

Faculty of physics

General relativity

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Disclaimer: these notes are not official and were not endorsed by professor. The content is almost entirely based on the lectures, but occasionally the notes were modified or extended slightly to improve clarity or add extra information. In particular, the mistakes are almost surely mine. If you find any, please let me know at simon.bukovsek@tum.de or Simon.Bukovsek@campus.lmu.de.

Invitation

There are four fundamental forces: *electromagnetic*, *strong*, *weak* and *gravitational*. Particles are described as excitations of fields. Each particle is characterized by three numbers: mass, charge and spin.

Let's first take a look at an example in electromagnetism. Coulomb's law gives the force between two charges

$$\mathbf{F} = -\frac{q_1 q_2}{r^2}\frac{\mathbf{r}}{|r|}.$$

The electric field due to the first charge is then defined as $\mathbf{E} = \mathbf{F}/q_2$. Now, the equation of motion in an electric field is given by the Newton's second law

$$m^{\text{in}}\mathbf{a} = q\mathbf{E}.$$

Note that the mass m^{in} is the *inertial* mass. The acceleration is $\mathbf{a} = e\mathbf{E}/m^{\text{in}}$. In gravitational field $\mathbf{g}$ the following holds:

$$m^{\text{gr}}\mathbf{g} = m^{\text{in}}\mathbf{a}.$$

The mass m^{gr} is the *gravitational* mass. It so happens that $m^{\text{gr}} = m^{\text{in}}$. The first to show that was Galileo.

Assume now particles $1, 2, ..., N$ in gravitational field $\mathbf{g}$. The equation of motion for k-th particle is

$$m_k^{\text{in}}\frac{\mathrm{d}^2\mathbf{x}_k}{\mathrm{d}t^2} = m_k^{\text{gr}}\mathbf{g} + \sum_{i=1}^{N}\mathbf{F}(\mathbf{x}_k - \mathbf{x}_i),$$

where the sum skips $i = k$. Let's see what happens if we introduce a coordinate change $x, t \to x', t'$ with

$$\mathbf{x}' = \mathbf{x} - \frac{1}{2}\mathbf{g}t^2 \quad \text{and} \quad t' = t,$$

assuming $\mathbf{g}$ is constant. The equation of motion in the new coordinates is

$$m_k^{\text{in}}\frac{\mathrm{d}^2\mathbf{x}'_k}{\mathrm{d}t'^2} = (m_k^{\text{gr}} - m_k^{\text{in}})\sum_{i=1}^{N}\mathbf{F}(\mathbf{x}'_k - \mathbf{x}'_i) = \sum_{i=1}^{N}\mathbf{F}(\mathbf{x}'_k - \mathbf{x}'_i).$$

This tells us that free-falling bodies in a gravitational field are actually in an inertial frame of reference. This is called the equivalence principle. But the professor doesn't really like it.

Chapter 1

Differential geometry

All geometrical objects (vectors, tensors, forms) will be written in bold from this section.

1.1 Manifolds

A *manifold* is a collection of points. An n-dimensional manifold, each point can be described by n coordinates $\xi^1, \xi^2, \dots \xi^n$. The coordinate system M can be arbitrarily complicated. The coordinates can be written in condensed notation

$$(\xi^1, \xi^2, \dots, \xi^n)_M = \xi^\alpha.$$

We can also have a function x valued on a manifold, which can be written as $x^\alpha(\xi^\beta)$. Indices α and β run from 1 to n. Such a function can be thought of as a change of coordinates. We will mostly deal with infinitely-differentiable functions with inverses $\xi^\alpha(x^\beta)$. If we have two coordinate transformations $x(\xi)$ and $\tilde{x}(\xi)$, then there is also a transformation $\tilde{x}(x)$.

A *scalar* is a function that doesn't change under coordinate transformations. The notation is $f(M) = f(x_M) = f(\tilde{x}_M)$. A transformation would look like $f(\xi^\alpha) = f(\xi^\alpha(x^\beta)) = \tilde{f}(x^\beta)$. A *curve* $\lambda(t)$ can be parametrized as

$$\xi^\alpha_{\lambda(t)} \equiv \xi^\alpha(\lambda(t)) = \xi^\alpha(t).$$

Let there be points M and N on the curve at parameters t_0 and $t_0 + \Delta t$. Denote

$$\xi^\alpha_{MN} = \xi^\alpha(\lambda(t_0 + \Delta t)) - \xi^\alpha(\lambda(t_0)) = \xi^\alpha(t_0 + \Delta t) - \xi^\alpha(t_0).$$

The scalar function f can be written as

$$\begin{aligned}
\Delta f_{MN} &= f_N - f_M \\
&= f(\xi^\alpha(t_0 + \Delta t)) - f(\xi^\alpha(t_0)) \\
&= \sum_{\alpha=1}^{n} \left.\frac{\mathrm{d}\xi^\alpha(t)}{\mathrm{d}t}\right|_{t=t_0} \left.\frac{\partial f}{\partial \xi^\alpha}\right|_M \Delta t + \mathcal{O}(\Delta t^2) \\
&= \left(\frac{\mathrm{d}\xi^\alpha(t)}{\mathrm{d}t}\frac{\partial}{\partial \xi^\alpha}\right)_{\lambda(t_0)} f(\xi^\alpha)\Delta t.
\end{aligned}$$

Here, the *Einstein summation convention* is used:

$$\sum_{\alpha=1}^{n} \frac{\mathrm{d}\xi^\alpha(t)}{\mathrm{d}t}\frac{\partial}{\partial\xi^\alpha} \equiv \frac{\mathrm{d}\xi^\alpha(t)}{\mathrm{d}t}\frac{\partial}{\partial\xi^\alpha}.$$

The time derivative is thus rewritten as

$$\mathbf{X} = \left(\frac{\partial}{\partial t}\right)_{\lambda(t_0)} = \left(\frac{\mathrm{d}\xi^\alpha(t)}{\mathrm{d}t}\frac{\partial}{\partial\xi^\alpha}\right)_{t_0} = {}^{(\xi)}X^\alpha \mathbf{E}_\alpha.$$

Two new symbols were used:

$$\mathbf{E}_\alpha \equiv \left.\frac{\partial}{\partial\xi^\alpha}\right|_{\lambda(t_0)} \quad \text{and} \quad {}^{(\xi)}X^\alpha = \left.\frac{\mathrm{d}\xi^\alpha(t)}{\mathrm{d}t}\right|_{t_0}.$$

If there are two different coordinate systems $x^\alpha(\xi)$ and $\tilde{x}^\alpha(\xi)$, then

$$\mathbf{X} = \left.\frac{\mathrm{d}x^\alpha(t)}{\mathrm{d}t}\right|_{t_0}\frac{\partial}{\partial x^\alpha} = \left.\frac{\mathrm{d}\tilde{x}^\beta}{\mathrm{d}t}\right|_{t_0}\frac{\partial}{\partial\tilde{x}^\beta}.$$

Introducing $\mathbf{e}_\alpha = \partial/\partial x^\alpha$ and $\tilde{\mathbf{e}}_\beta = \partial/\partial\tilde{x}^\beta$ (unit vectors), the above equation can be written as

$$\mathbf{X} = X^\alpha\mathbf{e}_\alpha = \tilde{X}^\beta\tilde{\mathbf{e}}_\beta.$$

The transformation between the two coordinate systems is then

$$\mathbf{e}_\alpha = \frac{\partial}{\partial x^\alpha} = \frac{\partial\tilde{x}^\beta}{\partial x^\alpha}\frac{\partial}{\partial\tilde{x}^\beta} = \frac{\partial\tilde{x}^\beta}{\partial x^\alpha}\mathbf{e}_\beta$$

and hence

$$\tilde{x}^\beta = \frac{\partial\tilde{x}^\beta}{\partial x^\alpha}x^\alpha.$$

All vectors at a point M form a linear space called *tangent space* T_M. We get a number when we apply $\mathbf{X}$ on a scalar function f. When applied to a vector function, is there an object that gives a scalar? Yes, it is called a 1-form or a *covector* (usually denoted by $\boldsymbol{\eta}$). It is linear

$$\boldsymbol{\eta}(\alpha\mathbf{X} + \beta\mathbf{Y}) = \alpha\boldsymbol{\eta}(\mathbf{X}) + \beta\boldsymbol{\eta}(\mathbf{X}).$$

It can also be denoted as $\boldsymbol{\eta}(\mathbf{X}) = \langle\boldsymbol{\eta}, \mathbf{X}\rangle$. From the scalar product, it is more obvious that the result is scalar and that $\langle\boldsymbol{\eta}, \mathbf{X}\rangle = \langle\mathbf{X}, \boldsymbol{\eta}\rangle$. Furthermore, we introduce a basis $\boldsymbol{\omega}^\alpha$ of 1-forms, such that

$$\langle\boldsymbol{\omega}^\alpha, \mathbf{X}\rangle = \langle\boldsymbol{\omega}^\alpha, X^\beta\mathbf{e}_\beta\rangle = X^\beta\langle\boldsymbol{\omega}^\alpha, \mathbf{e}_\beta\rangle,$$

hence

$$\boldsymbol{\omega}^\alpha(\mathbf{e}_\beta) = \langle\boldsymbol{\omega}^\alpha, \mathbf{e}_\beta\rangle = \delta^\alpha{}_\beta.$$

As a consequence, it holds $\boldsymbol{\eta} = \eta_\alpha\boldsymbol{\omega}^\alpha$. The basis $\boldsymbol{\omega}^\alpha$ is called the *dual basis* to $\mathbf{e}_\alpha$. We introduce another notation:

$$\mathbf{d}f(\mathbf{X}) = \mathbf{X}(f).$$

Now

$$\langle\boldsymbol{\omega}, \mathbf{e}_\beta\rangle = \langle\mathbf{d}x^\alpha, \partial/\partial x^\beta\rangle = \frac{\partial}{\partial x^\beta}x^\alpha = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta.$$

For the coordinate transformation $x^\alpha \to \tilde{x}^\alpha(x^\beta)$ it holds

$$\mathbf{d}x^\alpha = \frac{\partial x^\alpha}{\partial \tilde{x}^\beta}\mathbf{d}\tilde{x}^\beta$$

and

$$\boldsymbol{\eta} = \eta_\alpha \mathbf{d}x^\alpha = \tilde{\eta}_\beta \mathbf{d}\tilde{x}^\beta.$$

The transformation between 1-forms is then

$$\tilde{\eta}_\beta = \frac{\partial x^\alpha}{\partial \tilde{x}^\beta}\eta_\alpha.$$

The transformation matrix is exactly the inverse of the transformation matrix for vectors. Vectors are elements of tangent space $\mathbf{X} \in T_M$, and 1-forms (or *covectors*) are elements of the *cotangent space* T_M^*. The cotangent space is the dual space to the tangent space.

1.2 Tensors

Define a *tensor* of rank $(1,1)$ as the map $\mathbf{T} : \Pi_1^1 \to \mathbb{R}$, where

$$\Pi_1^1 = T_M^* \times T_M \ni (\boldsymbol{\eta}, \mathbf{X}).$$

They "eat" a vector and a covector and produce a number:

$$\mathbf{T}(\text{covector}, \text{vector}) = \text{scalar}.$$

A tensor is a bilinear map, which means it is linear in both arguments:

$$\mathbf{T}(\alpha\boldsymbol{\eta} + \beta\boldsymbol{\eta}', \gamma\mathbf{X} + \delta\mathbf{X}') = \alpha\gamma\mathbf{T}(\boldsymbol{\eta}, \mathbf{X}) + \alpha\delta\mathbf{T}(\boldsymbol{\eta}, \mathbf{X}') + \beta\gamma\mathbf{T}(\boldsymbol{\eta}', \mathbf{X}) + \beta\delta\mathbf{T}(\boldsymbol{\eta}', \mathbf{X}').$$

In local coordinates, this reads

$$\mathbf{T}(\boldsymbol{\eta}, \mathbf{X}) = \mathbf{T}(\eta_\alpha \boldsymbol{\omega}^\alpha, X^\beta \mathbf{e}_\beta) = \eta_\alpha X^\beta \mathbf{T}(\boldsymbol{\omega}^\alpha, \mathbf{e}_\beta) = \eta_\alpha X^\beta T^\alpha{}_\beta.$$

Now we introduce the basis of a tensor space $\mathbf{e}_\alpha \otimes \boldsymbol{\omega}^\beta$ such that $(\mathbf{e}_\alpha \odot \boldsymbol{\omega}^\beta)(\boldsymbol{\eta}, \mathbf{X}) = \eta_\alpha X^\beta$. So, a general tensor can be written as

$$\mathbf{T}(\boldsymbol{\eta}, \mathbf{X}) = \eta_\alpha X^\beta T^\alpha{}_\beta = \mathbf{e}_\alpha \otimes \boldsymbol{\omega}^\beta(\boldsymbol{\eta}, \mathbf{X}) T^\alpha{}_\beta = T^\alpha{}_\beta \mathbf{e}_\alpha \otimes \boldsymbol{\omega}^\beta(\boldsymbol{\eta}, \mathbf{X}).$$

From this it follows that $\mathbf{T} = T^\alpha{}_\beta \mathbf{e}_\alpha \otimes \boldsymbol{\omega}^\beta$. The coordinate changes are given as

$$\mathbf{T} = T^\alpha{}_\beta \mathbf{e}_\alpha \otimes \boldsymbol{\omega}^\beta = \tilde{T}^\gamma{}_\delta \tilde{\mathbf{e}}_\gamma \otimes \tilde{\boldsymbol{\omega}}^\delta.$$

With the known transformations for $\mathbf{e}$ and for $\boldsymbol{\omega}$ we can write the transformation of the whole tensor:

$$\mathbf{T} = T^\alpha{}_\beta \mathbf{e}_\alpha \otimes \boldsymbol{\omega}^\beta = T^\alpha{}_\beta \frac{\partial \tilde{x}^\gamma}{\partial x^\gamma}\tilde{\mathbf{e}}_\alpha \otimes \frac{\partial x^\beta}{\partial \tilde{x}^\delta}\boldsymbol{\omega}^\delta = T^\alpha{}_\beta \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^\delta}\mathbf{e}_\gamma \otimes \boldsymbol{\omega}^\delta.$$

Hence,

$$T^\gamma{}_\delta = T^\alpha{}_\beta \frac{\partial x^\beta}{\partial \tilde{x}^\delta}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}.$$

Now define a general tensor of rank (s, r) as the map $\mathbf{T} : \Pi_s^r \to \mathbb{R}$, where

$$\Pi_s^r = \underbrace{T_M^* \times \cdots \times T_M^*}_{s \text{ times}} \times \underbrace{T_M \times \cdots \times T_M}_{r \text{ times}}.$$

They "eat" r vectors and s covectors and produce numbers. If we write ${}^{(1)}\boldsymbol{\eta} = {}^{(i)}\eta_{\alpha_i}\boldsymbol{\eta}^{\alpha_i}$ and ${}^{(i)}\mathbf{X} = {}^{(1)}X^{\beta_i}\boldsymbol{\eta}_{\beta_i}$. The tensor written in components is

$$\begin{aligned}\mathbf{T}({}^{(1)}\boldsymbol{\eta},\dots,{}^{(1)}\boldsymbol{\eta},{}^{(1)}\mathbf{X},\dots,{}^{(1)}\mathbf{X}) &= {}^{(1)}\eta_{\alpha_1}\dots{}^{(s)}\eta_{\alpha_s}\,{}^{(1)}X^{\beta_1}\dots{}^{(r)}X^{\beta_r}\mathbf{T}(\boldsymbol{\omega}^{\alpha_1},\dots,\boldsymbol{\omega}^{\alpha_s},\mathbf{e}_{\beta_1},\dots,\mathbf{e}_{\beta_r})\\ &= {}^{(1)}\eta_{\alpha_1}\dots{}^{(s)}\eta_{\alpha_s}\,{}^{(1)}X^{\beta_1}\dots{}^{(r)}X^{\beta_r}T^{\alpha_1\dots\alpha_s}{}_{\beta_1\dots\beta_r}.\end{aligned}$$

The transformation of the components goes as

$$\tilde{T}^{\alpha'_1\dots\alpha'_s}{}_{\beta'_1\dots\beta'_r} = \frac{\partial\tilde{x}^{\alpha'_1}}{\partial x^{\alpha_1}}\dots\frac{\partial\tilde{x}^{\alpha'_s}}{\partial x^{\alpha_s}}\frac{\partial x^{\beta_1}}{\partial\tilde{x}^{\beta'_1}}\dots\frac{\partial x^{\beta_r}}{\partial\tilde{x}^{\beta'_r}}T^{\alpha_1\dots\alpha_s}{}_{\beta_1\dots\beta_r}.$$

We can express the tensor as a sum of basis tensors (the sum is implicit)

$$\mathbf{T} = T^{\alpha_1\dots\alpha_s}{}_{\beta_1\dots\beta_r}\,\mathbf{e}_{\alpha_1}\otimes\dots\otimes\mathbf{e}_{\alpha_s}\otimes\boldsymbol{\eta}^{\beta_1}\otimes\dots\otimes\boldsymbol{\eta}^{\beta_r}.$$

Such tensor has n^{r+s} unit tensors and components. For example, consider a tensor:

$$\mathbf{T}(\text{covector},\text{vector},\text{covector}) \longrightarrow T^{\alpha}{}_{\beta}{}^{\gamma}\,\mathbf{e}_\alpha\otimes\boldsymbol{\omega}^\beta\otimes\mathbf{e}_\gamma.$$

A vector is a tensor of rank $(1,0)$, and a covector is a tensor of rank $(0,1)$. What can we do with tensors?

1. We can add them. Two tensors T, S of rank (s,r) add together as
$$(T\pm S)^{\alpha_1\dots\alpha_s}{}_{\beta_1\dots\beta_r} = T^{\alpha_1\dots\alpha_s}{}_{\beta_1\dots\beta_r} \pm S^{\alpha_1\dots\alpha_s}{}_{\beta_1\dots\beta_r}.$$

2. We can multiply them by a scalar. A tensor T of rank (s,r) and a scalar a multiply as
$$(aT)^{\alpha_1\dots\alpha_s}{}_{\beta_1\dots\beta_r} = aT^{\alpha_1\dots\alpha_s}{}_{\beta_1\dots\beta_r}.$$

3. We can multiply them. A tensor T of rank (s,r) and a tensor S of rank (s',r') multiply as
$$(T\cdot S)^{\alpha_1\dots\alpha_s\alpha'_1\dots\alpha'_s}{}_{\beta_1\dots\beta_s\beta'_1\dots\beta'_s} = T^{\alpha_1\dots\alpha_s}{}_{\beta_1\dots\beta_r}\cdot S^{\alpha'_1\dots\alpha'_s}{}_{\beta'_1\dots\beta'_r}.$$

4. We can contract them. A tensor T of rank $(s+1,r+1)$ contracts to a tensor S of rank (s,r):
$$T^{\alpha_1\dots\psi\dots\alpha_s}{}_{\beta_1\dots\psi\dots\beta_r} = S^{\alpha_1\dots\alpha_s}{}_{\beta_1\dots\beta_r}.$$

Remark. Generally, a rank $(2,0)$ tensor cannot be constructed as a product of two rank-$(1,0)$ tensors (vectors) – the number of degrees of freedom doesn't match.

Let's summarize the discussion about tensors to clear up any remaining confusion. A rank (p,q) tensor is a map that maps p covectors and q vectors to a scalar. The tensor itself is a p-times vector and q-times covector (the numbers are exactly the opposite since a vector must act on a covector to produce a scalar and vice versa). A tensor component has p upper indices and q lower indices.

