr}(\mathbb{1})}.$$

With this, we obtain

$$\begin{aligned} I &= 2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}(\not{k}_\perp \not{k}_\perp)(-4i\varepsilon^{\rho\sigma\mu\nu} p_{1\rho} q_\sigma)}{\mathrm{Tr}(\mathbb{1}) k^2 (k - p_1)^2 (k - q)^2} \\ &= 8i\varepsilon^{\rho\sigma\mu\nu} p_{1\rho} p_{2\sigma} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_\perp^2}{k^2 (k - p_1)^2 (k - q)^2} \\ &= 8i\varepsilon^{\rho\sigma\mu\nu} p_{1\rho} p_{2\sigma} g_\perp^{\mu\nu} \underbrace{\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_\mu k_\nu}{k^2 (k - p_1)^2 (k - q)^2}}_{I^{\mu\nu}}. \end{aligned}$$

We perform the tensor decomposition of $I^{\mu\nu}$:

$$I^{\mu\nu} = C_0 g^{\mu\nu} + C_{11} \frac{p_1^\mu p_1^\nu}{q^2} + C_{12} \frac{p_1^\mu p_2^\nu}{q^2} + C_{21} \frac{p_2^\mu p_1^\nu}{q^2} + C_{22} \frac{p_2^\mu p_2^\nu}{q^2}.$$

To get the coefficients C_{ij}, we contract with $p_i^\mu p_j^\nu$. Actually, we only need C_0 for our computation, because all other terms will be zero when contracted with $\varepsilon^{\mu\nu\rho\sigma}$. It holds

$$C_0 = \frac{4}{2-d} \frac{1}{q^2} \int \frac{\mathrm{d}^d k}{(2\pi)^k} \frac{(k - p_1)(k - p_2)}{k^2 (k - p_1)^2 (k - q)^2} = \frac{2}{d-2} \frac{1}{4} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 (k - q)^2} = -\frac{2}{d-2} \frac{i}{16\pi^2} \frac{1}{\varepsilon} + \dots$$

The full term is

$$\begin{aligned} I &= 8i\varepsilon^{\rho\sigma\mu\nu} p_{1\rho} p_{2\sigma} g_{\mu\nu}^\perp g^{\mu\nu} \frac{2}{d-2} \frac{1}{4} \left(-\frac{i}{16\pi^2} \frac{1}{\varepsilon} + \dots \right) \\ &= 2i\varepsilon^{\rho\sigma\mu\nu} p_{1\rho} p_{2\sigma} (d-2) \frac{2}{d-2} \left(-\frac{i}{16\pi^2} \frac{1}{\varepsilon} + \dots \right) \\ &= -\frac{1}{4\pi^2} \varepsilon^{\rho\sigma\mu\nu} p_{1\rho} p_{2\sigma}. \end{aligned}$$

Collecting all the terms, the final result is

$$q_\rho \Delta^{\mu\nu\rho} \propto A + B = -\frac{1}{2\pi^2}\varepsilon^{\rho\sigma\mu\nu} p_{1\rho} p_{2\sigma} \neq 0.$$

Hence, the Ward identity for axial current is not satisfied. This result can also be written in a different way:

$$\langle p_1 p_2 | \partial_\mu J_A^\mu | \Omega \rangle = -\frac{1}{2\pi^2}\varepsilon^{\rho\sigma\mu\nu}(-ip_3)_\rho \epsilon_\mu(p_1)(-ip_2)_\sigma \epsilon_\nu(p_2) = -\frac{1}{16\pi^2}\langle p_1 p_2 | \varepsilon^{\rho\mu\sigma\nu} F_{\rho\mu} F_{\sigma\nu} | \Omega \rangle .$$

From here it is evident that the divergence of the axial current not zero:

$$\partial_\mu J_A^\mu = -\frac{1}{16\pi^2}\varepsilon^{\rho\mu\sigma\nu} F_{\rho\mu} F_{\sigma\nu}.$$

This is called the *Adler-Bell-Jackiw anomaly*. What Adler and Bardeem showed is that this anomaly is 1-loop exact. The exact calcualtion for vector currents shows (trust me) that the vector current *is* conserved.

Next, we will try to understand the chiral anomaly using the path integral. The generating functional is

$$Z[\mathcal{J}_n] = C \int \mathcal{D}[\phi_n] \exp\left(iS[\phi_n] + i \int \mathrm{d}^4x \; J_n \phi_n \right).$$

We introduce an infinitesimal transformation

$$\phi_n(x) \to \tilde{\phi}_n(x) = \phi_n(x) + \varepsilon \theta^a(x) F_n^a[\phi_{n'}(x)].$$

The generating functional is changed as follows

$$\tilde{Z}[\mathcal{J}_n] - Z[\mathcal{J}_n] = \int \mathcal{D}[\phi_n] \exp\left(iS[\phi_i] + i \int \mathrm{d}^4x \; \mathcal{J}_n \phi_n \right) \int \mathrm{d}^4x \; \theta^a(x) \left[\sum_j \frac{\delta F_j^a[\phi_{n'}]}{\delta \phi_j} + i \left(\frac{\delta S}{\delta \phi_j} + \mathcal{J}_j(x) \right) F_j^a[\phi_{n'}(x)] \right] =$$

