

Faculty of physics

Cosmology

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Disclaimer: these notes are not official and were not endorsed by professor. The content is almost entirely based on the lectures, but occasionally the notes were modified or extended slightly to improve clarity or add extra information. In particular, the mistakes are almost surely mine. If you find any, please let me know at simon.bukovsek@tum.de or Simon.Bukovsek@campus.lmu.de.

Invitation

Let's start with a question: how is general relativity different to the classical Maxwell theory? With Maxwell theory, we can be given the sources, and we can calculate the fields using the Green functions and the equations are linear. In GR, the Einstein equations are non-linear, and the solutions are very complicated. We cannot just be given the energy-momentum tensor and calculate the metric. The energy-momentum tensor already depends on the metric. Hence, we need an awful lot of physical intuition and symmetry considerations to find any solutions. In this course, we will be studying one of the very simple solutions of the Einstein equations.

The symmetries of the Minkowski space are isotropy and homogeneity. They are outrageously violated on the small scales, but we have hope that on the certain scale, we can reasonably use these assumptions. The measurements that give us the most hope are the observations of the cosmic microwave background (CMB). It is the radiation that was emitted when the universe was about 300 000 years old, and it is remarkably isotropic. We are trying to measure the tiny fluctuations in this isotropy, but these deviations are of the order of 10^{-5}, and they can tell us a lot about the universe.

The literature for this course is:

- Mukhanov: Physical Foundations of Cosmology,
- Weinberg: Cosmology,
- Baumann: Cosmology.

What is cosmology? It is the study of the dynamics of a specific geometry and physical processes performed by those (effective) degrees of freedom that couple to this geometry.

Chapter 1

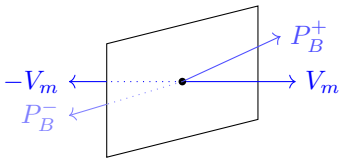
Prefect fluids

1.1 Heuristic considerations

The idea is to describe perfect (ideal) fluids with the *stress-energy tensor* (SET) also called the energy-momentum tensor. We will assume that we have a *smooth* manifold M and also a smooth source. Instead of considering every point-like particle, we will average over a small volume and consider only the average flow of the fluid. Let m be a point on the manifold M and let $V_m \in T_m(M)$ be a vector in the tangent space at m. We will assume the vector to be normalized to 1 according to the metric g, $\|V\|_g = 1$. Next, consider

$$V_m^\perp = \{X_m \in T_m(M) : X_m \perp_g V_m, \text{ i.e. } g(X_m, V_m) = 0\}.$$

The dimension of this space is $\dim(V_m^\perp) = \dim(M) - 1$. We can picture it as a cube (box), centred at m.



box $B(V_m)$

The total nett momentum transferred through the box is $P_B \coloneqq P_B^+ - P_B^-$. Let $W_m \in T_m(M)$. The component of the nett total momentum P_B in the direction of W_m is $|\langle W, P_B\rangle_m|$. We have denoted the scalar product between two vectors as $g_m(A, B) = \langle A, B\rangle$. The (ultra) local formulation of the energy-momentum tensor is

$$T(V, W) = \lim_{\mathrm{vol}(B(V))\to 0} \frac{g(W, P_B)}{\mathrm{vol}(B(V))} \in \mathrm{Ten}_2^0(M).$$

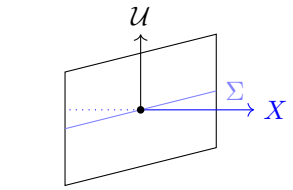
For instance, consider an observer $\mathcal{U}$ ($\mathcal{U} = W$) that moves with the flow, $P_B \sim \mathcal{U}$. The rest frame of the observer is $\mathcal{U}^\perp$. Use $P_B^- = 0$, so we can decompose the nett momentum as

$$p_B = p_B^+ = E_B\mathcal{U} + P_B^\perp$$

where $P_B^{\perp} \perp_g \mathcal{U}$ and E_B is the projection of p_B on $\mathcal{U}$. The energy-momentum tensor is