1.3 Metric tensor

Imagine two points in an n-dimensional space with Euclidean coordinates ξ. Let the points A and B have coordinates ξ^α and ξ^β, respectively. The distance between the points is

$$(\Delta s)^2 = (\Delta\xi^1)^2 + \cdots + (\Delta\xi^n)^2 = \delta_{\alpha\beta}\Delta\xi^\alpha\Delta\xi\beta.$$

If we do a coordinate transformation $\xi^\alpha = x^\alpha(\xi^\gamma)$ with the inverse $\xi^\alpha = \xi^\alpha(x^\beta)$. Let A have coordinates x^α and B have coordinates $x^\alpha + \Delta x^\alpha$. The distance between the points is

$$(\Delta s)^2 = \delta_{\alpha\beta}\Delta\xi^\alpha\Delta\xi^\beta = \delta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\gamma}\frac{\partial\xi^\beta}{\partial x^\delta}\Delta x^\gamma\Delta x^\delta + \mathcal{O}(\Delta x^3).$$

We want to write

$$(\Delta s)^2 = g_{\gamma\delta}\Delta x^\gamma\Delta x^\delta + \mathcal{O}(\Delta x^3)$$

or even simpler

$$\mathrm{d}s^2 = g_{\gamma\delta}\mathrm{d}x^\gamma\,\mathrm{d}x^\delta.$$

Hence, we define a *metric tensor*:

$$g_{\gamma\delta} = \delta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\gamma}\frac{\partial\xi^\beta}{\partial x^\delta}.$$

It is left as an exercise to the reader to prove that $g_{\gamma\delta}$ is a rank $(0,2)$ tensor, i.e.

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta = \tilde{g}_{\gamma\delta}\mathrm{d}\tilde{x}^\gamma\mathrm{d}\tilde{x}^\delta, \quad g_{\alpha\beta} = \frac{\partial\tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial\tilde{x}^\delta}{\partial x^\beta}\tilde{g}_{\gamma\delta}.$$

Metric tensor is also symmetric: $g_{\alpha\beta} = g_{\beta\alpha}$. In one dimension, the metric tensor is just a number $g_{00} = 1$, so there cannot be any curvature. In two dimensions, the metric tensor is

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix}.$$

It has only 3 degrees of freedom, as $g_{01} = g_{10}$. In the n dimensional space, there are $n(n+1)/2$ different metric tensor components. We can write

$$\mathbf{g} = g_{\alpha\beta}\boldsymbol{\omega}^\alpha\boldsymbol{\omega}^\beta = g_{\alpha\beta}\mathbf{d}x^\alpha\mathbf{d}x^\beta.$$

If we write the vector between A and B as $\mathbf{X}_{AB} = \Delta X^\alpha\mathbf{e}_\alpha$, the distance between the points is

$$\Delta s^2 = \mathbf{g}(\Delta X^\alpha\mathbf{e}_\alpha, \Delta X^\beta\mathbf{e}_\beta) = \mathbf{g}(\mathbf{e_\alpha}, \mathbf{e_\beta})\Delta X^\alpha\Delta x^\beta = g_{\alpha\beta}\Delta X^\alpha\Delta x^\beta.$$

The scalar product is defined as

$$\mathbf{X}\cdot\mathbf{Y} = \mathbf{g}(\mathbf{X},\mathbf{Y}).$$

There is a correspondence between each covector $\boldsymbol{\eta}$ and certain vector $\mathbf{X}_{\boldsymbol{\eta}}$, such that

$$\boldsymbol{\eta}(\mathbf{Y}) = \langle\boldsymbol{\eta},\mathbf{Y}\rangle = \mathbf{X}_{\boldsymbol{\eta}}\cdot\mathbf{Y}$$

for all $\mathbf{Y}$. In the coordinate representation, the product is

$$\eta_\alpha Y^\alpha = g_{\alpha\beta}X^\beta_\eta Y^\alpha,$$

so we define

$$\eta_\alpha = g_{\alpha\beta}X^\beta_\eta \equiv x_\alpha.$$

The component of a vector with an upper index is called *contravariant*, and the component of a covector with a lower index is called *covariant*. Using the metric tensor, we can raise and lower indices:

$$T^{\alpha}{}_{\beta} = g^{\alpha\sigma} T_{\sigma\beta}.$$

There is also the inverse metric $g^{\alpha\beta}$, which is defined such that

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}.$$

Due to the action of rising and lowering indices, we can do all the proceeding calculations using just the index notation.

1.4 Parallel transport

Let's consider two vectors at slightly different positions: $\mathbf{A}(x_0)$ and $\mathbf{A}(x_0 + \Delta x)$. Using the consideration of the previous section, we can uniquely represent with the covariant or contravariant components A_α or A^α. The problem in curved space is that the expression $\Delta A_\alpha = A_\alpha(x_0 + \Delta x) - A_\alpha(x_0)$ doesn't tell the difference between two vectors. The only way to compare the two vectors is to parallel transport one to the other. Let's denote the difference between the two vectors as $\mathrm{D}\mathbf{A}$, so $\mathrm{D}\mathbf{A} = \mathbf{A}(x_0 + \Delta x) - \mathbf{A}^{\mathrm{PT}}(x_0 + \Delta x)$, where $A^{\mathrm{PT}}_\alpha(x_0 + \Delta x) = A_\alpha + \Delta A_\alpha$. However, in ξ coordinate system, the representations of $\mathbf{A}^{\mathrm{PT}}(x_0 + \Delta x)$ and $\mathbf{A}(x_0)$ should be the same ${}^{\xi}A_\alpha$. The representation in the x coordinate system is then

$$A_\alpha(x_0) = \left.\frac{\partial \xi^\beta}{\partial x^\alpha}\right|_{x_0} {}^{\xi}A_\beta.$$

Writing the same for the parallel transported vector, we get

$$\begin{aligned}
A^{\mathrm{PT}}_\alpha(x_0) &= \left.\frac{\partial \xi^\beta}{\partial x^\alpha}\right|_{x_0+\Delta x} {}^{\xi}A_\beta \\
&= \left(\left.\frac{\partial \xi^\beta}{\partial x^\alpha}\right|_{x_0} + \frac{\partial^2 \xi^\beta}{\partial x^\alpha \partial x^\gamma} \Delta x^\gamma \right) {}^{\xi}A_\beta \\
&= A_\alpha(x_0) + \frac{\partial^2 \xi^\beta}{\partial x^\alpha \partial x^\gamma} \frac{\partial x^\delta}{\partial \xi^\beta} A_\delta(x_0) \Delta x^\gamma \\
&\equiv A_\alpha(x_0) + \Gamma^{\delta}{}_{\alpha\gamma} A_\delta(x_0) \Delta x^\gamma.
\end{aligned}$$

The quantity $\Gamma^{\delta}{}_{\alpha\gamma}$ is called the *Christoffel symbol* and is defined as

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{\partial^2 \xi^\delta}{\partial x^\beta \partial x^\gamma} \frac{\partial x^\alpha}{\partial \xi^\delta}.$$

It has 40 independent components in 4 dimensions. The coordinate transformation of the Christoffel symbol is

$$\tilde{\Gamma}^{\alpha}{}_{\beta\gamma} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\beta} \frac{\partial x^\sigma}{\partial \tilde{x}^\gamma} \Gamma^{\mu}{}_{\rho\sigma} + \frac{\partial^2 x^\delta}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma} \frac{\partial \tilde{x}^\alpha}{\partial x^\delta}.$$

So we see that the Christoffel symbol is *not* a tensor. The difference $\mathrm{D}\mathbf{A} = \mathbf{A}(x_0 + \Delta x) - \mathbf{A}^{\mathrm{PT}}(x_0 + \Delta x)$ can be expressed in the x coordinate system as

$$\mathrm{D}A_\alpha = A_\alpha(x_0 + \Delta x) - A_\alpha(x_0) - \Gamma^{\delta}{}_{\alpha\beta} A_\delta(x_0) \Delta x^\beta.$$

In the limit, when $\Delta x \to 0$, the difference ΔA_α becomes

$$\lim_{\Delta x^\beta \to 0} \frac{\mathrm{D}A_\alpha}{\Delta x^\beta} = \frac{\partial A_\alpha}{\partial x^\beta} - \Gamma^\delta{}_{\alpha\beta} A_\delta,$$

which is a tensor, because it transforms a vector Δx^β to a vector $\mathrm{D}A_\alpha$. To simplify the notation, we introduce the *covariant derivative* of a covariant vector A_α as

$$A_{\alpha;\beta} = \partial_\beta A_\alpha - \Gamma^\delta{}_{\alpha\beta} A_\delta = A_{\alpha,\beta} - \Gamma^\delta{}_{\alpha\beta} A_\delta.$$

If we have a curve $\lambda(s)$, how does the vector change on this curve? The change of the vector is given by

$$\frac{\mathrm{D}A_\alpha}{\mathrm{d}s} = A_{\alpha;\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}s}.$$

The same can be done for a contravariant vector A^α:

$$A^\alpha{}_{;\beta} = A^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\gamma} A^\gamma$$

and

$$\frac{\mathrm{D}A^\alpha}{\mathrm{d}s} = A^\alpha{}_{;\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}s}.$$

Now that we have found a covariant derivative for a vector and a covector, we can find it for a tensor $T^\alpha{}_\beta$. The easier case would be if we can assume that the tensor is a product of a covector and a vector, so $T^\alpha{}_\beta = A^\alpha B_\beta$ (which is certainly not always the case). We write $T^\alpha{}_\beta{}^{\mathrm{PT}}(x_0 + \Delta x) = T^\alpha{}_\beta(x_0) + \delta T^\alpha{}_\beta$. Furthermore,

$$\begin{aligned}
\delta T^\alpha{}_\beta &= (\delta X^\alpha) Y_\beta + X^\alpha (\delta Y_\beta) \\
&= -\Gamma^\alpha{}_{\gamma\delta} X^\gamma \Delta x^\delta Y_\beta + \Gamma^\gamma{}_{\beta\delta} \Delta x^\delta Y_\gamma X^\alpha \\
&= (T^\alpha{}_\gamma \Gamma^\gamma{}_{\beta\delta} - T^\gamma{}_\beta \Gamma^\alpha{}_{\gamma\delta}) \Delta x^\delta.
\end{aligned}$$

Adopting the same notation as before, $\mathrm{D}T^\alpha{}_\beta = T^\alpha{}_\beta(x_0 + \Delta x) - T^\alpha{}_\beta{}^{\mathrm{PT}}(x_0 + \Delta x) \equiv T^\alpha{}_{\beta;\gamma} \Delta x^\gamma$, we define

$$T^\alpha{}_{\beta;\gamma} = T^\alpha{}_{\beta,\gamma} - T^\alpha{}_\delta \Gamma^\delta{}_{\beta\gamma} + T^\delta{}_\beta \Gamma^\alpha{}_{\delta\gamma}.$$

Using this convenient notation, the derivative of a tensor along a curve is

$$\frac{\mathrm{D}T^\alpha{}_\beta}{\mathrm{d}s} = T^\alpha{}_{\beta;\gamma} \frac{\mathrm{d}x^\gamma}{\mathrm{d}s}$$

and the derivative is distributed over the product

$$(A^\alpha B_\beta)_{;\gamma} = A^\alpha{}_{;\gamma} B_\beta + A^\alpha B_{\beta;\gamma}.$$

Since the Christoffel symbol is not a tensor, rising and lowering indices don't work as usual; however, it works for the first index:

$$\Gamma_{\alpha\beta\gamma} = g_{\alpha\delta} \Gamma^\delta{}_{\beta\gamma}.$$

1.5 Local internal derivative

The goal is to locally find a coordinate system where $\Gamma^{\alpha}{}_{\beta\gamma}(x_0)$ disappears. There is such coordinate transform only when $\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta}$. Remember that Γ only vanishes at $\mathcal{O}$. Let's consider a transformation $x^\alpha \to \tilde{x}^\alpha$, such that

$$\tilde{x}^\alpha \equiv \xi^\alpha = (x^\alpha - x_0^\alpha) + \frac{1}{2}\Gamma^{\alpha}{}_{\beta\gamma}(x^\beta - x_0^\beta)(x^\gamma - x_0^\gamma).$$

It is an exercise to show that $(\tilde{\Gamma}^{\alpha}{}_{\beta\gamma})_{\mathcal{O}} = ({}^{\xi}\Gamma^{\alpha}{}_{\beta\gamma})_{\mathcal{O}} = 0$ and hence that the components of the tensor at $\mathcal{O}$ don't change $\frac{\partial \xi^\alpha}{\partial x^\beta}\big|_{\mathcal{O}} = \delta^{\alpha}{}_{\beta}$. Introducing the coordinate system $\tilde{\xi}^\alpha = a^{\alpha}{}_{\beta}\xi^\beta$ with the inverse $\xi^\beta = b^{\beta}{}_{\alpha}\tilde{\xi}^\alpha$, we can write the transformation of the metric tensor as

$$\tilde{g}_{\alpha\beta}(\mathcal{O}) = \frac{\partial \xi^\mu}{\partial \tilde{\xi}^\alpha}\frac{\partial \xi^\nu}{\partial \tilde{\xi}^\beta} g_{\alpha\beta}(\mathcal{O}).$$

The metric tensor in flat space-time is just the *Minkowski metric*:

$$g_{\alpha\beta}(\mathcal{O}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta_{\alpha\beta}.$$

Furthermore,

$$\mathrm{D}A_\alpha = g_{\alpha\beta}\mathrm{D}A^\beta = \mathrm{D}(g_{\alpha\beta}A^\beta) = A^\beta \mathrm{D}g_{\alpha\beta} + g_{\alpha\beta}\mathrm{D}A^\beta$$

or equivalently

$$g_{\alpha\beta}\frac{\mathrm{D}A^\beta}{\mathrm{d}s} = \frac{\mathrm{D}(g_{\alpha\beta}A^\beta)}{\mathrm{d}s} = (g_{\alpha\beta}A^\beta)_{;\gamma}\dot{x}^\gamma = (g_{\alpha\beta;\gamma}A^\beta + g_{\alpha\beta}A^{\alpha}{}_{;\gamma})\dot{x}^\gamma,$$

where $\dot{x}^\gamma = \mathrm{d}x^\gamma/\mathrm{d}s$. It follows $g_{\alpha\beta;\gamma} = 0$. From that, we can deduce

$$g_{\alpha\beta,\gamma} = \Gamma^{\delta}{}_{\alpha\gamma}g_{\delta\beta} + \Gamma^{\delta}{}_{\beta\gamma}g_{\alpha\delta} = \Gamma_{\beta\alpha\gamma} + \Gamma_{\alpha\beta\gamma}.$$

We can do the same with the following two combinations

$$g_{\alpha\gamma,\beta} = \Gamma_{\gamma\alpha\beta} + \Gamma_{\alpha\gamma\beta}$$

and

$$-g_{\gamma\beta,\alpha} = -\Gamma_{\beta\gamma\alpha} - \Gamma_{\gamma\beta\alpha}.$$

Adding all three equations, we get

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(g_{\gamma\delta,\beta} + g_{\beta\delta,\gamma} - g_{\beta\gamma,\delta}).$$

1.6 Geodesic equation

Let M be a point in a curved space-time with coordinates ξ^α. The straight line from the origin to M is parametrized as $\xi^\alpha(s) = \xi^\alpha_M s/s_0$. In curved coordinates x^α the curve has parametrization $x^\alpha(\xi(s)) = x^\alpha(s)$, and equivalently we denote $\xi^\alpha(s) = \xi^\alpha(x^\gamma(s))$. Let's look at the derivative of the vector along the curve

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}\xi^\alpha}{\mathrm{d}s}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \xi^\alpha(x(s))}{\partial x^\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\right) = \frac{\partial \xi^\alpha}{\partial x^\beta}\frac{\mathrm{d}^2 x^\beta(s)}{\mathrm{d}s^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\gamma \partial x^\beta}\frac{\mathrm{d}x^\gamma}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0.$$

Multiplying by $\frac{\partial x^\delta}{\partial \xi^\alpha}$, we get

$$\frac{\mathrm{d}^2 x^\delta(s)}{\mathrm{d}s^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\gamma \partial x^\beta}\frac{\partial x^\delta}{\partial \xi^\alpha}\frac{\mathrm{d}x^\gamma}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0.$$

The middle term is just the Christoffel symbol, so

$$\boxed{\frac{\mathrm{d}^2 x^\delta}{\mathrm{d}s^2} + \Gamma^\delta{}_{\gamma\beta}\frac{\mathrm{d}x^\gamma}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0.}$$

We can denote the derivative of the vector along the curve as a four-velocity $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}s$. The *geodesic equation* is then

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}s} + \Gamma^\alpha{}_{\beta\gamma}u^\beta u^\gamma = 0.$$

Another option to rewrite it is

$$\left(\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}u^\gamma\right)\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = u^\alpha{}_{;\beta}u^\beta = \frac{\mathrm{D}u^\alpha}{\mathrm{d}s} = 0.$$

This equation also tells us that if $\mathbf{v}(x)$ is a unit tangent vector on a geodesic at x and $\mathbf{v}(x+\Delta x)$ is a unit tangent vector on the geodesic curve at $x+\Delta x$, then the latter is the same to the former under parallel transport $\mathbf{v}^{\mathrm{PS}}(x) = \mathbf{v}(x+\Delta x)$. There is a much shorter derivation in Landau and Lifshitz. The last useful form of the geodesic equation is

$$\frac{\mathrm{d}u_\alpha}{\mathrm{d}s} - \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}u^\beta u^\gamma = 0$$

This equation sacrifices the covariance but is useful for finding conserved quantities if the metric has specific symmetries.