The part $\delta F_j^a[\phi_{n'}]/\delta\phi_j$ come from the Jacobian of the functional integral:

$$\mathrm{Jac} = \left| \det \frac{\delta \phi'_{j'}(y)}{\phi_j(x)} \right| = \exp \mathrm{tr} \log \left(1 + \varepsilon \frac{\delta(\theta^a(y) F_{j'}^a[\phi_n(y)])}{\delta \phi_j(x)} \right) \approx \exp\left(\varepsilon \,\mathrm{tr} \left(\theta^a(y) \frac{\delta F_{j'}^a[\phi_n(y)]}{\delta \phi_j(x)} \right) \right) \approx 1 + \varepsilon \int \mathrm{d}^4x \; \theta^a(x) \sum_j$$

If we had a global symmetry ($\theta^a(x) = \theta^a$), the Noether' current would have the following divergence:

$$\partial^\mu J_\mu^a(x) = -\frac{\delta S}{\delta \phi_j(x)} F_j^a[\phi_n(x)].$$

However, if the symmetry is not global, we get

$$\partial^\mu J_\mu^a[\phi_n(c)] = \mathcal{J}_j F_j^a[\phi_n(x)] + \sum_j \frac{\delta F_j^a[\phi_n(x)]}{\delta \phi_j(x)}.$$

We say that the symmetry is *anomalous* if the last term is not zero,

$$\sum_j \frac{\delta F_j^a[\phi_n(x)]}{\delta \phi_j(x)} \neq 0.$$

Now, we will go back to chiral anomaly in massless QED. We will employ the *Fujikawa method*. The massless QED Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\Psi}i\not{D}\Psi.$$

The symmetry in question is the axial symmetry $\Psi \to \Psi' = e^{i\theta\gamma^5}\Psi$. To compute the Jacobian of the transformation, we will exapnd Ψ in a complete set of eigenstates of $\not{D}$ for fixed $A_\mu(x)$. We define the left- and right-eigenfunctions $(\phi_n, \overline{\phi}_n)$ such that

$$i\not{D}\phi_n = \lambda_n\phi_n, \quad \overline{\phi}_n i\not{D} = \lambda_n\overline{\phi}_n.$$

The expansion of Ψ in eigenstates is

$$\Psi(x) = \sum_n a_n\phi_n(x), \quad \overline{\Psi}(x) = \sum_n \overline{a}_n\overline{\phi}_n(x)$$

where a_n and $\overline{a}_n$ are complex Grasmmaninan numbers. The path integral measure becomes

$$\mathcal{D}\Psi\,\mathcal{D}\overline{\Psi} = \prod_n \mathrm{d}a_n\,\mathrm{d}\overline{a}_n.$$

The field transforms as

$$\Psi'(x) = \sum_n a'_n\phi_n(x) = (1 + i\theta\gamma^5)\sum_n a_n\phi_n.$$

We find a'_n by using the orthogonality relation

$$\int \mathrm{d}^4x\ \phi_m(x)\overline{\phi}_n(x) = \delta_{mn}.$$

We get

$$a'_m = a_m + i\sum_n \int \mathrm{d}^4x\ \theta(x)\overline{\phi}_m(x)\gamma^5\phi_n(x)a_n = \sum_n(\delta_{mn} + C_{mn})a_n = (\mathbb{1} + C)_{mn}a_n.$$

The transformation of the measure is

$$\mathcal{D}\Psi'\,\mathcal{D}\overline{\Psi}' = |\mathrm{Jac}|^{-2}\mathcal{D}\Psi\mathcal{D}\overline{\Psi},$$

where

$$\mathrm{Jac} = \det(\mathbb{1} + C) = \exp\mathrm{tr}\log(\mathbb{1} + C).$$

Using the infinitesimal expression for C, we get

$$\log(\mathrm{Jac}) = i\int \mathrm{d}^4x\ \theta(x)\sum_n \overline{\phi}_n(x)\gamma^5\phi_n(x).$$

The expression that we have consists of two problematic parts, $\mathrm{Tr}(\gamma^5) = 0$ and $\delta(x - x) = \infty$. In order for the product to behave well, we have to regulate the expression. An option for the gauge invariant regulator

$$\begin{aligned}\lim_{\Lambda\to\infty}\sum_n \overline{\phi}_n(x)\gamma^5\phi_n(x)e^{\lambda_n^2/\Lambda^2} &= \lim_{\Lambda\to\infty}\sum_n \overline{\phi}_n(x)\gamma^5 e^{(i\not{D})^2/\Lambda^2}\phi_n(x)\\ &= \lim_{\Lambda\to\infty}\langle x|\,\mathrm{tr}(\gamma^5 e^{(i\not{D})^2/\Lambda^2})|x\rangle\\ &= \lim_{\Lambda\to\infty}\int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{ikx}\,\mathrm{tr}(\gamma^5 e^{(i\not{D})^2/\Lambda^2})e^{-ikx}.\end{aligned}$$