1.7 Riemann tensor

If we take two geodesics in flat space and look at acceleration between them, it should be zero $\mathrm{d}^2 l/\mathrm{d}s^2 = 0$. In curved space, this is generally not the case. Let's parametrize geodesics with parameter s and different geodesics by a continuous parameter θ. The vector between two nearby geodesics (separated by $\Delta\theta$) is

$$\eta^\alpha = \frac{\partial x^\alpha(s;\theta)}{\partial \theta}\Delta\theta = v^\alpha \Delta\theta.$$

We used the notation

$$v^\alpha \equiv \left(\frac{\partial x^\alpha(s;\theta)}{\partial \theta}\right)_s \quad \text{and} \quad u^\alpha \equiv \left(\frac{\partial x^\alpha(s,\theta)}{\partial s}\right)_v.$$

Next, we use the useful relation

$$u^\alpha{}_{;\beta}v^\beta = v^\alpha{}_{;\beta}u^\beta.$$

To prove this relation, we move to local inertial frame ξ, where the Christoffel symbols are equal to zero:

$$\frac{\partial u^\alpha}{\partial \xi^\beta}\frac{\partial \xi^\beta}{\partial v} = \frac{\partial v^\alpha}{\partial \xi^\beta}\frac{\partial \xi^\beta}{\partial s}.$$

It follows $\partial u^\alpha/\partial\theta = \partial v^\alpha/\partial s$, which is obviously true. The acceleration between two geodesics is then

$$\frac{\mathrm{D}^2\eta^\alpha}{\mathrm{d}s^2} = \frac{\mathrm{D}^2 v^\alpha}{\mathrm{d}s^2}\Delta\theta.$$

Writing the derivative, one obtains

$$\begin{aligned}
\frac{\mathrm{D}^2 v^\alpha}{\mathrm{d}s^2} &= \frac{\mathrm{D}}{\mathrm{d}s}\left(v^\alpha{}_{;\beta}u^\beta\right)\\
&= (v^\alpha{}_{;\beta}u^\beta)_{;\gamma}u^\gamma\\
&= (u^\alpha{}_{;\beta}v^\beta)_{;\gamma}u^\gamma\\
&= u^\alpha{}_{;\beta;\gamma}v^\beta u^\gamma + u^\alpha{}_{;\beta}v^\beta{}_{;\gamma}u^\gamma\\
&= u^\alpha{}_{;\beta;\gamma}v^\beta u^\gamma + u^\alpha{}_{;\gamma}v^\gamma{}_{;\beta}u^\beta\\
&= u^\alpha{}_{;\beta;\gamma}v^\beta u^\gamma + u^\alpha{}_{;\gamma}u^\gamma{}_{;\beta}v^\beta\\
&= (u^\alpha{}_{;\beta;\gamma} - u^\alpha{}_{;\gamma;\beta})u^\gamma v^\beta + (u^\alpha{}_{;\gamma}u^\gamma)_{;\beta}v^\beta\\
&= (u^\alpha{}_{;\beta;\gamma} - u^\alpha{}_{;\gamma;\beta})u^\gamma v^\beta.
\end{aligned}$$

The last term in the penultimate line is zero because that is exactly the definition of geodesic. We define the *Riemann tensor* R as

$$u^\alpha{}_{;\beta;\gamma} - u^\alpha{}_{;\gamma;\beta} = R^\alpha{}_{\delta\gamma\beta}u^\delta.$$

Written with Christoffel symbols, the Riemann tensor can be expressed as

$$R^\alpha{}_{\delta\gamma\beta} = \frac{\partial\Gamma^\alpha{}_{\delta\beta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha{}_{\delta\gamma}}{\partial x^\beta} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\delta\beta} - \Gamma^\alpha{}_{\sigma\beta}\Gamma^\sigma{}_{\delta\gamma}.$$

From the previous result, it follows that

$$\frac{\mathrm{D}^2\eta^\alpha}{\mathrm{d}s^2} = R^\alpha{}_{\delta\gamma\beta}u^\delta u^\gamma\eta^\beta.$$

In flat space it holds $R^\alpha{}_{\delta\gamma\beta} = 0$. The Riemann tensor is not zero in the local inertial frame in curved space since $\Gamma^\alpha{}_{\beta\gamma,\delta} \neq 0$. If we take a vector $\mathbf{A}$ in curved space-time and parallel transport it around a closed curve to $\mathbf{A}^{\mathrm{PT}}$, we get

$$\delta A_\alpha = \oint \Gamma^\beta{}_{\alpha\gamma}A_\beta \mathrm{d}x^\gamma = \frac{1}{2}R^\delta{}_{\alpha\beta\gamma}A_\delta \oint x^\beta \mathrm{d}x^\gamma.$$

Hence, parallelly transporting a vector (or any tensor) along two different paths with the same endpoint does not give the same vector. The only exemption is the metric tensor. Riemann tensor has 256 indices; however, they are not independent. We will now explore the symmetries of the tensor to figure out exactly how many independent components it has. First, we will write it more "democratically" as

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\sigma}R^\sigma{}_{\beta\gamma\delta}.$$

Expressed with metric tensors, it is

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\frac{\partial g_{\alpha\delta}}{\partial x^\beta\partial x^\gamma} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha\partial x^\delta} - \frac{\partial g_{\alpha\gamma}}{\partial x^\beta\partial x^\delta} - \frac{\partial g_{\beta\delta}}{\partial x^\alpha\partial x^\gamma}\right) + g_{\sigma\rho}\left(\Gamma^\sigma{}_{\alpha\delta}\Gamma^\rho{}_{\beta\delta} - \Gamma^\sigma{}_{\alpha\gamma}\Gamma^\rho{}_{\beta\delta}\right).$$

From this expression, we can see the following symmetries:

- $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}$,
- $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$,
- $R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$.

The last symmetry is also called the *Bianchi identity*. In 2 dimensions, only one independent component R_{1010} exists. In 3 dimensions, we have options $\alpha\beta = (01, 02, 12)$ and $\gamma\delta = (01, 02, 12)$. We get six independent components since the tensor is symmetric for the exchange $\alpha\beta \leftrightarrow \gamma\delta$. In 4 dimensions we have

$$\alpha\beta = (01, 02, 03, 12, 13, 23)$$

and

$$\gamma\delta = (01, 02, 03, 12, 13, 23).$$

The symmetric 6×6 matrix has 21 independent components. The Bianchi identity

$$R_{0123} + R_{0312} + R_{0231} = 0$$

removes one degree of freedom, so the number of independent components is 20. A *Ricci tensor* is defined as

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}.$$

It is symmetric $R_{\alpha\beta} = R_{\beta\alpha}$. In 1 dimension, it has one independent component (since the Riemann tensor only has one independent component). In 2 dimensions, it has six independent components, so the Riemann tensor is completely expressible by the Ricci tensor (a curious consequence is that in 3D, there is no "real" gravity). After shuffling the indices even more, we get

$$R^{\alpha}{}_{\beta\gamma\delta;\sigma} + R^{\alpha}{}_{\beta\sigma\gamma;\delta} + R^{\alpha}{}_{\beta\delta\sigma;\gamma} = 0.$$

Contracting the first and third indexes, we get

$$R_{\beta\delta;\sigma} - R_{\beta\sigma;\delta} + R^{\alpha}{}_{\beta\delta\sigma;\alpha} = 0.$$

Now we multiply by $g^{\beta\delta}$ and rename δ in the middle term to α and get

$$R_{;\sigma} - R^{\alpha}{}_{\sigma;\alpha} + (g^{\beta\delta} R^{\alpha}{}_{\beta\delta\sigma})_{;\alpha} = 0.$$

Simplifying even further, it holds

$$\left(R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta} R\right)_{;\alpha} = 0.$$

This is the *contracted Bianchi identity* for the Ricci tensor. The term inside the brackets is called the *Einstein tensor*:

$$G^{\alpha}{}_{\beta} = \left(R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta} R\right).$$

Chapter 2

Einstein's equations

2.1 Special relativity

An *event* $\mathcal{P}(x^\alpha)$ is a point in space-time, defined by its coordinates (x, y, z, t) (when and where). The invariant distance between events is

$$\mathrm{d}s^2 = \eta_{\alpha\beta}\mathrm{d}\xi^\alpha \mathrm{d}\xi^\beta,$$

where $\eta_{\alpha\beta}$ is Minkowski metric

$$\eta_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

If $\mathrm{d}s^2 > 0$, the events are *time-like* separated and can be causally connected. If $\mathrm{d}s^2 < 0$, the events are *space-like* separated and cannot be causally connected. Lastly, if $\mathrm{d}s^2 = 0$, the events are *light-like* separated and can be connected by a light ray. We have six parameters to describe the group of linear isometries in 3 dimensions: 3 translations and three rotations (around every plane). Such transformations preserve the metric. In 4 dimensions, there are again three rotations in the "space" and four pseudo-rotations. In total, there are thus 10 parameters. This is called the Poincaré group (Λ, a), which transform the coordinates as

$$\tilde{\xi}^\alpha = \Lambda^\alpha{}_\beta \xi^\beta + a^\alpha$$

such that

$$\mathrm{d}s^2 = \eta_{\alpha\beta}\mathrm{d}\xi^\alpha \mathrm{d}\xi\beta = \eta_{\alpha\beta}\mathrm{d}\tilde{\xi}^\alpha \mathrm{d}\tilde{\xi}^\beta.$$

Hence, for any Lorentz transformation Λ, it holds

$$\eta_{\alpha\beta} = \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta.$$

The invariant distance between two events is then

$$\mathrm{d}s_0^2 = \left(\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\gamma}\frac{\partial\xi^\beta}{\partial x^\delta}\right)\mathrm{d}x^\gamma \mathrm{d}x^\delta = g_{\gamma\delta}\mathrm{d}x^\gamma \mathrm{d}x^\delta.$$

Classically, the Newtonian gravity is described with two equations:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla\phi, \quad \Delta\phi = 4\pi G\mathcal{E}.$$

The last equation is called *Poisson equation*, and $\mathcal{E}$ denotes the energy density. Let's look at the low-velocity approximation. Let $u^\mu = \mathrm{d}x^\mu/\mathrm{d}s$. The invariant interval is then

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \sum_{i=1}^{3}(\mathrm{d}\xi^i)^2 = (1 - |\mathbf{v}|^2)\mathrm{d}t^2.$$

So

$$u^\mu = \frac{1}{\sqrt{1-|\mathbf{v}|^2}}v^\mu.$$

The zeroth component goes to 1, and the spatial components go to v^i in the limit $v \ll c$. Take the geodesic equation

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}s} + \Gamma^\alpha{}_{\beta\gamma}u^\beta u^\gamma = 0.$$

Using the fact that $u^0 = 1$ and $u^i \ll 1$, we can simplify the equation to

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}s} + \Gamma^\alpha{}_{00} \approx 0.$$

What is $\Gamma^\alpha{}_{00}$? We know that

$$\Gamma^\alpha{}_{00} = \frac{1}{2}g^{\alpha\delta}\left(\frac{\partial g_{\delta 0}}{\partial x^0} + \frac{\partial g_{\delta 0}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\delta}\right).$$

Assume that the metric tensor is static and just a small perturbation of the Minkowski metric $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$. The Christoffel symbols are then

$$\Gamma^\alpha{}_{00} = -\frac{1}{2}\eta^{\alpha\delta}\frac{\partial h_{00}}{\partial x^\delta} + \mathcal{O}(h^2).$$

The result is $\Gamma^0{}_{00} \approx 0$ and $\Gamma^i{}_{00} \approx \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}$. For the slowly varying field it holds $v^i \approx u^i$, $\mathrm{d}s \approx \mathrm{d}t$ and hence

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i} = -\nabla\phi.$$

Setting $h_{00} = 2\phi$ gives us the right Newtonian limit. The metric tensor is then $g_{00} = 1 + 2\phi$. Using that, we can also prove that

$$R_{00} = \nabla^2\phi.$$

2.2 Redshift in static gravitational field

The static metric is such that there exists a frame x^α such that the metric tensor doesn't depend on time. For the light ray, the interval is zero $\mathrm{d}s^2 = 0$, so

$$g_{00}\mathrm{d}t^2 + 2g_{0i}\mathrm{d}x^i\mathrm{d}t + g_{ik}\mathrm{d}x^i\mathrm{d}x^k = 0.$$

The middle terms can be equal to zero after some coordinate rotation. The time interval can be expressed as

$$\mathrm{d}t = \sqrt{-\frac{g_{ij}\mathrm{d}x^i\mathrm{d}x^j}{g_{00}}}.$$

Let's say that the beginning of the wave was emitted at x_1 and t_1 and the end of the wave at $t_1 + \Delta t_1$. The wave is then received at x_2 and t_2 and the end of the wave at $t_2 + \Delta t_2$. The time interval $t_2 - t_1$ is

$$t_2 - t_1 = \int_{x_1}^{x_2} \sqrt{-\frac{g_{ij}\mathrm{d}x^i \mathrm{d}x^j}{g_{00}}} = I.$$

But

$$(t_2 + \Delta t_2) - (t_1 - \Delta t_1) = I$$

and so $\Delta t_2 = \Delta t_1 = \Delta t$. But this doesn't mean that the redshift is zero. The *proper time* that observer at x_1 measures is $\Delta\tau_1 = \Delta s = \sqrt{g_{00}(x_1)}\Delta(t)$ and the observer at x_2 measures $\Delta\tau_2 = \sqrt{g_{00}(x_2)}\Delta(t)$. The "redshift" is then

$$\frac{\nu_2}{\nu_1} = \frac{\Delta\tau_1}{\Delta\tau_2} = \sqrt{\frac{g_{00}(x_1)}{g_{00}(x_2)}}.$$

In a weak field, $g_{00} = 1 + 2\phi$, so

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_1)}{g_{00}(x_0)}} = \sqrt{\frac{1 + 2\phi(x_1)}{1 + 2\phi(x_2)}} \approx 1 + \phi(x_1) - \phi(x_2).$$

Considering the relative difference and the gravitational potential of the Earth, we get

$$z = \frac{\nu_2 - \nu_1}{\nu_1} = \frac{\Delta\nu}{\nu_1} \approx \phi(x_1) - \phi(x_2) = \frac{GM_\oplus}{c^2}\frac{R_1 - R_2}{R_1 R_2} \approx \frac{GM_\oplus}{c^2}\frac{\Delta R}{R_\oplus^2} = \frac{g\Delta h}{c^2}.$$

2.3 Einstein's equations

The metric has 10 independent components, so we would naively expect 10 equations to represent the theory. However, moving into a different coordinate system, the metric tensor changes, but the solution should be the same. This means that we are left with six independent equations. The intuitive way to derive the equations is to start with the Poisson equation $\Delta\phi = 4\pi G\mathcal{E}$. Writing it in tensor form, the left side could be

$$G_{\alpha\beta} = C_1 R_{\alpha\beta} + C_2 R g_{\alpha\beta} + \Lambda g_{\alpha\beta}.$$

We could also have terms like $R^\delta{}_\beta R_{\delta\alpha}$, but they would be quadratic in curvature and hence negligible in weak fields. We will take a short detour to classical hydrodynamics for the other side of the equation. The fluid is described by the function $\mathcal{E}(x^i, t)$, which is the energy density, the pressure $p(\mathcal{E})$ and the velocity field $v^j(x^i, t)$. The energy-momentum tensor $T_{\alpha\beta}$ is symmetric, which is defined so that equating its derivative with zero gives all equations of motion:

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0.$$

This gives four equations of motion in 4 dimensions. The change of the mass is

$$\Delta M(t) = \int \mathcal{E}(x^i, t)\,\mathrm{d}V.$$

Then

$$\Delta M(t + \Delta t) - \Delta M(t) = \int (\mathcal{E}(x^i, t + \Delta t) - \mathcal{E}(x^i, t)), \mathrm{d}V = \int \frac{\partial \mathcal{E}}{\partial t}\Delta t\,\mathrm{d}V.$$

We can derive this also with a different approach:

$$\Delta M(t+\Delta t) - \Delta M(t) = -\oint \mathcal{E}\mathbf{v}\Delta t \mathrm{d}\boldsymbol{\sigma} = -\nabla(\mathcal{E}\mathbf{v})\Delta t\,\mathrm{d}V.$$

Combining these two results, we get

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \nabla(\mathcal{E}\mathbf{v}) = 0}$$

which is the *continuity equation*. The second Newton's law is

$$\Delta M \frac{\mathrm{d}\mathbf{v}(\mathbf{x}(t),t)}{\mathrm{d}t} = \mathbf{F} = \mathbf{F}_{\text{press.}} + \mathbf{F}_{\text{grav.}}.$$

The two acting forces can be rewritten as

$$\mathbf{F}_{\text{press.}} = -\oint p\,\mathrm{d}\boldsymbol{\sigma} = -\int \nabla p\,\mathrm{d}V \approx -\nabla p\,\Delta V$$

and

$$\mathbf{F}_{\text{grav.}} = -\nabla\phi\,\Delta M.$$

Combining this and writing the total derivative as a sum of partial and advective derivatives, we get

$$\boxed{\frac{\partial \mathbf{v}(x^i,t)}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\mathcal{E}}\nabla p + \nabla\phi = 0.}$$

This is called the *Euler equation* (which are three equations). So, we have our four equations. Now, we can try to construct the energy-momentum tensor. It is a rather difficult task, so we will state the result, and the reader can check that the non-relativistic limit of the derivatives of the energy-momentum actually gives the Euler and continuity equations. The energy-momentum tensor is

$$\boxed{T^{\alpha\beta} = (\mathcal{E}+p)u^\alpha u^\beta - p g^{\alpha\beta}}$$

and the equation of motion is

$$\boxed{T^{\alpha\beta}{}_{;\beta} = 0.}$$

The energy density $\mathcal{E}$ is measured in the frame co-moving with the fluid. We already know that $T^{\alpha\beta} = T^{\beta\alpha}$, but it also holds that $T^{\alpha}{}_{\beta,\alpha} = 0$. If we assume a more general form of the energy-momentum tensor

$$T^{\alpha}{}_{\beta} = \phi^{;\alpha}\phi_{;\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}\phi^{;\gamma}\phi_{;\gamma}$$

the condition $T^{\alpha}{}_{\beta,\alpha} = 0$ gives the following equation of motion

$$\Box\phi \equiv \phi^{;\alpha}{}_{;\alpha} = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) = 0.$$

For a single particle moving on a geodesic, the energy-momentum tensor is

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} m_0 \int \mathrm{d}s \, \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \delta(x^\sigma - x^\sigma(s)).$$

Combining our previous guess for the left-hand side of the equation and the energy-momentum tensor on the left, we get

$$G^\alpha{}_\beta \equiv C_1 R^\alpha{}_\beta + C_2 \delta^\alpha{}_\beta R + \Lambda \delta^\alpha{}_\beta = 8\pi G T^\alpha{}_\beta \,.$$

We take the covariant derivative with respect to α. The right side gives zero, so

$$C_1 R^\alpha{}_{\beta;\alpha} + C_2 \frac{1}{2} R_{;\beta} = 0.$$

The definition of Ricci tensor gives $R^\alpha{}_{\beta;\alpha} = \frac{1}{2} R_{;\beta}$, so

$$\left(\frac{C_1}{2} + C_2 \right) R_{;\beta} = 0.$$

Taking the trace of the Einstein equation (and setting $\Lambda = 0$),

$$(C_1 + 4C_2) R_{;\beta} = 8\pi G T^\alpha{}_{\alpha;\beta} = 8\pi G (\mathcal{E} - 3p)_{;\beta}.$$

With this, we see that if $\mathcal{E} \neq 0$ and $p = 0$, it cannot happen that $R_{;\beta} = 0$. Hence, $C_2 = C_1/2$. Using the fact that $R_{00} - \frac{1}{2} R = \Delta g_{00} = 8\pi G \mathcal{E}$ we fix $C_1 = 1$ and get the final form of the Einstein's equations

$$\boxed{R^\alpha{}_\beta - \frac{1}{2} \delta^\alpha{}_\beta R + \Lambda \delta^\alpha{}_\beta = 8\pi G T^\alpha{}_\beta \,.}$$

The Λ term was added later; Einstein added it to explain the static universe. If we take $p = -\mathcal{E}$, we get $T^\alpha{}_\beta = -p\delta^\alpha{}_\beta = \mathcal{E}\delta^\alpha{}_\beta$. The cosmological constant term can then be included in the energy-momentum tensor. It acts as a "negative mass" that prevents the mass in the universe from collapsing. Hence, we will write the Einstein's equations as

$$G^\alpha{}_\beta = 8\pi G T^\alpha{}_\beta \,.$$

The trace of Einstein's equations gives

$$-R = 8\pi G T,$$

where $T = T^\alpha{}_\alpha$ is the trace of the energy-momentum tensor. Using this, we can rewrite the Einstein's equations as

$$R^\alpha{}_\beta = 8\pi G (T^\alpha{}_\beta - \delta^\alpha{}_\beta T).$$

Bianchi identity gives $(T^\alpha{}_\beta - \frac{1}{2}\delta^\alpha{}_\beta R)_{;\alpha} = 0$, which immediately gives $T^\alpha{}_{\beta;\alpha} = 0$.

The *Cauchy problem* asks for the solution of a partial differential equation that satisfies certain conditions given on a hyper-surface in the domain. The PDE is typically written as a generalization of $\dot{q} = F(q)$. *Cauchy-Kovalevskaya theorem* guarantees the existence and uniqueness of a solution given enough initial conditions and that F is analytic.

Let's count some degrees of freedom. The number of degrees of freedom is defined as half of the necessary initial conditions. If we take vacuum Maxwell equations, we have in total eight

equations for six variables (3 for $\mathbf{E}$ and 3 for $\mathbf{B}$). Is the system over-deterministic? No! If we take the divergence of the last two equations, we get the time derivatives of the first two equations. If the solution satisfies first two equations initially ($\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$), it will satisfy them always. The number of degrees of freedom then seems to be 3. However, two of the dimensions of initial conditions are cancelled by those two constraints, so we are actually left with only 2 degrees of freedom (the same as the number of polarizations of a photon). This can be more easily seen if we look at the equations as $\Box A^\mu = 0$ and consider the gauge freedom.

Consider now the Einstein equations in vacuum:

$$G^{\mu\nu} = 8\pi G T^{\mu\nu} = 0.$$

The Einstein tensor $G^{\mu\nu}$ contains metric tensors and their derivative. Four equations are associated with the terms $G^{0\alpha}$ and another six equations associated with G^{ij}. Let's check what order these equations are. The Bianchi identity says $G^{\alpha\beta}{}_{;\beta} = 0$. From this, it follows

$$\frac{\partial G^{\alpha 0}}{\partial t} = -\frac{\partial G^{\alpha i}}{\partial x^i} - \Gamma^\alpha{}_{\lambda\sigma} G^{\lambda\sigma} - \Gamma^\sigma{}_{\sigma\lambda} G^{\alpha\lambda}.$$

The terms on the right side include up to second derivatives; therefore, the left side will only include second-order derivatives. But one derivative is already explicitly shown; hence, $G^{\alpha 0}$ only has first-order derivatives, and the equation $G^{\alpha 0} = 0$ contributes *4 first-order differential equations*. The term $G^{ij} = 0$ contributes *6 second order differential equations*. Combining that, we have $4 + 12 = 16$ initial conditions and hence 8 degrees of freedom. But that is still too much! Clearly, we have the redundancy of the coordinate changes

$$\tilde{g}_{\alpha\beta} = \frac{\partial x^\rho}{\partial \tilde{x}^\alpha} \frac{\partial x^\sigma}{\partial \tilde{x}^\beta} g_{\rho\sigma}.$$

The coordinate system has 4 degrees of freedom, so we need to fix 4 of them to fix the coordinate system. If we denote $g^{00} = N$ and $g^{0i} = N_i$ we have the following implicit equations:

$$G^{0\alpha}(\dot{g}_{ik}, g_{ik}, N, N_i) = 0,$$
$$G^{mn}(\ddot{g}_{ik}, \dot{g}_{ik}, g_{ik}, N, N_i) = 0.$$

We have to determine six functions g_{ik}, but we have 10 differential equations. The system seems (as in electro-magnetism) over-deterministic. However, the first four equations are constraints – if they are satisfied once, the remaining six equations will keep them satisfied. Namely, if we look at the Bianchi identity, the condition $G^{mn} = 0$ implies $\dot{G}^{\alpha 0} \propto G^{\alpha 0}$, so having $G^{\alpha 0} = 0$ at time t_0 it will hold at any other time. We are left with six differential equations and 12 initial conditions for $G^{mn} = 0$. The constraint $G^{\alpha 0} = 0$ has four initial conditions, which reduces the total number of initial conditions to 8 and the number of degrees of freedom to 4. But there was also the freedom in the choice of N and N_i, which removes another four initial conditions to give us, finally, that the equations have 2 degrees of freedom. In other words, a graviton only has two polarizations, the same as a photon.

Chapter 3

Schwarzschild solution

3.1 Spherically symmetrical gravitational field

An infinitesimal distance on a sphere can be expressed as $\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$. In the spherically symmetric case, we expect the metric to only depend on t and r:

$$\mathrm{d}s^2 = g_{ab}(r,t)\mathrm{d}x^a\mathrm{d}x^b + \Omega^2(r,t)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2).$$

We have $a, b = 0, 1$ and $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$. There are 2 degrees of freedom in the choice of metric, so we decide to set $g_{01} = 0$ and $\Omega^2(r,t) = r^2$. Thus, we write

$$\mathrm{d}s^2 = e^{\vartheta(r,t)}\mathrm{d}t^2 - e^{\lambda(r,t)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega,$$

where ϑ and λ are unknown. We have the following non-zero elements of the metric:

$$g_{00} = e^{\vartheta(r,t)}, \quad g_{11} = -e^{\lambda(r,t)}, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta,$$
$$g^{00} = e^{-\vartheta(r,t)}, \quad g^{11} = -e^{-\lambda(r,t)}, \quad g^{22} = -\frac{1}{r^2}, \quad g^{33} = -\frac{1}{r^2\sin^2\theta}.$$

Calculating some Christoffel symbols, we get

$$\Gamma^0{}_{00} = \frac{1}{2}\dot{\vartheta}, \quad \Gamma^0{}_{01} = \Gamma^0{}_{10} = \frac{1}{2}\vartheta' \quad \Gamma^0{}_{11} = -\frac{\dot{\lambda}}{2}e^{\lambda-\vartheta}, \quad \Gamma^1{}_{22} = -r^2e^{-\lambda}.$$

The vacuum solution $T^\mu{}_\nu = 0$ yields three equations:

$$\begin{aligned}
0 = 8\pi G T^0{}_0 &= -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} \\
0 = 8\pi G T^1{}_0 &= -e^{-\lambda}\frac{\dot{\lambda}}{2} \\
0 = 8\pi G T^1{}_1 &= -e^{-\lambda}\left(\frac{1}{r^2} + \frac{\vartheta'}{r}\right) + \frac{1}{r^2}.
\end{aligned}$$

The first equation gives $\dot{\lambda} = 0$, so $\lambda = \lambda(r)$ and the other two equations form a system

$$\begin{aligned}
e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \\
e^{-\lambda}\left(\frac{\vartheta'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0.
\end{aligned}$$

Adding the two equations gives $\lambda' + \vartheta' = 0$, so $\lambda + \vartheta = f(t)$. Introduce new variable $\tilde{\vartheta}$, such that $\lambda + \tilde{\vartheta} = 0$ and $\tilde{\vartheta} + f(t) = \vartheta$. The first element of the metric is

$$g_{00}\mathrm{d}t^2 = e^{\vartheta}\mathrm{d}t^2 = e^{\tilde{\vartheta}}e^{f(t)}\mathrm{d}t^2 = e^{\tilde{\vartheta}}\mathrm{d}\tilde{t}^2.$$

With this redefinition of time, we removed all time dependence from the metric. The metric is then

$$\mathrm{d}s^2 = e^{-\lambda(r)}\mathrm{d}t^2 - e^{\lambda(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2.$$

We get the following vacuum equation

$$\frac{(e^{-\lambda})'}{r} + \frac{e^{-\lambda}}{r^2} = \frac{1}{r^2}.$$

Use the ansatz $e^{-\lambda} = y$ and the equation reads

$$ry' + y = 1 \quad \Longrightarrow \quad (yr)' = 1.$$

Integrating this gives $yr = r + C$ or $y = 1 + C/r$. Noticing $g_{00} = e^{-\lambda}$ can also be written as $g_{00} = 1 - 2\phi$ with $\phi = -GM/r$, we get the Schwarzschild metric

$$\boxed{\mathrm{d}s^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{\left(1 - \frac{2GM}{rc^2}\right)} - r^2\mathrm{d}\Omega^2.}$$

We can simplify the notation by introducing the *Schwarzschild radius* $r_\mathrm{s} = 2GM/c^2$ and setting $c = 1$. The metric then reads:

$$\mathrm{d}s^2 = (1 - r_\mathrm{s}/r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{(1 - r_\mathrm{s}/r)} - r^2\mathrm{d}\Omega^2.$$

Inside the centre of the spherical symmetry, $M = 0$ so there is Minkowski spacetime. Denoting $f(r) = 1 - r_\mathrm{s}/r$, the metric in matrix form is

$$g_{\mu\nu} = \begin{bmatrix} f(r) & 0 & 0 & 0 \\ 0 & -1/f(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}.$$

Next, we will try to write the geodesic equations. Denote $x^\mu(\lambda) = (x^1, x^2, x^3, x^4) = (t, r, \theta, \phi)$. We can replace the parameter λ with proper time τ for massive particles. The *contravariant velocity* is

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$$

and the *covariant velocity* is

$$u_\mu = g_{\mu\nu}u^\nu = (f\dot{t}, -\frac{\dot{r}}{f}, -r^2\dot{\theta}, -r^2\sin^2\theta\dot{\phi}).$$

The geodesic is

$$\dot{u}^\alpha + \Gamma^\alpha{}_{\mu\nu}u^\mu u^\nu = 0$$

or $u^{\alpha}{}_{;\beta}u^{\beta} = 0$. In particular, it holds $(u^{\alpha}u_{\alpha})_{;\beta}u^{\beta} = 0$. From this, we see

$$u^{\alpha}u_{\alpha} = g_{\alpha\beta}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}\tau} = \text{const.} = \kappa = \begin{cases} 1, & \text{for } m \neq 0 \\ 0, & \text{for } m = 0. \end{cases}$$

This equation in components reads

$$f(r)\dot{t} - \dot{r}^2/f(r) - r^2\dot{\theta}^2 - r^2\sin^2(\theta)\dot{\phi}^2 = \kappa. \tag{3.1}$$

Continuing with the geodesic equation

$$\dot{u}_{\alpha} + \frac{1}{2}\frac{\partial g_{\beta\delta}}{\partial x^{\alpha}}u^{\beta}u^{\delta} = 0$$

we get for the r-component

$$(f(r)\dot{t})\dot{} = 0 \tag{3.2}$$

for the θ-component

$$-(r^2\dot{\theta})\dot{} + r^2\sin\theta\cos\theta\dot{\phi}^2 = 0 \tag{3.3}$$

and for the ϕ-component

$$(r^2\sin^2\theta\dot{\phi})\dot{} = 0. \tag{3.4}$$

Equation Eq. (3.3) has an easy solution $\theta = \pi/2$. Equation Eq. (3.4) gives a conserved quantity $r^2\dot{\phi} = \mathcal{J}$ which is the angular momentum. The equation Eq. (3.2) gives another conserved quantity $f(r)\dot{t} = C$ which is the energy. The equation Eq. (3.1) can be then rewritten as

$$f(r)C^2 - \dot{r}^2/f(r) - r^2\mathcal{J}^2 = \kappa.$$

To get the equation for r in dependence of ϕ, we rewrite the differential of $\mathrm{d}\tau$ as

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\phi} = \frac{\mathcal{J}}{r^2}\frac{\mathrm{d}}{\mathrm{d}\phi}$$

and hence

$$\dot{r}^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \mathcal{J}^2\left(\frac{1}{r}^2\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \mathcal{J}^2\left(\frac{\mathrm{d}}{\mathrm{d}\phi}\frac{1}{r}\right)^2.$$

Defining $v = 1/r$ simplifies the equation for r to

$$C^2 = \mathcal{J}^2\left(\frac{\mathrm{d}v}{\mathrm{d}\phi}\right)^2 + (\mathcal{J}^2v^2 - \kappa)(1 - r_{\mathrm{s}}v).$$

To solve this equation, we will first differentiate:

$$0 = 2\mathcal{J}^2\frac{\mathrm{d}v}{\mathrm{d}\phi}\frac{\mathrm{d}^2v}{\mathrm{d}\phi^2} + 2\mathcal{J}^2v\frac{\mathrm{d}v}{\mathrm{d}\phi}(1 - r_{\mathrm{s}}v) - r_{\mathrm{s}}\frac{\mathrm{d}v}{\mathrm{d}\phi}(\mathcal{J}^2v^2 + \kappa).$$

Divide the whole equation by $\mathrm{d}v/\mathrm{d}\phi$ (with this we lose solutions where r doesn't depend on ϕ.) and get

$$\frac{\mathrm{d}^2v}{\mathrm{d}\phi^2} + v = \frac{\kappa r_{\mathrm{s}}}{2\mathcal{J}^2} + \frac{3}{2}r_{\mathrm{s}}v^2.$$

The contribution of general relativity to Newtonian gravity is only in the last term. The last term is usually tiny, so we can solve this equation perturbatively. For $\kappa = 0$ we have

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\phi^2} + v = \frac{3}{2} r_\mathrm{s} v^2.$$

In Newtonian gravity, the last term doesn't exist, and the solution is $v(\phi) = A \sin(\phi + \phi_0)$. Due to rotational symmetry, we can set $\phi_0 = 0$ and the solution is

$$r = \frac{1/A}{\sin \phi}.$$

The value of $1/A$ is called *impact parameter* and is denoted b. The general solution includes the last term. Write the solution as the sum of the Newtonian solution v_0 and the perturbation $v_1 \ll v_0$:

$$\frac{\mathrm{d}^2 (v_0 + v_1)}{\mathrm{d}\phi^2} + (v_0 + v_1) = \frac{3 r_\mathrm{s}}{2} (v_0 + v_1)^2.$$

Dropping all terms on the right that don't include v_1 gives

$$\frac{\mathrm{d}^2 v_1}{\mathrm{d}\phi^2} + v_1 = \frac{3 r_\mathrm{s}}{2 b^2} \sin^2 \phi = \frac{3 r_\mathrm{s}}{2 b^2} \frac{1 - \cos(2\phi)}{2}.$$

The solution is the sum of homogenous and particular solutions:

$$v = \frac{1}{b} \sin \phi + \frac{r_\mathrm{s}}{4 b^2} (3 + \cos(2\phi)).$$

This equation describes a slight bend in the straight-line trajectory of the photon. To get the bending angle $\Delta\phi$, we must find the zeros of the equation: $v(\Delta\phi) = 0$. Using $\cos(\Delta\phi) \approx 1$ and $\sin(\Delta\phi) \approx \Delta\phi$, we get $\Delta\phi = r_\mathrm{s}/b$. The whole bending angle is then $2\Delta\phi = 2 r_\mathrm{s}/b$:

$$\delta = \frac{4GM}{c^2 b}.$$

For the Sun it has the value $\delta = 1{,}75'' R_\odot / b$. This was first measured in the year 1919 by Eddington and Dyson. Let's return to the massive particles ($\kappa = 1$). The Newtonian approximation gives the equation

$$\frac{\mathrm{d}^2 v_0}{\mathrm{d}\phi^2} + v_0 = \frac{r_\mathrm{s}}{2\mathcal{J}^2}$$

with a solution

$$\frac{1}{r} = v_0 = \frac{r_\mathrm{s}}{2\mathcal{J}^2} + A \cos(\phi - \phi_0) = \frac{1}{p}(1 + e \cos\theta),$$

where we defined $p = 2\mathcal{J}^2 / r_\mathrm{s}$ and $e = Ap$. This is either circle ($e = 0$), an ellipse ($0 < e < 1$), parabola ($e = 1$) or hyperbola ($e > 0$):

$$r = \frac{p}{1 + e \cos\theta}.$$

In general relativity, the first perturbation is $v = v_0 + v_1$. We do the same steps as for the $\kappa = 0$ case (dropping any v_1 dependence on the right side) and get

$$\frac{\mathrm{d}^2 v_1}{\mathrm{d}\phi^2} v_1 = \frac{3 r_\mathrm{s}}{2 p^2} (1 + e \cos\theta)^2 = \frac{3 r_\mathrm{s}}{2 p^2} (1 - e^2/2) + \frac{3 r_\mathrm{s}}{2 p^2} \cos\phi + \frac{3 r_\mathrm{s}}{4 p^2} \cos(2).$$

The solution is

$$v_1 = \frac{3r_\mathrm{s}}{2p^2}(1 + e^2/2) + \frac{3r_\mathrm{s}}{2p^2}\phi\sin\phi - \frac{r_\mathrm{s}}{4p^2}\cos(2\phi).$$

Combined with the Newtonian solution, we get

$$v = v_0 + v_1 = p^{-1}\left(1 + e\cos\phi + \frac{3r_\mathrm{s}}{2p^2}\phi\sin\phi + \dots\right) \approx \tilde{p}^{-1}\left(1 + \tilde{e}\cos\left(\left(1 - \frac{3r_\mathrm{s}}{2p}\right)\phi\right)\right).$$

This causes the precession of the perihelion of

$$\Delta\phi \approx \frac{3\pi r_\mathrm{s}}{p}$$

per orbit.

Chapter 4

Weak gravitational field

In this chapter, we will consider the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $|h_{\mu\nu}| \ll 1$. The Christoffel symbols are then

$$\Gamma^{\delta}{}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\delta}(h_{\nu\alpha,\mu} + h_{\mu\alpha,\mu} - h_{\mu\nu,\alpha}).$$

The Ricci tensor is

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu;\alpha} - \Gamma^{\alpha}{}_{\mu\alpha;\nu}$$

and the Ricci scalar is

$$R = g^{\mu\nu}R_{\mu\nu} = \eta^{\mu\nu}R_{\mu\nu}.$$

The Einstein tensor becomes

$$G^{\mu}{}_{\nu} = \frac{1}{2}\left(-\Box\overline{h}^{\mu}{}_{\nu} - \delta^{\mu}{}_{\nu}\overline{h}^{\alpha}{}_{\beta}{}^{,\beta}{}_{,\alpha} + \overline{h}^{\mu}{}_{\alpha}{}^{,\alpha}{}_{,\nu} + \overline{h}^{\alpha}{}_{\nu}{}^{,\mu}{}_{,\alpha}\right),$$

where $\overline{h}^{\mu}{}_{\nu} = h^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}h^{\alpha}{}_{\alpha}$. This equation is pretty (and Lorentz invariant), but in order to actually solve it we will have to break this invariance. The tensor $h_{\mu\nu}$ has 10 components. Under spatial rotations, the component g_{00} behaves as a scalar, components g_{0i} as a vector and g_{ij} as a tensor. We have already seen, that h_{00} can be written as 2ϕ. The components g_{0i} can be written as $g_{0i} = -B_{,i} + S_i$ with S being divergence-less $S_i{}^{,i} = 0$. The tensor part is

$$h_{ik} = 2\psi\delta_{ik} - 2E_{,ik} + F_{i,k} + F_{k,i} + {}^{\mathrm{TT}}h_{ik},$$

where $F_i{}^{,i} = 0$, ${}^{\mathrm{TT}}h^{i}{}_{i} = 0$ and ${}^{\mathrm{TT}}h^{i}{}_{k,i} = 0$. The notation TT means "traceless and transverse". There are 10 free parameters in these functions as shown in the following table. We will decompose the metric tensor into the "scalar" part, the "vector" part and the "tensor" part:

$$h_{\alpha\beta} = h^{(\mathrm{s})}_{\alpha\beta} + h^{(\mathrm{v})}_{\alpha\beta} + h^{(\mathrm{t})}_{\alpha\beta}.$$

In the first order, the Einstein tensor decomposes accordingly:

$$G^{\mu}{}_{\nu}(h^{(\mathrm{s})}_{\alpha\beta} + h^{(\mathrm{v})}_{\alpha\beta} + h^{(\mathrm{t})}_{\alpha\beta}) = G^{\mu}{}_{\nu}(h^{(\mathrm{s})}) + G^{\mu}{}_{\nu}(h^{(\mathrm{v})}) + G^{\mu}{}_{\nu}(h^{(\mathrm{t})}).$$

Function	Type	Free parameters
ϕ	scalar	1
B	scalar	1
S_i	vector	2
ψ	scalar	1
E	scalar	1
F_i	vector	2
${}^{\mathrm{TT}}h_{ik}$	tensor	2