Now we use

$$(i\not{\mathrm{D}})^2 = -\mathrm{D}^2 - i\sigma^{\mu\nu}[i\mathrm{D}_\mu, i\mathrm{D}_\nu] = -\mathrm{D}^2 + \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}.$$

After expanding the exponential, the first non-zero trace will come from the second order in the exapnsion (the zeroth and first order vanish, because $\mathrm{tr}(\gamma^5) = \mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$) and all higher terms will be supperssed by the regulator. We have

$$\begin{aligned}
\lim_{\Lambda\to\infty}\sum_n \overline{\phi}_n(x)\gamma^5\phi_n(x)e^{\lambda_n^2/\Lambda^2} &= \lim_{\lambda\to\infty}\int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{ikx}\,\mathrm{tr}\left[\gamma^5\frac{1}{2}\left(\frac{e}{2}\sigma^{\mu\nu}\frac{F_{\mu\nu}}{\Lambda^2}\right)^2\right]e^{-\partial^2/\Lambda^2}e^{-ikx}\\
&= \lim_{\lambda\to\infty}\left(\frac{e^2}{8\Lambda^2}\right)\left(\int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{k^2/\Lambda^2}\right)F^{\mu\nu}F^{\mu'\nu'}\,\mathrm{tr}(\gamma^5\sigma_{\mu\nu}\sigma_{\mu'\nu'})\\
&= \frac{i}{16\pi^2}\frac{e^2}{8}i^2\,\mathrm{tr}(\gamma^5\gamma_\mu\gamma_\nu\gamma_{\mu'}\gamma_{\nu'})F^{\mu\nu}F^{\mu'\nu'}\\
&= \frac{e^2}{32\pi^2}\varepsilon^{\mu\nu\mu'\nu'}F_{\mu\nu}F_{\mu'\nu'}.
\end{aligned}$$

The Jacobian of the transformation is

$$\mathrm{Jac} = \exp\left(-i\int\mathrm{d}^4x\;\theta(x)\frac{e^2}{32}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right).$$

The transformed generating functional becomes

$$Z' = \int\mathcal{D}\Psi\mathcal{D}\overline{\Psi}\;\exp\left[i\int\mathrm{d}^4x\;\overline{\Psi}i\not{\partial}\Psi + \theta\left(\partial_\mu j^{5,\mu} + \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right)\right].$$

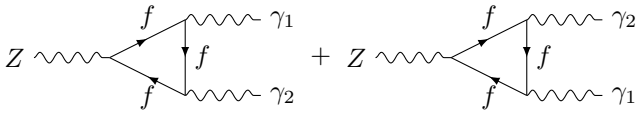
After the variation with respect to θ we find

$$\partial_\mu j^{5,\mu} = -\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \neq 0.$$

If we interpret this anomaly as a term in the Lagrangian

$$\delta\mathcal{L} \propto \theta(x)\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

it can expain the decay of the neutral pion to two pions. An important point is that anomaies break the gauge invarianve. This means that we really want our gauge symmetries to be anomaly-free. Many proofs depend on the fact that that the gauge currents are conserved: renormalizability depends on this and the decoupling of ghosts as well. The standard model is chiral, so should we expect anomalies? The answer is yes, for individual particles. But when all particles are collected togehter, the anomalies do indeed cancel. This requirement also gives as a strong constraint to how each generation of particles is constructed. Let's consider the $Z \to \gamma\gamma$ decay.



As we know, the Z boson couples to the fermions as $g_V\gamma^\mu + g_A\gamma^\mu\gamma^5$. We can ignore the vector part, since this part vanishes by *Furry's theorem*. The photon couplings to the loop are

$$V_\gamma^\mu = \sum_f Q_f\overline{\Psi}_f\gamma^\mu\Psi_f$$

and the nonzero contribution of the Z vertex is

$$V_Z^\mu = \propto \sum_f I_f \overline{\Psi}_f \gamma^\mu \gamma^5 \Psi_f,$$

where I_f is the third component of the weak isospin. Summing over all possible fermions, we get that the anomaly of $Z \to \gamma\gamma$ diagram has to be proportional to

$$\sum_f I_f Q_f^2 N_{c,f}$$

where $N_{c,f}$ is the number of colors of the fermion f (3 for quarks and 1 for leptons). The total anomaly for this diagram is thus

$$\underbrace{-\frac{1}{2}\cdot(-1)^2\cdot 1}_{e_\mathrm{L}} + \underbrace{\frac{1}{2}\cdot(2/3)^2\cdot 3}_{u_\mathrm{L}} - \underbrace{\frac{1}{2}\cdot(-1/3)^2\cdot 3}_{d_\mathrm{L}} = 0.$$

Fortunately, the anomaly cancels. We could also add other generations and the anomaly would still cancel within every generation. We are not away of the danger yet. There are nine other diagrams that bring anomalies, and one has to check all of them. Fortunately, all diagrams are anomaly-free and hence the entire electroweak theory is anomaly-free.

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