Let's first solve the scalar part. Write

$$h = h^0{}_0 + h^i{}_i = h_{00} - h_{ii} = 2\phi - 6\psi - 2\nabla^2 E.$$

Furthermore, we have

$$\begin{aligned}
(\overline{h}^{(\mathrm{s})})^0{}_0 &= \eta^{0\alpha} h^{(\mathrm{s})}_{\alpha 0} = \frac{1}{2}\delta^0{}_0 h^{(\mathrm{s})} = \phi + 3\psi - \nabla^2 E, \\
(\overline{h}^{(\mathrm{s})})^0{}_i &= -(h^{(\mathrm{s})})^i{}_0 = h_{0i} = -B_{,i}, \\
(\overline{h}^{(\mathrm{s})})^i{}_j &= (\psi - \phi)\delta_{ij} - \nabla^2 E \delta_{ij} + 2E_{,ij}.
\end{aligned}$$

The Einstein tensor components are

$$\begin{aligned}
(G^{(\mathrm{s})})^0{}_0 &= -2\nabla^2 \psi, \\
(G^{(\mathrm{s})})^0{}_i &= 2\frac{\partial}{\partial t}\psi_{,i}, \\
(G^{(\mathrm{s})})^i{}_j &= -D_{,ij} - \delta_{ij}\left(\nabla^2 D + 2\frac{\partial^2 \psi}{\partial t^2}\right),
\end{aligned}$$

where

$$D = \phi - \psi + \frac{\partial}{\partial t}\left(B - \frac{\partial E}{\partial t}\right).$$

For the vector part, we have

$$(\overline{h}^{(\mathrm{v})})^0{}_0 = 0, \quad (\overline{h}^{(\mathrm{v})})^i{}_0 = -(\overline{h}^{(\mathrm{v})})^i{}_0 = h^{(\mathrm{v})}_{0i} = S_i, \quad (\overline{h}^{(\mathrm{v})})^i{}_j = -F_{i,j} - F_{j,i}.$$

The Einstein tensor components are

$$\begin{aligned}
(G^{(\mathrm{v})})^0{}_0 &= 0, \\
(G^{(\mathrm{v})})^0{}_i &= \frac{1}{2}\nabla^2\left(S_i - \frac{\partial}{\partial t}F_i\right), \\
(G^{(\mathrm{v})})^i{}_j &= -\frac{1}{2}\left[\left(S_i - \frac{\partial}{\partial t}F_i\right)_{,j} + \left(S_j - \frac{\partial}{\partial t}F_j\right)_{,i}\right].
\end{aligned}$$

For the tensor part, it holds

$$\begin{aligned}
(G^{(\mathrm{t})})^0{}_0 &= 0, \\
(G^{(\mathrm{t})})^0{}_i &= 0, \\
(G^{(\mathrm{t})})^i{}_j &= -\frac{1}{2}\Box^{\mathrm{TT}} h^i{}_j = \frac{1}{2}\Box^{\mathrm{TT}} h_{ij}.
\end{aligned}$$

4.1 Vacuum solution

For $T^{\mu}{}_{\nu} = 0$ we have the following equations

$$\begin{aligned}
0 &= 2\nabla^2\psi \\
0 &= 2\frac{\partial}{\partial t}\psi_{,i} + \frac{1}{2}\nabla^2\left(S_i - \frac{\partial}{\partial t}F_i\right) \\
0 &= -D_{,ik} - \delta_{ik}\left(\nabla^2 D + 2\frac{\partial^2\psi}{\partial t^2}\right) - \frac{1}{2}\left[\left(S_i - \frac{\partial}{\partial t}F_i\right)_{,k} + \left(S_k - \frac{\partial}{\partial t}F_k\right)_{,i}\right] + \frac{1}{2}\Box^{\mathrm{TT}}h_{ik}.
\end{aligned}$$

The solution to the first equation is $\psi = 0$. For the second one, we have

$$S_i - \frac{\partial}{\partial t}F_i = 0$$

and for the third one

$$D_{,ik} - \delta_{ik}\nabla^2 D + \frac{1}{2}\Box^{\mathrm{TT}}h_{ik} = 0.$$

Taking the trace of the last equation gives $\nabla^2 D = 0$ and hence

$$0 = D = \phi + \frac{\partial}{\partial t}\left(B - \frac{\partial E}{\partial t}\right).$$

Additionally, we have $\Box^{\mathrm{TT}}h_{ik} = 0$.

- Scalars:
$$\psi = 0, \quad D = 0, \quad \phi = -\frac{\partial}{\partial t}\left(B - \frac{\partial E}{\partial t}\right).$$
- Vectors:
$$S_i = \frac{\partial}{\partial t}F_i.$$
- Tensor:
$$\Box^{\mathrm{TT}}h_{ik} = 0.$$

Let's consider a change of coordinates $x^\mu \to \tilde{x}^\mu(x^\nu)$. Then $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}$. We want to find $\tilde{h}_{\mu\nu}(h_{\alpha\beta})$. First, we will assume small changes of coordinates, so we can write $\tilde{x}^\alpha = x^\alpha + \xi^\alpha$. The metric in the new coordinates is then

$$\tilde{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}g_{\alpha\beta}$$

and

$$\eta_{\mu\nu} + \tilde{h}_{\mu\nu} = (\delta^\alpha{}_\mu - \xi^\alpha{}_{,\mu})(\delta^\beta{}_\nu - \xi^\beta{}_{,\nu})(\eta_{\alpha\beta} + h_{\alpha\beta}).$$

For the first order in h, we have

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}.$$

Write $\xi = (\xi_0, \xi_i)$, where

$$\xi_i = \zeta_{,i} + {}^{\mathrm{T}}\xi_i,$$

where ${}^{\mathrm{T}}\xi_i{}^{,i} = 0$. Let's look at how functions change:

$$\begin{aligned} 2\tilde{\phi} = \tilde{h}_{00} = h_{00} - 2\dot{\zeta}_0 = 2\phi - 2\dot{\zeta}_0, \implies \tilde{\phi} = \phi - \dot{\xi}_0, \\ \tilde{h}_{0i} = -\tilde{B}_{,i} + \tilde{S}_i = h_{0i} - \xi_{0,i} - \dot{\zeta}_{,i} - {}^{\mathrm{T}}\dot{\xi}_i \\ = -B_{,i} + S_i - \xi_{0,i} - \dot{\zeta}_{,i} - {}^{\mathrm{T}}\dot{\xi}_i \end{aligned}$$

This means that B transforms as

$$\tilde{B} = B + \xi_0 + \dot{\zeta}$$

and S_i transforms as

$$\tilde{S}_i = S_i - \xi_{0,i} - \dot{\zeta}_{,i} - {}^{\mathrm{T}}\dot{\xi}_i.$$

When we look at $\tilde{h}_{ik}$ we also get the transformations for other functions. All transformations are summarized here:

$$\begin{aligned} \tilde{\phi} &= \phi - \dot{\xi}_0 \\ \tilde{B} &= B + \xi_0 + \dot{\zeta} \\ \tilde{S}_i &= S_i - {}^{\mathrm{T}}\dot{\xi}_i \\ \tilde{\psi} &= \psi \\ \tilde{E} &= E + \zeta \\ \tilde{F}_i &= F_i - {}^{\mathrm{T}}\xi_i \\ {}^{\mathrm{TT}}\tilde{h}_{ik} &= {}^{\mathrm{TT}}h_{ik}. \end{aligned}$$

If we assume that in the coordinate system x, the values of B and E are not equal to zero, we can choose $\zeta = -E$. With this, we have $\tilde{E} = E + \zeta = 0$. Then we chose $\xi_0 = -B + \dot{E}$. With this, we also have $\tilde{B} = 0$. It can be shown that $D = 0$ for all coordinate systems, so

$$\tilde{D} = \tilde{\phi} + (\tilde{B} - \dot{\tilde{E}})\dot{} = 0.$$

This implies also $\tilde{\phi} = 0$. So, just by changing the coordinate system, we can set all scalars to 0. For the vectors, choose ${}^{\mathrm{T}}\xi_i = F_i$. With this $\tilde{F}_i = 0$ and $\tilde{S}_i = \dot{\tilde{F}}_i = 0$. All the vector functions also vanished! So we are left with only

$$\Box\,{}^{\mathrm{TT}}h_{ik} = 0, \quad {}^{\mathrm{TT}}h^i{}_i = h^i{}_{k,i} = 0.$$

Perform the Fourier transformation of ${}^{\mathrm{TT}}h_{ik}$ and get

$${}^{\mathrm{TT}}h_{ij} = \int \pi_{ij}(\omega, \mathbf{k}) e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}\, \mathrm{d}\omega\, \mathrm{d}^3k.$$

The equation above becomes:

$$\int (\omega^2 - |\mathbf{k}|^2)\pi_{ij}(\omega, \mathbf{k}) e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}\, \mathrm{d}\omega\, \mathrm{d}^3k = 0.$$

The solution is $\omega^2 = |\mathbf{k}|^2$. Denote $\pi_{ij}(|\mathbf{k}|, \mathbf{k}) = \pi_{ij}(\mathbf{k})$. The two additional conditions become:

$${}^{\mathrm{TT}}h^i{}_i = 0 \implies \pi^i{}_i(\mathbf{k}) = \pi^x{}_x(\mathbf{k}) + \pi^y{}_y(\mathbf{k}) + \pi^z{}_z(\mathbf{k}) = 0$$

and

$$^{\mathrm{TT}}h^{i}{}_{j,k} = 0 \implies \pi^{z}{}_{z}(\mathbf{k}) = \pi^{z}{}_{y}(\mathbf{k}) = \pi^{z}{}_{x}(\mathbf{k}) = 0.$$

The only non-zero components are $\pi^{x}{}_{x}$, $\pi^{x}{}_{y}$, $\pi^{y}{}_{x}$ and $\pi^{y}{}_{y}$. The additional constraint is $\pi^{x}{}_{x} = -\pi^{y}{}_{y}$ and from the fact that π is symmetric, we also know $\pi^{x}{}_{y} = \pi^{y}{}_{x}$. So we can write

$$^{\mathrm{TT}}h_{ik} = \begin{pmatrix} h_{xx} & h_{xy} & 0 \\ h_{yx} & -h_{xx} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The two solutions for $\hat{\mathbf{k}} = \hat{z}$ are

$$^{\mathrm{TT}}h_{xx} = -{}^{\mathrm{TT}}h_{yy} = A_{+}e^{i\omega(t+z)}, \quad {}^{\mathrm{TT}}h_{xy} = 0$$

and

$$^{\mathrm{TT}}h_{xy} = {}^{\mathrm{TT}}h_{yx} = A_{\times}e^{i\omega(t+z)}, \quad {}^{\mathrm{TT}}h_{xx} = 0.$$

The two solutions correspond to the two polarizations of the gravitational wave.

4.2 Weak field in non-empty space

If the energy-tensor is not zero, the equations for ψ, S, F, B and h are

$$\begin{aligned} 4\pi G T^{0}{}_{0} &= 2\nabla^2\psi \\ 4\pi G T^{0}{}_{i} &= 2\frac{\partial}{\partial t}\psi_{,i} + \frac{1}{2}\nabla^2\left(S_i - \frac{\partial}{\partial t}F_i\right) \\ -4\pi G T_{ik} &= -D_{,ik} - \delta_{ik}\left(\nabla^2 D + 2\frac{\partial^2\psi}{\partial t^2}\right) - \frac{1}{2}\left[\left(S_i - \frac{\partial}{\partial t}F_i\right)_{,k} + \left(S_k - \frac{\partial}{\partial t}F_k\right)_{,i}\right] + \frac{1}{2}\Box\,{}^{\mathrm{TT}}h_{ik}. \end{aligned}$$

Taking the difference between the first equation's time derivative and the second equation's spatial derivative, we get $T^{\alpha}{}_{0,\alpha} = 0$. Taking the sum of the second equation's time derivative and the first equation's spatial derivative, we get $T^{\alpha}{}_{i,\alpha} = 0$. Under the SO(3) transformation, $T^{0}{}_{0}$ transforms as a scalar, $T^{0}{}_{i}$ as a vector and $T^{i}{}_{k}$ as a tensor. Let's write $T^{0}{}_{i} = {}^{(\mathrm{v})}T^{0}{}_{i} + \partial_i\alpha$, with ${}^{(\mathrm{v})}T^{0}{}_{i}{}^{,i}$. To express α, take the divergence of the equation and get $T^{0}{}_{i,i} = \Delta\alpha$, where Δ is the Laplacian. The equation for α is then

$$\alpha = \frac{1}{\Delta}T^{0}{}_{i,i} = -\frac{1}{4\pi}\int \frac{T^{0}{}_{i,i}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3\mathbf{x}'.$$

So, the vector part of the energy-momentum tensor can be rewritten as

$$T^{0}{}_{i} = {}^{(\mathrm{v})}T^{0}{}_{i} + \partial_i\left(\frac{1}{\Delta}T^{0}{}_{k,k}\right).$$

Decomposition of the tensor part $T^{i}{}_{k}$ is a bit more complicated. We can write

$$T^{i}{}_{k} = {}^{\mathrm{TT}}T^{i}{}_{k} + \lambda_{,ik} + \mu\delta_{ik} + {}^{(\mathrm{v})}\sigma_{i,k} + {}^{(\mathrm{v})}\sigma_{k,i}.$$

The additional conditions are ${}^{(\mathrm{TT})}T^i_{\ i} = {}^{(\mathrm{TT})}T^i_{\ k,i} = 0$ and ${}^{(\mathrm{v})}\sigma_i{}^{,i} = 0$. Taking the trace of the equation, we get $T^i_{\ i} = \Delta\lambda + 3\mu$. The divergence gives $T^i_{\ k,i} = \partial_k\Delta\lambda + \partial_k\mu + \Delta\,{}^{(\mathrm{v})}\sigma_k$. Two divergences give $T^i_{\ k,ik} = \Delta^2\lambda + \Delta\mu$. Using these three results, we can write

$$\begin{aligned}
\mu &= \frac{1}{2}\left(T^i_{\ i} - \frac{1}{\Delta}T^i_{\ k,ik}\right),\\
\lambda &= \frac{1}{2\Delta}\left(-T^i_{\ i} + \frac{3}{\Delta}T^i_{\ k,ik}\right),\\
{}^{(\mathrm{v})}\sigma_k &= \frac{1}{\Delta}\left(T^i_{\ k,i} - \partial_k\left(\frac{1}{\Delta}T^l_{\ m,lm}\right)\right).
\end{aligned}$$

We can simplify the equations a bit using $T^i_{\ k,ik} = -T^0_{\ k,k0} = T^k_{\ 0,k0} = -\ddot{T}^0_{\ 0}$. In special case of the ideal fluid, this says $(\mathcal{E}v^iv^k)_{,ik} = -\ddot{\mathcal{E}}$. We have

$$\begin{aligned}
\psi &= \frac{4\pi G}{\Delta}T^0_{\ 0},\\
S_i - \dot{F}_i &= \frac{16\pi G}{\Delta}\,{}^{(\mathrm{v})}T^0_{\ i} = \frac{16\pi G}{\Delta}\left(T^0_{\ i} + \partial_i\frac{1}{\Delta}\dot{T}^0_{\ 0}\right),\\
D &= 8\pi G\lambda = -\frac{4\pi G}{\Delta}\left(\frac{3}{\Delta}\ddot{T}^0_{\ 0} + T^i_{\ i}\right)\\
\Delta D + 2\ddot{\psi} &= 8\pi G\mu\\
(S_i - \dot{F}_i)^{\cdot} &= -16\pi G\,{}^{(\mathrm{v})}\sigma_i\\
\Box^{\mathrm{TT}}h_{ik} &= -16\pi G\,{}^{\mathrm{TT}}T_{ik}.
\end{aligned}$$

The first and second equations imply the fourth and fifth equations. Let's first tackle the first equation:

$$\begin{aligned}
\psi(\mathbf{r},t) &= -G\int\frac{T^0_{\ 0}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'\\
&= -G\int T^0_{\ 0}(\mathbf{r}',t)\left(\frac{1}{|\mathbf{r}|} - r'_j\frac{\partial}{\partial r_j}\frac{1}{|\mathbf{r}|} + \frac{1}{2}r'_ir'_j\frac{\partial^2}{\partial r_j\partial r_i}\frac{1}{|\mathbf{r}|} + \dots\right)\mathrm{d}^3\mathbf{r}'\\
&= -G\left\{\frac{1}{|\mathbf{r}|}\int T^0_{\ 0}(\mathbf{r}',t)\,\mathrm{d}^3\mathbf{r}' - \frac{\partial}{\partial r_j}\frac{1}{|\mathbf{r}|}\int r'_iT^0_{\ 0}(\mathbf{r}',t)\,\mathrm{d}^3\mathbf{r}' + \frac{1}{2}\frac{\partial^2}{\partial r_i\partial r_j}\frac{1}{|\mathbf{r}|}\int r'_ir'_jT^0_{\ 0}(\mathbf{r}',t)\,\mathrm{d}^3\mathbf{r}' + \dots\right\}.
\end{aligned}$$

In the non-relativistic approximation, we have $p \ll \mathcal{E}$ and $|\mathbf{v}| \ll c$. Then $T^0_{\ 0} \approx \mathcal{E}$ and

$$\int T^0_{\ 0}(\mathbf{r}',t)\,\mathrm{d}^3\mathbf{r}' = M(t).$$

The mass is conserved, so $M(t) = M$. The dipole moment is

$$d_j(t) = \int T^0_{\ 0}(\mathbf{r}',t)r'_j\,\mathrm{d}^3\mathbf{r}' = \int\mathcal{E}r'_j\,\mathrm{d}^3\mathbf{r}'.$$

The dipole moment also doesn't change with time since the momentum is conserved:

$$\dot{d}_j(t) = \int\dot{T}^0_{\ 0}\,r'_j\,\mathrm{d}^3\mathbf{r}' = \int\dot{T}^i_{\ 0,i}\,r'_j\,\mathrm{d}^3\mathbf{r}' = \int\dot{T}^0_{\ j}\,\mathrm{d}^3\mathbf{r}' = \int\mathcal{E}v_j\,\mathrm{d}^3\mathbf{r}'.$$

The quadrupole moment is the first moment that is not conserved:

$$Q_{ij}(t) = \int \left(r'_i r'_j - \frac{1}{3}|\mathbf{r}'|^2 \delta_{ij} \right) T^0{}_0(\mathbf{r}', t)\, \mathrm{d}^3\mathbf{r}'.$$

Up to the quadrupole moment, the value of ψ is

$$\psi(\mathbf{r}, t) = -G\left(\frac{M}{|\mathbf{r}|} + \frac{\mathbf{r}\cdot\mathbf{d}}{|\mathbf{r}|^3} + \frac{3}{2}\frac{r_i r_j Q_{ij}(t)}{|\mathbf{r}|^5} \right).$$

By choosing an appropriate coordinate system, we can set $B = E = 0$ so that $D = \phi - \psi$. If we look at the equation for D, we see that both terms are tiny:

$$T^i{}_i = (\mathcal{E} + p)u^i u_i - 3p \ll T^0{}_0$$

and

$$\frac{1}{\Delta}\ddot{T}^0{}_0 \approx \frac{1}{|\mathbf{r}|}\frac{\partial^2}{\partial t^2}\int T^0{}_0(\mathbf{r}, t)\, \mathrm{d}^3\mathbf{r}' \approx 0.$$

Hence, $D \approx 0$ and $\phi \approx \psi$. The invariant interval is then

$$\mathrm{d}s^2 = (1 + 2\phi)\mathrm{d}t^2 - (1 - 2\phi)\delta_{ik}\mathrm{d}x^i \mathrm{d}x^j.$$

Now, we will look at $S_i - \dot{F}_i$:

$$\begin{aligned} S_i - \dot{F}_i &= \frac{16\pi G}{\Delta}\left(T^0{}_i - \partial_i \frac{1}{\Delta}\dot{T}^0{}_0 \right) \\ &= -4G\int \frac{T^0{}_i - \partial_i(\frac{1}{\Delta}\dot{T}^0{}_0)}{|\mathbf{r} - \mathbf{r}'|}\mathrm{d}^3\mathbf{r}' \\ &\approx -4G\frac{r_j}{|\mathbf{r}|^3}\int r'_j T^0{}_i(\mathbf{r}', t)\, \mathrm{d}^3\mathbf{r}' \end{aligned}$$

Setting $\dot{F}_i = 0$, the equation for S_i becomes

$$S_i = -4G\frac{r_j}{|\mathbf{r}|^3}M_{ij}, \qquad M_{ij} = \int r'_j T^0{}_i(\mathbf{r}', t)\, \mathrm{d}^3\mathbf{r}'.$$

The term S contributes a term $2S_i \mathrm{d}x_i \mathrm{d}t$ to the metric. Let's look at the geodesic equation in the non-relativistic limit. We have $u^i \ll 1$, $u^0 \approx 1$ and $\mathrm{d}s \approx \mathrm{d}t$. Hence,

$$\frac{\mathrm{d}u^i}{\mathrm{d}t} + \Gamma^i{}_{00}u^0 u^0 + 2\Gamma^i{}_{j0}u^j u^0 + \Gamma^i{}_{kl}u^k u^l = 0.$$

The last term has the factor of u^i twice, so we can neglect it. Furthermore,

$$\Gamma^i{}_{00} = \frac{1}{2}\left(h^i{}_{0,0} + h^i{}_{0,0} - h^0{}_{0,i} \right) \approx 0$$

and

$$\Gamma^i{}_{j0} = \frac{1}{2}\left(h^i{}_{j,0} + h^i{}_{0,j} - h^j{}_{0,i} \right) \approx \frac{1}{2}\left(h_{j0,i} - h_{i0,j} \right) = \frac{1}{2}(S_{j,i} - S_{i,j}).$$

Using this, we get

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} \approx (S_{i,j} - S_{j,i})v^j = \mathbf{v} \times \nabla \times \mathbf{S}$$

or, equivalently,

$$\mathbf{F} = m(\mathbf{v} \times \nabla \times \mathbf{S}).$$

This is some sort of Coriolis force that is present only in general relativity. Now, we will return to the calculation of gravitational waves. Using $\Box^{\mathrm{TT}} h_{ij} = -16\pi G\, {}^{\mathrm{TT}}T_{ij}$, we have

$$\begin{aligned}
{}^{\mathrm{TT}}h_{ij}(\mathbf{r},t) &= -\frac{16\pi G}{\Box}\, {}^{\mathrm{TT}}T_{ij}(\mathbf{r},t) \\
&= 4G \int \frac{{}^{\mathrm{TT}}T_{ij}(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}\, \mathrm{d}^3\mathbf{r}' \\
&\overset{\mathbf{r}\gg\mathbf{r}'}{=} 4G \int \frac{{}^{\mathrm{TT}}T_{ij}(\mathbf{r}', t-|\mathbf{r}|/c)}{|\mathbf{r}|}\, \mathrm{d}^3\mathbf{r}' \\
&= -\frac{4G}{|\mathbf{r}|}\left(\int T^i{}_k(\mathbf{r}', t-|\mathbf{r}'|)\, \mathrm{d}^3\mathbf{r}\right)^{\mathrm{TT}}.
\end{aligned}$$

Let's now look at the integral of the metric tensor:

$$\begin{aligned}
\int T^i{}_k\, \mathrm{d}^3\mathbf{r} &= \frac{1}{2}(\delta^i{}_m \delta^k{}_l + \delta^i{}_l \delta^k{}_m) \int T^m{}_l\, \mathrm{d}^3\mathbf{r} \\
&= \frac{1}{2}\int \partial_l \partial_m (r^i r^k) T^m{}_l\, \mathrm{d}^3\mathbf{r} \\
&= \frac{1}{2}\int r^i r^k T^l{}_{m,lm}\, \mathrm{d}^3\mathbf{r} \\
&= -\frac{1}{2}\frac{\partial^2}{\partial t^2}\int r^i r^k T^0{}_0\, \mathrm{d}^3\mathbf{r}.
\end{aligned}$$

We denote

$$X^{ik}(t) = X_{ik}(t) = \int r^i r^k T^0{}_0(\mathbf{r}', t)\, \mathrm{d}^3\mathbf{r}'.$$

To get the transverse traceless part of this tensor, we consider the projection operator

$$P_{lm} = \delta_{lm} - \frac{r_l r_m}{|\mathbf{r}|^2}, \quad P_{lm}P_{mn} = P_{ln}.$$

The transverse traceless part of the tensor is then

$${}^{\mathrm{TT}}X_{ik} = P_{im}X_{ml}P_{lk} - \frac{1}{2}P_{ik}P_{lm}X_{lm}.$$

We can check that ${}^{\mathrm{TT}}X_{ii} = 0$ and for $i \neq j$

$$\lim_{|\mathbf{r}|\to\infty}\left|\frac{{}^{\mathrm{TT}}X_{ik,i}}{{}^{\mathrm{TT}}X_{ik,j}}\right| = 0.$$

Furthermore, it can be shown that ${}^{\mathrm{TT}}X_{ik} = {}^{\mathrm{TT}}Q_{ik}$ where

$${}^{\mathrm{TT}}Q_{ik} = P_{im}Q_{ml}P_{lk} - \frac{1}{2}P_{ik}P_{lm}Q_{lm},$$

$$Q_{ik} = \int (r^i r^k - |\mathbf{r}|^2\delta^{lm}/3) T^0{}_0\, \mathrm{d}^3\mathbf{r}.$$

Using all these results, the transverse traceless part of the metric tensor can be written as

$$\boxed{{}^{\mathrm{TT}}h_{ik}(\mathbf{r},t) = \frac{2G}{|\mathbf{r}|}\,{}^{\mathrm{TT}}\ddot{Q}_{ik}(t-|\mathbf{r}|/c).}$$

This is called the *quadrupole formula*. Remember from the electromagnetism that the rate of energy emitted by a moving electric dipole is proportional to the square of the second derivative of the dipole moment (Larmor formula):

$$P = \frac{\mu_0|\ddot{\mathbf{p}}|^2}{6\pi c}.$$

By this parallel, we can conclude that the power emitted by gravitational waves is proportional to the square of the third time derivative of the mass quadrupole moment.

4.3 Energy-momentum tensor of gravitation waves

Assume that there is small perturbation $h_{\alpha\beta}$ on top of the background metric $g^{(0)}_{\alpha\beta}$. We will also assume that the amplitude and wavelength of the perturbation is much smaller than the characteristic length scale of the background metric:

$$\lambda \ll L, \qquad |h_{\alpha\beta}| \ll g^{(0)}_{\alpha\beta}.$$

The Einstein equation reads

$$G^{\alpha}{}_{\beta}(g^{(0)}+h) = 8\pi G T^{\alpha}{}_{\beta}\,.$$

Expanding the Einstein tensor to the second order and averaging over a large distance L gives

$$\left\langle G^{(0)}(g^{(0)})\right\rangle + \left\langle G^{(1)}(h)\right\rangle + \left\langle G^{(2)}(h)\right\rangle + \cdots = 8\pi G T.$$

Averaging $g^{(0)}$ over the length L changes nothing (as $g^{(0)}$ is essentially constant at that scale). The second term, however, is averaged to zero. So we write

$${G^{(0)}}^{\alpha}{}_{\beta}(g^{(0)}) = 8\pi G\left(T^{\alpha}{}_{\beta} - \frac{1}{8\pi G}\left\langle (G^{(2)})^{\alpha}{}_{\beta}(h^2)\right\rangle\right).$$

Hence, we denote

$${}^{\mathrm{GW}}T^{\alpha}{}_{\beta} = -\frac{1}{8\pi G}\left\langle (G^{(2)})^{\alpha}{}_{\beta}(h^2)\right\rangle$$

To get the power emitted, we have to consider the ^{0i} components of the energy-momentum tensor:

$${}^{\mathrm{GW}}T^{0}{}_{j} = \frac{1}{32\pi G}\,{}^{\mathrm{TT}}h^{i}{}_{k,0}\,{}^{\mathrm{TT}}h^{k}{}_{i,j} = -\frac{1}{32\pi G}\frac{4G^2}{|\mathbf{r}|^2}\,{}^{\mathrm{TT}}\dddot{Q}_{ik}\,{}^{\mathrm{TT}}\dddot{Q}_{ik}\frac{r_j}{|\mathbf{r}|}.$$

The power emitted is

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \int T^{0}{}_{j}\frac{r^j}{|\mathbf{r}|}|\mathbf{r}|^2\,\mathrm{d}\Omega = \frac{G}{5c^5}\dddot{Q}_{ik}\dddot{Q}_{ik}.}$$

This is very tedious to derive. One needs to resort to the following relations:

$$\int n^l n^m \,\mathrm{d}\Omega = \frac{4\pi}{3}\delta^{lm}, \qquad \int n^l n^m n^k n^r \,\mathrm{d}\Omega = \frac{4\pi}{15}(\delta^{lm}\delta^{kr} + \delta^{lk}\delta^{mr} + \delta^{lr}\delta^{mk}).$$

In the next discussion, we will use Planck's units:

$$\begin{aligned} l_\mathrm{P} &= \sqrt{\frac{G\hbar}{c^3}} = 1{,}616 \cdot 10^{-33}\,\mathrm{cm}, \\ m_\mathrm{P} &= \sqrt{\frac{\hbar c}{G}} = 2{,}176 \cdot 10^{-8}\,\mathrm{kg}, \\ t_\mathrm{P} &= \sqrt{\frac{G\hbar}{c^5}} = 5{,}391 \cdot 10^{-44}\,\mathrm{s}, \\ T_\mathrm{P} &= \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1{,}416 \cdot 10^{32}\,\mathrm{K}. \end{aligned}$$

Now we set $\hbar = G = c = k_B = 1$. Assume we have two masses M circulating each other in a circular orbit with radius L and period P. The quadrupole moment is in the first approximation just $Q_{ik} \sim ML^2$ and $\ddot{Q}_{ik} \sim ML^2P^{-3}$. The power emitted is then

$$\frac{\mathrm{d}E_\mathrm{GW}}{\mathrm{d}t} \sim \omega^6 L^4 M^2 \sim (\omega M)^{10/3}$$

where we used the virial theorem to show $L \sim M^{1/3}\omega^{-2/3}$. The total energy of binary system is $E_\mathrm{BS} \sim -M^2/L \sim \omega^{2/3}M^{5/3}$. The energy conservation gives

$$\begin{gathered} \frac{1}{E_\mathrm{BS}}\frac{\mathrm{d}E_\mathrm{BS}}{\mathrm{d}t} = \frac{1}{E_\mathrm{BS}}\frac{\mathrm{d}E_\mathrm{GW}}{\mathrm{d}t}, \\ \frac{2}{3}\frac{\dot{\omega}}{\omega} \sim \frac{(\omega M)^{10/3}}{\omega^{2/3}M^{5/3}}, \\ \frac{\mathrm{d}P}{\mathrm{d}t} = (M\omega)^{5/3}. \end{gathered}$$

For a system with $P = 7$ h 45min 7sec, $M = 1.4M_\odot$, we get $\mathrm{d}P/\mathrm{d}t \sim -10^{-13}$. Calculating this result much more precisely gives $\mathrm{d}P/\mathrm{d}t = -2{,}4 \cdot 10^{-12} = -7{,}2 \cdot 10^{-5}\,\mathrm{s/year}$.

Chapter 5

Cosmology

5.1 Friedmann-Lemaître-Robertson-Walker metric

We start with two assumptions about the universe's large-scale structure – there exist hypersurfaces of equal time, such that the universe is *homogenous* and *isotropic*. In two dimensions, the possible isotropic metrics are

$$\begin{aligned}
\mathrm{d}s^2 &= a^2(\mathrm{d}\theta^2 + \mathrm{d}\sin^2(\theta)\varphi^2),\\
\mathrm{d}s^2 &= a^2(\mathrm{d}\theta^2 + \theta^2\mathrm{d}\varphi^2) = r^2 + r^2\mathrm{d}\varphi^2,\\
\mathrm{d}s^2 &= a^2(\mathrm{d}\theta^2 + \sinh^2(\theta)\mathrm{d}\varphi^2).
\end{aligned}$$

In three dimensions, the possible isotropic metrics are

$$\begin{aligned}
\mathrm{d}s^2 &= a^2(\mathrm{d}\chi^2 + \sin^2(\chi)(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2)),\\
\mathrm{d}s^2 &= a^2(\mathrm{d}\chi^2 + \chi^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2)) = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2,\\
\mathrm{d}s^2 &= a^2(\mathrm{d}\chi^2 + \sinh^2(\chi)(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2))
\end{aligned}$$

In general, we can write

$$\mathrm{d}l^2_{3\mathrm{D}} = a^2(\mathrm{d}\chi^2 + \Phi^2(\chi)\mathrm{d}\Omega^2)$$

with $\Phi(\chi) = \{\sin\chi, \chi, \sinh\chi\}$. We want to express the function Φ as one thing, not in three separate cases. Observe that if $r = \sin\chi$ then $\mathrm{d}\chi = \mathrm{d}r/\sqrt{1-r^2}$ and if $r = \sinh\chi$ then $\mathrm{d}\chi = \mathrm{d}r/\sqrt{1+r^2}$. So we write

$$\mathrm{d}l^2_{3\mathrm{D}} = a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2)\right),$$

where $k = +1$ for sphere, $k = 0$ for flat space and $k = -1$ for pseudo-sphere (saddle). Notice that the only parameter that can change with time in the above equation for it to remain isotropic is $a(t)$. In 4 dimensions, we write

$$\boxed{\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right].}$$

This is called *Friedmann-Lemaître-Robertson-Walker metric*. Let's look at different cases.

- In *spherical geometry* ($k = +1$) in 3 dimensions the surface metric is $\mathrm{d}l2\mathrm{D}^2 = a^2 \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)$. The area of the sphere is $S_{2\mathrm{D}}(\chi) = 4\pi a \sin^2\chi$, hence the possible values for χ are on the interval $(0, \pi)$. The volume of a ball with radius χ_0 is

$$V(\chi_0) = \int_0^{\chi_0} 4\pi a^3 \sin^2\chi\,\mathrm{d}\chi = 2\pi a^3\left(\chi_0 - \frac{1}{2}\sin(2\chi_0)\right).$$

  The volume of the whole universe is $V = 2\pi^2 a^3$.

- The volume metric in *hyperbolic space* is $\mathrm{d}l_{2\mathrm{D}}^2 = a^2(\mathrm{d}\chi^2 + \sinh^2(\chi)\mathrm{d}\Omega^2)$. The surface of a sphere is $S_{2\mathrm{D}} = 4\pi a^2 \sinh^2\chi$ which grows exponentially with χ.

5.2 Friedmann equations

Let's assume we are at the coordinate $\chi = 0$. The objects on constant χ are moving along geodesics. Assume a galaxy A is positioned at χ_A. The actual distance to the galaxy is $D_A(t) = a(t)\chi_A$. The velocity of the galaxy

$$v_A = \frac{\mathrm{d}D_A(t)}{\mathrm{d}t} = \frac{\dot{a}}{a}\chi_A a = \frac{\dot{a}}{a}D_a = H(t)D_A.$$

We introduced a parameter $H(t) = \dot{a}/a$, called *Hubble constant*. Indeed, it is not constant, but it changes rather slowly. So if we want to be really fastidious, what we measure today as H_0 would be called *Hubble parameter* (measured today). The unit of the Hubble constant is $[t^{-1}]$, so the reciprocal of the Hubble constant is a rough measure of the age of the universe (called the Hubble time). To describe the early universe, we have three functions: the scaling factor $a(t)$, the energy density $\mathcal{E}(t)$ and the pressure as a function of the energy density:

$$p(\mathcal{E}) = \begin{cases} \frac{1}{3}\mathcal{E}, & \text{radiation,} \\ 0, & \text{dust (baryonic and CDM),} \\ -\mathcal{E}, & \text{dark energy (DE).} \end{cases}$$

This is called the *equation of state*. The acronym CDM refers to the cold dark meter. With these functions, we can construct the energy-momentum tensor

$$T^\alpha{}_\beta = (\mathcal{E} + p)v^\alpha v_\beta - p\delta^\alpha{}_\beta.$$

Let us assume we have the equation of motion for a scalar field φ given by

$$\Box\varphi + V_{,p} = 0$$

where V is some potential. The energy-momentum tensor for the scalar field is

$$T^\alpha{}_\beta = \varphi^{,\alpha}\varphi_{,\beta} - (\varphi^{,\gamma}\varphi_{,\gamma} - V)\delta^\alpha{}_\beta.$$

The equation of motion exactly gives the condition $T^\alpha{}_{\beta;\alpha} = 0$. We can define pressure and energy density as

$$\mathcal{E} = \frac{1}{2}\varphi^{,\gamma}\varphi_{,\gamma} + V, \quad p = \frac{1}{2}\varphi^{,\gamma}\varphi_{,\gamma} - V.$$

If we also define $u^\alpha = \varphi^{,\alpha}/\sqrt{\varphi^{,\gamma}\varphi_{,\gamma}}$. If we assume φ is only function of time and that $V \gg \dot{\varphi}^2$, we get $p \approx -\mathcal{E}$.

Assume now a spherically symmetric universe filled with matter and a particle at a comoving coordinate χ. The mass that affects the particle is the mass M inside the sphere of radius $r = a(t)\chi$. The Newton's law says

$$\ddot{r} = \ddot{a}\chi = -\frac{GM}{r^2} = -\frac{4\pi G}{3}\mathcal{E}a\chi.$$

When we derive this equation from the Einstein equations, we get

$$\boxed{\ddot{a} = -\frac{4\pi}{3}G(\mathcal{E} + 3p)a.}$$

This is called the *second Friedmann equation*. To see why we get the additional term $3p$, we can consider the infinitesimal change in energy:

$$\mathrm{d}E = \mathrm{d}(\mathcal{E}V) = -p\mathrm{d}V = -p\mathrm{d}(a\chi)^3 = -3\chi^3 pa^2\mathrm{d}a.$$

So we get $\mathrm{d}\mathcal{E} = -3(\mathcal{E} + p)\mathrm{d}a/a$ and so

$$\boxed{\dot{\mathcal{E}} = -3(\mathcal{E} + p)H.}$$

Taking the second Friedmann equation, multiplying it with $\dot{a}$ and using the 00-th Einstein equation to determine the integration constant, we get

$$\boxed{H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\mathcal{E}.}$$

This is called *first Friedmann equation*. Let's denote $p = w\mathcal{E}$. The equation that relates energy and pressure can be written as

$$\frac{\dot{\mathcal{E}}}{\mathcal{E}} = -3(1 + w)\frac{\dot{a}}{a}.$$

The solution to this equation is $\mathcal{E} \propto a^{-3(1+w)}$. In dust dominated universe, $w = 0$ and $\mathcal{E} \propto a^{-3}$. In radiation dominated universe, $w = 1/3$ and $\mathcal{E} \propto a^{-4}$. In dark energy dominated universe, $w = -1$ and $\mathcal{E} = \text{const}$. In flat space-time, we have

$$H^2 = \mathcal{E}_0 \left(\frac{a_0}{a}\right)^{3(1+w)}.$$

Since $H^2 \propto t^{-2}$. Hence,

$$a(t) = t^{\frac{2}{3(1+w)}}.$$

For $w = 0$, we have $a \propto t^{2/3}$, if $w = 1/3$ then $a \propto t^{1/2}$ and if $w = -1$ $a \propto e^{Ht}$. The Hubble parameter is

$$H = \frac{2}{3(1 + w)}\frac{1}{t}$$

and the energy density is

$$\mathcal{E} = \frac{1}{6\pi G(1 + w)^2 t^2}.$$

Chapter 6

Conformal diagrams

We will introduce the concept of *conformal time* η. It is defined such that $\mathrm{d}t^2 = a^2(\eta)\,\mathrm{d}\eta^2$, so

$$\eta = \int \frac{\mathrm{d}t}{a(t)}.$$

With this, we can write

$$\mathrm{d}s^2 = a(\eta)(\mathrm{d}\eta^2 - \mathrm{d}\chi^2 - \Phi^2(\eta)\,\mathrm{d}\Omega^2).$$

We also denote a' to be the derivative of a with respect to η. The first Friedmann equation can be written as

$$\left(\frac{a'}{a^2}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\mathcal{E}$$

or equivalently

$$a'^2 + ka^2 = \frac{8\pi G}{3}\mathcal{E}a^4$$

and the energy-pressure relation as

$$\mathcal{E}' = -3\frac{a'}{a}(\mathcal{E} + p).$$

Differentiating the first Friedmann equation with respect to η and inserting the equation above gives

$$a'' + ka = \frac{4\pi G}{3}(\mathcal{E} - 3p)a^3.$$

6.1 Milne universe

Consider $\mathcal{E} = 3p$. We will examine the solutions for two different curvatures:

- For $k = 1$ the equation becomes $a'' + a = 0$ and the solution is $a = A\sin(\eta)$. The time is $t = A(1 - \cos\eta)$. The value of a is maximal at $\eta = \pi/2$ and zero again at $\eta = \pi$.
- For $k = -1$ the solutions are $a = A\sinh\eta$ and $t = A(1 - \cosh\eta)$.

In case $p = 0$ we have:

- for $k = 1$, the equation is
$$a'' + a = \frac{4\pi G}{3}\mathcal{E}a^3 = \text{const.}$$
The solution is $a = A(1 - \cos\eta)$ and time is $t = A(\eta - \sin\eta)$. The maximal value of a is at $\eta = \pi$, and it comes back to zero at $\eta = 2\pi$.
- For $k = -1$ the solutions are $a = A\sinh\eta$ and $t = A(\cosh\eta - \eta)$.

The last case we will check is $\mathcal{E} \to 0$. In empty space, we have $\dot{a}^2 = -k$.

- If $k = 0$, then the solution is $a = \text{const}$ and the metric is Minkowski
$$\mathrm{d}s^2 = \mathrm{d}\tau^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2.$$
- If $k = -1$, the solution is $a = t$ and
$$\mathrm{d}s^2 = \mathrm{d}t^2 - t^2(\mathrm{d}\chi^2 + \sinh^2\chi\,\mathrm{d}\Omega^2).$$
This is called *Milne universe*. We can change the metric to Minkowski using the transformation
$$\tau = t\cosh\chi, \qquad r = t\sinh\chi,$$
$$\tau^2 - r^2 = r^2, \qquad \frac{r}{t} = \tanh\chi.$$

This means that the metrics are locally the same, but not globally. Similarly, the equal-time slices in the universe can have zero curvature, but the whole universe can have non-zero curvature. The converse also holds – there can be equal-time slices with non-zero curvature in a universe with zero curvature. In other words, flat space doesn't have to be Minkowski.

6.2 De Sitter universe

For the dark matter, we have $\mathcal{E}_\Lambda = -p_\Lambda$. This implies $\mathcal{E}_\Lambda = \text{const}$. The first Friedmann equation then gives
$$\ddot{a} = -\frac{4\pi G}{3}(\mathcal{E}_\Lambda + 3p_\Lambda)a = H_0^2 a$$
where $H_0^2 = 8\pi G\mathcal{E}_\Lambda/3$. The solution is
$$a = C_1 = \exp(H_0 t) + C_2\exp(-H_0 t).$$
The first Friedmann equation gives another constraint on the constants, namely $4H_0^2 C_1 C_2 = k$. We will again separate the discussion for different curvatures:

- For $k = 0$, one of the constants must be zero, so $a \propto e^{H_0 t}$. In the case $k \neq 0$, we can always make $|C_1| = |C_2|$ by shifting time.
- For $k = 1$, we have $C_1 = C_2 = 1/2H_0$ and
$$a = \frac{1}{H_0}\cosh(H_0 t).$$
- For $k = -1$, we have $C_1 = -C_2 = 1/2H_0$ and
$$a = \frac{1}{H_0}\sinh(H_0 t).$$

We got three different solutions for the same energy density. The metric reads

$$\mathrm{d}s^2 = \mathrm{d}t^2 - H_0^2 \begin{Bmatrix} \sinh^2(H_0 t) \\ \exp(2H_0 t) \\ \cosh^2(H_0 t) \end{Bmatrix} \left(\mathrm{d}\chi^2 + \begin{Bmatrix} \sinh^2 \chi \\ \chi^2 \\ \sin^2 \chi \end{Bmatrix} \mathrm{d}\Omega^2 \right).$$

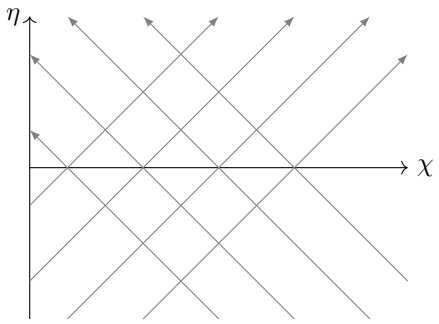
The three cases converge for big t to the same metric – the flat one. The last is called *de Sitter space*, and the first is the *anti-de Sitter space*. They are, respectively, the maximally symmetric solutions to the space with constant negative and positive curvature.

6.3 Conformal diagrams

Now, return to conformal time. The metric is

$$\mathrm{d}s^2 = a^2(\eta)(\mathrm{d}\eta^2 - \mathrm{d}\chi^2 - \Phi^2(\chi)\mathrm{d}\Omega^2).$$

Let's draw a diagram with η on the y-axis and χ on the x-axis (we will assume that Ω is not changing as we consider straight lines). To draw geodesics on the diagram, consider $\mathrm{d}s^2 = 0$ and hence $\mathrm{d}\eta^2 = \mathrm{d}\chi^2$. The curves are $\chi(\eta) \pm \eta + c$.



To get the dependence of η on time, we integrate over the reciprocal of the scale factor. The dependence of χ on η can be written

$$\chi_p(\eta) = \eta - \eta_0 = \int_{t_i}^{t} \frac{\mathrm{d}t}{a(t)}.$$

The size of the causally connected region is

$$d_p(t) = a(t)\chi_p(\eta) = a(t) \int_0^t \frac{\mathrm{d}t}{a(t)} = \begin{cases} 2t, & p = \mathcal{E}/3, \\ 3t, & p = 0. \end{cases}$$

The definition of horizon is $\chi_e(\eta) = \eta_{\mathrm{end}} - \eta$ and so

$$d_e(t) = a(t) \int_t^{t_{\max}} \frac{\mathrm{d}t}{a(t)}.$$

The time t_{end} can go to infinity if $a(t)$ increases fast enough, faster than t. In the de Sitter space, we get

$$d_e(t) = a_0 e^{H_0 t} \int_t^{\infty} \frac{\mathrm{d}t}{a_0} e^{-H_0 t} = \frac{1}{H_0}.$$

Let's write the spherically symmetric metric

$$\mathrm{d}s^2 = g_{ab}(x^c)\,\mathrm{d}x_a\mathrm{d}x^b + R^2(x^c)\mathrm{d}\Omega^2$$

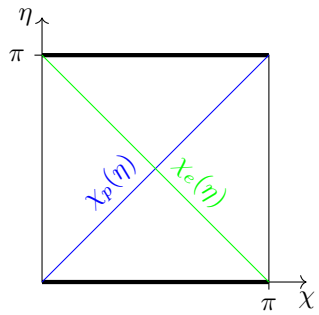
where $a, b = 0, 1$. Now we define

$$\tilde{x}^a = (\tilde{\eta}(x^a), \tilde{\chi}(x^a)).$$

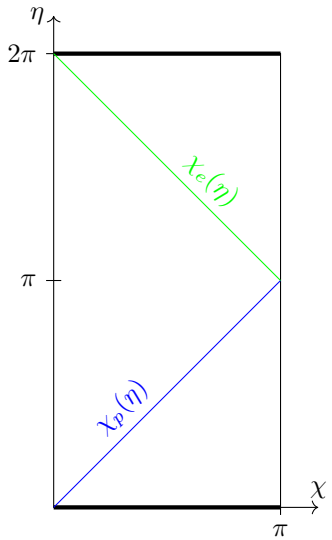
Let $\tilde{g}_{01} = 0$ and $\tilde{g}_{00} = \tilde{g}_{11} = a^2(\tilde{\eta}, \tilde{\chi})$. We get

$$\mathrm{d}s^2 = a^2(\tilde{\eta}, \tilde{\chi})(\mathrm{d}\tilde{\eta}^2 - \mathrm{d}\tilde{\chi}^2 - \Phi^2(\tilde{\eta}, \tilde{\chi})\mathrm{d}\Omega^2).$$

There are two defining characteristics of the *conformal diagrams*: light travels at 45° angles, and diagrams are finite. This means we need to have some function that maps infinite intervals to finite ones; a typical choice would be hyperbolic tangent $\eta = \tanh\tilde{\eta}$. For $k = +1$ and $p = \mathcal{E}/3$ we have $0 \leq \eta \leq \pi$ and $0 \leq \chi \leq \pi$, so the conformal diagram looks like this.



The upper thick line represents the *event horizon*, and the lower thick line represents the *particle horizon*. As we calculated before we have $\chi_p(\eta) = \eta - \eta_i$ and $\chi_e(\eta) = \pi - \eta$. Let's look at the conformal diagram for $p = 0$.



The observers move along the y axis as we stand at the origin. $\chi = 0$ and $\chi = \text{const}$ are geodesics. In this case, the line $\chi = 0$ represents the North Pole, and $\chi = \pi$ represents the South Pole. Each point on the diagram represents an appropriately large sphere.

6.4 De Sitter space

Let's now consider the dark energy-dominated universe ($\varepsilon = -p$). Again, we distinguish three cases.

- In the case $k = +1$, the interval is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{1}{H_0^2}\cosh^2(H_0 t)(\mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\Omega^2).$$

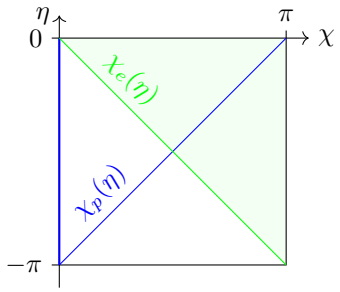
The range for t is $-\infty < t < \infty$ and for χ is $0 \leq \chi \leq \pi$. Introducing the conformal time

$$\eta = \int \frac{\mathrm{d}t}{a} = \int_{-\infty}^{t} \frac{H_0\,\mathrm{d}t}{\cosh(H_0 t)} = \arctan(\sinh(H_0 t)) - \frac{\pi}{2}$$

gives ($\sinh(H_0 t) = \cot(\eta)$)

$$\mathrm{d}s^2 = \frac{1}{H_0^2 \sin^2\eta}\left(\mathrm{d}\eta^2 - \mathrm{d}\chi^2 - \sin^2\chi\,\mathrm{d}\Omega^2\right)$$

which is defined for $-\pi < \eta < 0$ and $0 \leq \chi \leq \pi$. We have $\chi_p = \eta - \eta_i = \eta + \pi$ (denoted blue) and $\chi_e = \eta_{\mathrm{end}} - \eta = -\eta$ (denoted green). The conformal diagram looks like this.



Our location is always on the vertical blue line. The light from the green-shaded area cannot reach us at any time because the universe is expanding too fast.

- The next case is $k = 0$. The invariant interval is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{1}{H_0^2}e^{2H_0 t}(\mathrm{d}\overline{\chi}^2 + \overline{\chi}^2\,\mathrm{d}\Omega^2)$$

$$\mathrm{d}s^2 = \frac{1}{H_0^2\overline{\eta}^2}(\mathrm{d}\overline{\eta}^2 - \mathrm{d}\overline{\chi}^2 - \overline{\chi}^2\,\mathrm{d}\Omega^2)$$

with ranges $-\infty < t < \infty$ and $0 \leq \overline{\chi} < \infty$. However, when we introduce the conformal time $\overline{\eta}$, we express $\overline{\eta}$ and $\overline{\chi}$ with different parameters η and χ. First, we get The ranges for $\overline{\chi}$ and $\overline{\eta}$ are $0 \leq \overline{\chi} < \infty$ and $-\infty < \overline{\eta} < \infty$. Infinite intervals are not desired for conformal diagrams. So, we use the transformation

$$\overline{\eta} = \frac{\sin\eta}{\cos\eta + \cos\chi}, \qquad \overline{\chi} = \frac{\sin\chi}{\sin\eta + \cos\chi}.$$

We can see that $\overline{\chi}$ and $\overline{\eta}$ only cover half of the space ($\eta \times \chi$) (the triangle $(0,0)$-$(0,\pi)$-$(\pi,0)$).

- For $k = -1$, we have

$$\mathrm{d}s^2 = \mathrm{d}\tilde{t}^2 - \frac{1}{H_0^2}\sinh^2(H_0 t)(\mathrm{d}\tilde{\chi}^2 + \sinh^2\tilde{\chi}\,\mathrm{d}\Omega^2).$$

Introducing the conformal time

$$\tilde{\eta} = \int_{-\infty}^{t} \frac{H_0\,\mathrm{d}t}{\sinh(H_0 t)} = \ln(\tanh(H_0|t|/2)).$$

When we invert this expression, we get $\tanh(H_0|t|/2) = e^{\tilde{\eta}}$. The domains are $0 \leq t < \infty$, $0 \leq \tilde{\chi} < \infty$ and $0 < \tilde{\eta} < \infty$. The new invariant interval is

$$\mathrm{d}s^2 = \frac{1}{H_0^2 \sinh^2\tilde{\eta}}(\mathrm{d}\tilde{\eta}^2 - \mathrm{d}\tilde{\chi}^2 - \sinh^2\tilde{\chi}^2\,\mathrm{d}\Omega^2).$$

To get finite coordinates, we use the transformation

$$\tanh\tilde{\eta} = \frac{\sin\eta}{\cos\chi}, \qquad \tanh\tilde{\chi} = \frac{\sin\chi}{\cos\eta}.$$

Coordinates $\tilde{\chi}$ and $\tilde{\eta}$ cover only one eighth of the whole coordinate space ($\eta \times \chi$) (the triangle $(0,0)$-$(0,\pi/2)$-$(\pi/2,0)$).

6.5 Minkowski space

Now, return to Minkowski space. The metric is

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 - \mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2.$$

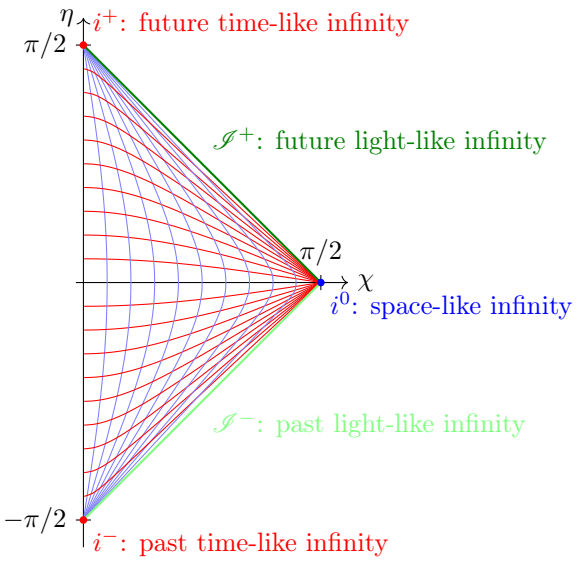
The bounds for τ and r are $-\infty < \tau < \infty$ and $0 \leq r < \infty$. We want to introduce the conformal time η and the conformal distance χ such that they will be bounded. One option is

$$\tanh\tau = \frac{\sin\eta}{\cos\chi}, \qquad \tanh r = \frac{\sin\chi}{\cos\eta}.$$

The bounds are $-\pi/2 < \eta < \pi/2$ and $0 \leq \chi \leq \pi/2$. The metric is

$$\mathrm{d}s^2 = \frac{1}{\cos^2\chi - \sin^2\eta}(\mathrm{d}\eta^2 - \mathrm{d}\chi^2 - \Phi^2(\eta,\chi)\,\mathrm{d}\Omega^2).$$

The conformal diagram looks like this.



The lines of constant τ are drawn in red, lines of constant r are drawn in blue. Line $\mathscr{I}^+$ in the future infinity where $r, t \to \infty$ but $(r - t) = \text{const}$ is drawn in dark green, the line $\mathscr{I}^-$ in the past infinity where $r, t \to -\infty$ but $(r + t) = \text{const}$ is drawn in light green. The lines of constant r have a prescription

$$r = \tan(\eta + \chi) - \tan(\eta - \chi)$$

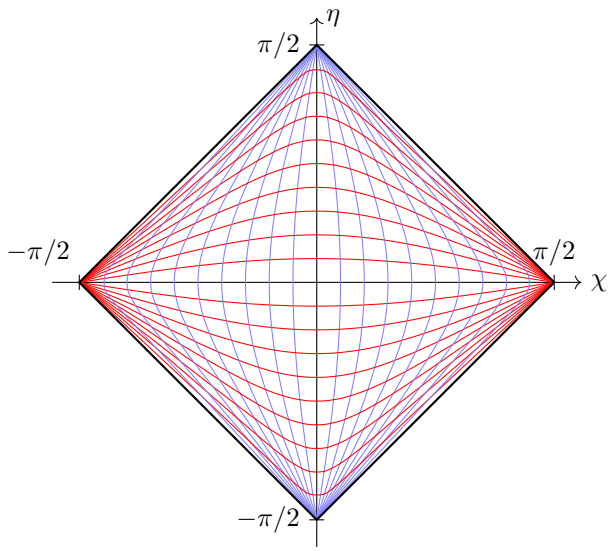
and lines of constant τ have a prescription

$$\tau = \tan(\eta + \chi) + \tan(\eta - \chi).$$

We can solve these equations explicitly, and we can get a relation between η and χ, for example, for constant τ:

$$\eta(\chi) = \frac{\pi}{2} - \arctan\left(\frac{\sqrt{(2\tau \sin\chi)^2 + 1} + 1}{2\tau \cos\chi}\right).$$

If we allow the variable r to go to $-\infty$ (so that every point on the diagram only represents half of the sphere), the diagram looks like this.



Let's quickly consider what the general shape of the conformal diagram would be in a few different conditions. For the open universe with the singularity in the past, we would have $0 < \tau < \infty$ and $0 \leq r < \infty$ with the shape of the diagram ◺. If the singularity is in the future, and we restrict ourselves to negative r, we have $-\infty < \tau < 0$ and $-\infty < r \leq 0$ and the diagram is ◹.

Chapter 7

Black holes

7.1 Free fall

Recall the Schwarzschild metric

$$\mathrm{d}s^2 = \left(1 - \frac{r_\mathrm{S}}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{r_\mathrm{S}}{r}} - r^2\mathrm{d}\Omega^2$$

with $r_\mathrm{S} = 2GM/c^2$. When light travels directly towards the black hole, its φ and θ are constant, so $\mathrm{d}\Omega = 0$. The condition $\mathrm{d}s^2 = 0$ gives

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\left(1 - \frac{r_\mathrm{S}}{r}\right).$$

Solving this differential equation gives

$$r + r_\mathrm{S}\ln(r - r_\mathrm{S}) = \pm t + C.$$

If we assume that light was at position r_0 at time t_0, we get

$$(r - r_0) + r_\mathrm{S}\ln\left(\frac{r - r_\mathrm{S}}{r_0 - r_\mathrm{s}}\right) = \pm(t - t_0).$$

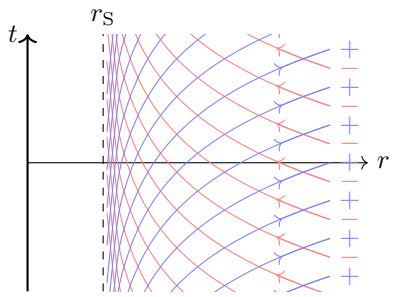
Asymptotically, when $r, r_0 \gg r_\mathrm{S}$ we have $r - r_0 = \pm(t - t_0)$ where the plus sign is for light moving outwards and the minus sign is for light moving inwards. If we start really close to the Schwarzschild radius, i.e. $r - r_\mathrm{S} = \Delta r \ll r_\mathrm{S}$ and denote Δt, we get $r_\mathrm{S}\ln\Delta r = \pm\Delta t$. The solution for the light moving outwards is

$$\Delta r \propto \Delta r_0 \exp(\Delta t/r_\mathrm{S}).$$

So, if it starts very close, the light takes a really long time to get away from the black hole. For the light moving inwards, we have

$$\Delta r \propto \Delta r_0 \exp(-\Delta t/r_\mathrm{S})$$

which means that the light never actually reaches the black hole.



So, the velocity of light in our coordinate slows down significantly. But in the frame of light, the velocity should always be c. Let's check this. We have

$$\frac{\Delta l}{\Delta \tau} = \frac{\sqrt{g_{rr}}}{\sqrt{g_{tt}}}\frac{\Delta r}{\Delta t} = \frac{1}{1-\frac{r_S}{r}} \cdot (\pm 1)\left(1-\frac{r_S}{r}\right) = \pm 1.$$

The redshift that the external observer sees is

$$\frac{\omega_1}{\omega_0} = \sqrt{\frac{g_{tt}(r_0)}{g_{tt}(r_1)}} = \sqrt{\frac{1-r_S/r_0}{1-r_S/r_1}}.$$

As the electron's position approaches the event horizon, the redshift goes to infinity ($\omega_1 \to 0$), so we get no signal.

Let's now consider a massive object falling into the black hole. Its four-velocity is

$$u^\alpha = \left(\frac{\mathrm{d}t}{\mathrm{d}s}, \frac{\mathrm{d}r}{\mathrm{d}s}, 0, 0\right).$$

We have a useful equation

$$g_{\alpha\beta}u^\alpha u^\beta = 1$$

for all massive objects (and it is equal to zero for photons). From this, we get

$$g_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = 1.$$

Since $g_{\alpha\beta,t} = 0$, we can use the geodesic equation to get

$$0 = \frac{\mathrm{d}u_t}{\mathrm{d}s} - \frac{1}{2}g_{\alpha\beta,t}u^\alpha u^\beta = \frac{\mathrm{d}u_0}{\mathrm{d}s}.$$

So

$$\text{const.} = u_t = u^t g_{tt} = g_{tt}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{E}{m_0}$$

where E is the energy of a particle and m_0 is its rest mass. From this, we define

$$\mathrm{d}s = g_{tt}\frac{m_0}{E}\mathrm{d}t.$$

Now we use the relation from before

$$\frac{1}{g_{tt}}\left(\frac{E}{m_0}\right)^2 + \left(\frac{E}{m_0}\right)^2 \frac{g_{rr}}{g_{tt}}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = 1$$

and we get

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\left(1 - \frac{r_\mathrm{S}}{r}\right)\frac{\sqrt{(E/m_0)^2 - 1 + r_\mathrm{S}/r}}{E/m_0}.$$

When r approaches r_S the equation simplifies the massless formula

$$\frac{\mathrm{d}r}{\mathrm{d}t} \approx \pm\left(1 - \frac{r_\mathrm{S}}{r}\right).$$

Hence, massive particles never reach the horizon for an outside observer. We can also see from the equation that the particle always moves within its light cone. It is left as an exercise to the reader that in the non-relativistic limit $(E - m_0)/m_0 \ll 1$, we get

$$\frac{1}{2}m\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{GMm}{r} = E - m_0.$$

For the falling body's proper coordinate time, we find that the time to fall to the horizon is

$$\Delta\tau = \int_{r_\mathrm{S}}^{r_0}\mathrm{d}s = \int_{r_\mathrm{S}}^{r_0}\sqrt{g_{tt}\,\mathrm{d}t^2 + g_{rr}\,\mathrm{d}r^2} = \int_{r_\mathrm{S}}^{r_0}\sqrt{\frac{g_{tt}}{\dot{r}^2} + g_{rr}}\,\mathrm{d}r.$$

If the body is originally stationary at r_0, then $E = m_0$ and the solution to the integral is

$$\Delta\tau = \frac{2}{3}r_\mathrm{S}\left(\left(\frac{r_0}{r_\mathrm{S}}\right)^{3/2} - 1\right).$$

For $t_0 = 100r_\mathrm{S} = 300\,\mathrm{km}$, the result is $\Delta\tau = 10^{-2}\,\mathrm{s}$. How would it feel to fall into the black hole? In empty space $R_{\mu\nu} = 0$ but $R_{\mu\nu\sigma\rho} \neq 0$. The tidal force is proportional to the invariant $I = R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$, which is in the vicinity of the event horizon

$$I = \frac{12r_\mathrm{S}^2}{r^6} \approx \frac{1}{r_\mathrm{S}^4} \approx \frac{1}{M^4}.$$

This means that big black holes have small tidal forces on their event horizon. But once you cross the horizon, I explodes, and the tidal force also explodes.

7.2 Coordinate systems

When $r_\mathrm{S} > r$ the metric becomes

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{r_\mathrm{S}/r - 1} - (r_\mathrm{S}/r - 1)\mathrm{d}t^2 - r^2\,\mathrm{d}\Omega^2.$$

The spatial and temporal part exchange and metric become time-dependent. Define $r \coloneqq \tau$ and $t \coloneqq R$. The metric is

$$\mathrm{d}s^2 = f(\tau)\,\mathrm{d}\tau^2 - \frac{1}{f(\tau)}\,\mathrm{d}R^2 - \tau^2\,\mathrm{d}\Omega^2$$

with $f(\tau) = \tau/(r_\mathrm{S} - \tau)$. The space inside the event horizon is bounded by $0 \leq \tau < r_\mathrm{S}$ and $-\infty < R < \infty$. So, the space is infinite, but the time is limited. For a very short time $\tau \ll r_\mathrm{S}$ the metric simplifies with $f(\tau) \approx \tau/r_\mathrm{S}$, so

$$\mathrm{d}s^2 = \frac{\tau}{r_\mathrm{S}}\,\mathrm{d}\tau^2 - \frac{r_\mathrm{S}}{\tau}\,\mathrm{d}R^2 - \tau^2\,\mathrm{d}\Omega^2.$$

If we change variables slightly to $\hat{t} = 2\tau^{3/2}/(3\sqrt{r_S})$ we get

$$\mathrm{d}s^2 = \mathrm{d}\hat{t}^2 - \left(\frac{\hat{t}}{\hat{t}_0}\right)^{-2/3} \mathrm{d}R^2 - \left(\frac{\hat{t}}{\hat{t}_0}\right)^{4/3} r_S^2 \,\mathrm{d}\Omega^2$$

with $0 < \hat{t} < 2r_S/3$. If we compare this to a *Kasner metric*

$$\mathrm{d}s^2 = \mathrm{d}t^2 - t^{2\alpha}\,\mathrm{d}x^2 - t^{2\beta}\,\mathrm{d}y^2 - t^{2\gamma}\,\mathrm{d}z^2$$

where $\alpha + \beta + \gamma = 0$ and $\alpha^2 + \beta^2 + \gamma^2 = 1$, we see that for small $\hat{t}$ the universe increases in one direction in decreases in the other two, the volume decreases and the shape is akin to a stick.

Another possible choice of coordinates is the following substitution

$$\mathrm{d}r^* \equiv \frac{\mathrm{d}r}{1 - r_S/r} \implies r^* = r + r_S \ln(r_S/r - 1).$$

This is called *tortoise coordinate*. In this case, the range of r^* is $-\infty < r^* < \infty$. The event horizon corresponds to $r^* = -\infty$. The advantage of this approach is that light cones don't get squashed when approaching the event horizon, but the downside is that we pushed the (interesting) event horizon to infinity. The metric is

$$\mathrm{d}s^2 = (1 - r_S/r(r^*))\,\mathrm{d}(t + r^*)\mathrm{d}(t - r^*).$$

The natural choice of change of coordinates is $t, r^* \to v, r$, where $v = t + r^*$. These are called (incoming) *Eddington-Finkelstein coordinates*. The transformation is

$$\mathrm{d}v = \mathrm{d}t + \frac{\mathrm{d}r}{1 - r_S/r} \implies \mathrm{d}(t - r^*) = \mathrm{d}v - \frac{2\,\mathrm{d}v}{1 - r_S/r}.$$

The metric is simplified to

$$\mathrm{d}s^2 = (1 - r_S/r)\mathrm{d}v^2 - 2\,\mathrm{d}v\,\mathrm{d}r.$$

When $r = r_S$, the metric is $\mathrm{d}s^2 = -2\,\mathrm{d}v\,\mathrm{d}r$. Introducing another new coordinates $v = (x - \tau)/\sqrt{2}$ and $r = (x + \tau)/\sqrt{2}$ we get $\mathrm{d}s^2 = \mathrm{d}\tau^2 - \mathrm{d}x^2$ which is Minkowski metric. For light geodesics, it holds $\mathrm{d}s^2 = 0$, so there are two options

- $\mathrm{d}v = 0$ (incoming light) and
- $\mathrm{d}v/\mathrm{d}r = 2/(1 - r_S/r)$ (outgoing light).

Far away from the r_S, these two conditions give perpendicular lines in the (r, v) plane.

Instead of v, we can choose $u = t - r^*$ (outgoing Eddington-Finkelstein coordinates). We have $\mathrm{d}u = \mathrm{d}t - \mathrm{d}r/(1 - r_S/r)$ and

$$\mathrm{d}s^2 = (1 - r_S/r)\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r.$$

For light, we again have $\mathrm{d}s^2 = 0$, so either

- $\mathrm{d}u = 0$ (outgoing light) or
- $\mathrm{d}u/\mathrm{d}r = -2/(1 - r_S/r)$ (incoming light).

Nothing can get into the black hole in these coordinates, but things can come out of it (but this is unphysical).

If we simultaneously use both v and u coordinates, we get

$$\mathrm{d}s^2 = (1 - r_\mathrm{S}/r(u,v))\,\mathrm{d}v\,\mathrm{d}u - r^2(u,v)\,\mathrm{d}\Omega^2.$$

Using the definition of tortoise coordinates, we get

$$1 - \frac{r_\mathrm{S}}{r} = \frac{r_\mathrm{S}}{r}\exp\left(\frac{r^* - r}{r_\mathrm{S}}\right) = \frac{r_\mathrm{S}}{r}e^{-r/r_\mathrm{S}}\exp\left(\frac{v-u}{r_\mathrm{S}}\right).$$

The metric is (ignoring the angular components)

$$\mathrm{d}s^2 = \frac{r_\mathrm{S}}{r}e^{-r/r_\mathrm{S}}\exp\left(\frac{v-u}{2r_\mathrm{S}}\right)\mathrm{d}u\,\mathrm{d}v = \frac{4r_\mathrm{S}^3}{r}e^{-r/r_\mathrm{S}}\,\mathrm{d}\left(e^{v/2r_\mathrm{S}}\right)\,\mathrm{d}\left(-e^{-u/2r_\mathrm{S}}\right) = \frac{4r_\mathrm{S}^3}{r}e^{-r/r_\mathrm{S}}\,\mathrm{d}\tilde{v}\,\mathrm{d}\tilde{u}.$$

We have defined $\tilde{v} = \exp(v/2r_\mathrm{S})$ and $\tilde{u} = -\exp(-u/2r_\mathrm{S})$. The domains for variables are $-\infty v < \infty$, $-\infty < u < \infty$, $0 < \tilde{v} < \infty$ and $-\infty < \tilde{u} < 0$. However, we can analytically continue $\tilde{u}$ and $\tilde{v}$ to be defined on the whole real line. The full metric is

$$\mathrm{d}s^2 = \frac{4r_\mathrm{S}^3}{r}e^{-r/r_\mathrm{S}}\,\mathrm{d}\tilde{v}\,\mathrm{d}\tilde{u} - r^2(\tilde{u},\tilde{v})\,\mathrm{d}\Omega^2, \quad -\infty < \tilde{u}, \tilde{v} < \infty.$$

We are still not finished introducing the new coordinates. This time, we use u and v (not the same as before) defined as

$$u = \frac{\tilde{v} - \tilde{u}}{2}, \quad v = \frac{\tilde{v} + \tilde{u}}{2}$$

so that $\mathrm{d}\tilde{v}\,\mathrm{d}\tilde{u} = \mathrm{d}v^2 - \mathrm{d}u^2$ which looks more like time and space. The metric is

$$\mathrm{d}s^2 = \frac{4r_\mathrm{S}^2}{r(u,v)}e^{-r(u,v)/r_\mathrm{S}}(\mathrm{d}v^2 - \mathrm{d}u^2) - r^2(u,v)\,\mathrm{d}\Omega^2.$$

The dependence of r on u and v can be explicitly found using the Lambert W function:

$$r = r_\mathrm{S}\left(1 + W_0\left(\frac{u^2 - v^2}{e}\right)\right).$$

We can find that for $r > r_\mathrm{S}$:

$$u = \pm\sqrt{\frac{r}{r_\mathrm{S}} - 1}e^{r/2r_\mathrm{S}}\cosh(t/2r_\mathrm{S}),$$

$$v = \pm\sqrt{\frac{r}{r_\mathrm{S}} - 1}e^{r/2r_\mathrm{S}}\sinh(t/2r_\mathrm{S})$$

and for $r < r_\mathrm{S}$

$$u = \pm\sqrt{1 - \frac{r}{r_\mathrm{S}}}e^{r/2r_\mathrm{S}}\sinh(t/2r_\mathrm{S}),$$

$$v = \pm\sqrt{1 - \frac{r}{r_\mathrm{S}}}e^{r/2r_\mathrm{S}}\cosh(t/2r_\mathrm{S}).$$

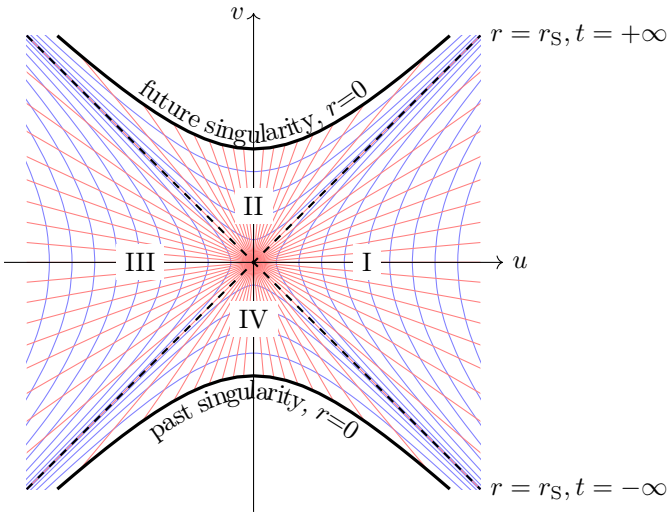
These are called the *Kruskal-Szekeres coordinates*. We can see that these equations parametrize hyperbolae in the (u, v) plane. Moreover, for $r = r_S$, the equations are $u^2 - v^2 = 0$ so $u = \pm v$. For $r = \text{const}$ the relation is

$$v = \pm\sqrt{f(r) + u^2}$$

and at $r = 0$ (at the singularity)

$$v = \pm\sqrt{1 + u^2}.$$

Equal-time lines are straight lines in the (u, v) plane.

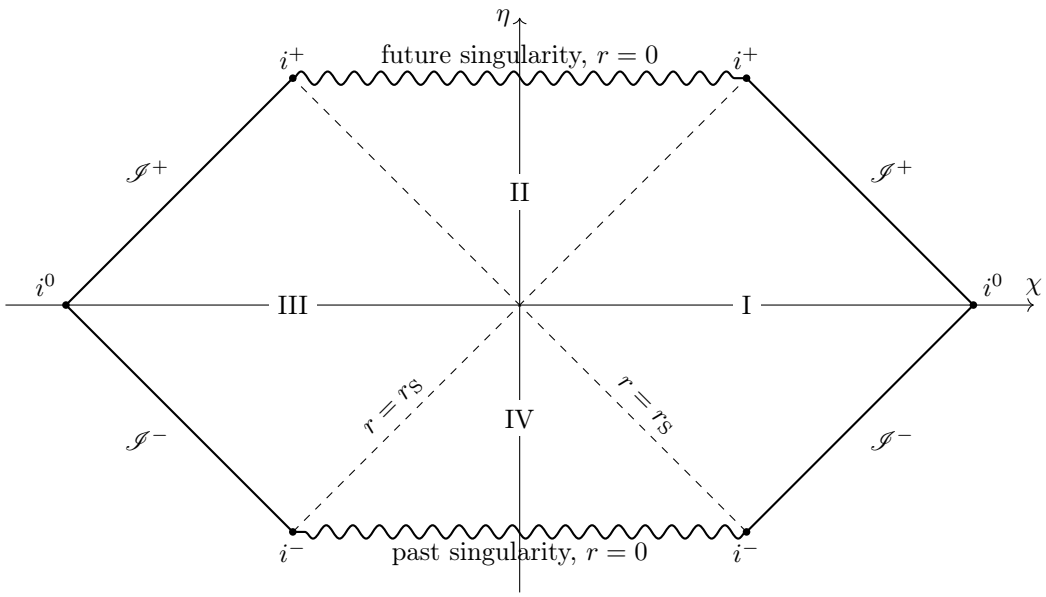


The lines of constant t are drawn in red, and the lines of constant r are drawn in blue. Region I is our universe outside the black hole, II is the space inside the event horizon, and regions III and IV are consequences of the analytical continuation. II represents the parallel universe, and region IV represents the white hole. We (the observers in region I) can only affect things in regions I and II, but if we go to region II, we can interact with objects from III.

Now apply the same transformation to Kruskal coordinates as we did on the Minkowski space:

$$\chi + \eta = \arctan(u + v), \quad \chi - \eta = \arctan(u - v).$$

What we get is this.



This is the conformal diagram for the Schwarzschild black hole. The region labels correspond to those in the Kruskal diagram.

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