$$T(\mathcal{U},\mathcal{U}) = \lim_{\mathrm{vol}(B(\mathcal{U}))\to 0} \frac{|g(\mathcal{U}, P_B)|}{\mathrm{vol}(B(\mathcal{U}))} = \lim_{\mathrm{vol}(B(\mathcal{U}))\to 0} E_B \frac{|g(\mathcal{U},\mathcal{U})|}{\mathrm{vol}(B(\mathcal{U}))} = \lim_{\mathrm{vol}(B(\mathcal{U}))\to 0} \frac{E_B}{\mathrm{vol}(B(\mathcal{U}))} = \rho(m),$$

which is called the *energy density* of the fluid. Now introduce two vectors X, Y that are perpendicular to $\mathcal{U}$, $X, Y \perp_g \mathcal{U}$. Hence, $B(X) \subset \Sigma \times I$ where Σ is a space-like hypersurface and I is the time interval.



box $B(X)$

The energy-momentum tensor is

$$T(X,Y) = \lim_{A\to 0} \frac{1}{A} \left(\lim_{\Delta t \to 0} \frac{g(P_B, Y)}{\Delta t} \right) = \text{pressure exerted by } P_B \text{ against } \Sigma.$$

1.2 Rigorous considerations

Definition 1.1. A *perfect fluid* on a spacetime (M, g) is a triple $(\mathcal{U}, \rho, \mathcal{P})$ where

1. $\mathcal{U}$ is a unit time-like future-pointing vector field on M called *flow vector field*,
2. $\rho \in \mathcal{C}^\infty(M, \mathbb{R})$ is the *energy density function* on M, $\mathcal{P}$ is called the *pressure function* on M.
3. The stress-energy tensor is

$$T = \rho \mathcal{U}^{\flat_g} \otimes \mathcal{U}^{\flat_g} + \mathcal{P} H^{\perp},$$

where $H^{\perp} = g + \mathcal{U}^{\flat_g} \otimes \mathcal{U}^{\flat_g}$ and $\mathcal{U}^{\flat_g} = g(\mathcal{U}, \bullet)$ is the dual of $\mathcal{U}$ and is metrically equivalent to $\mathcal{U}$.

For details on the $\flat$ notation, take a look at musical isomorphisms. We will often omit the subscript g and write $\mathcal{U}^{\flat}$ instead of $\mathcal{U}^{\flat_g}$.

Remark.

a) $\mathcal{U} \in \ker H^{\perp}$.

b) For $X, Y \perp_g \mathcal{U}$, it holds $T(\mathcal{U},\mathcal{U}) = \rho$, $T(X,Y) = \mathcal{P} g(X,Y)$ and $T(\mathcal{U}, X) = 0$.

If $(\mathcal{U}, \rho, \mathcal{P})$ is a perfect fluid, then the covariant conservation of the stress-energy tensor implies

(i) $\mathcal{U}(\rho) = -(\rho + \mathcal{P}) \operatorname{div} \mathcal{U}$ (energy balance),

(ii) $(\rho + \mathcal{P}) \mathrm{D}_{\mathcal{U}} \mathcal{U} = -\operatorname{grad}_{\perp} \mathcal{P}$ (force law)

This can be seen by considering $\overline{T}$, the stress-energy tensor metrically equivalent to T (via musical isomorphisms, both indices lifted). Elementary geometrical considerations give for $X \perp_g \mathcal{U}$

$$\operatorname{div}(\overline{T}) = \operatorname{div}\left(\overline{T}(\mathcal{U}^{\flat})\right) \mathcal{U} + \operatorname{div}\left(\overline{T}(X^{\flat})\right) X,$$

therefore the covariant conservation of energy implies $\operatorname{div}\big(\overline{T}(\mathcal{U}^\flat)\big) = 0$ and $\operatorname{div}\big(\overline{T}(X^\flat)\big) = 0$. By a direct computation, we can calculate the divergence of the stress-energy tensor

$$\operatorname{div}\big(\overline{T}\big) = (\mathrm{D}_{\mathcal{U}}\rho)\mathcal{U} + (\rho+\mathcal{P})\operatorname{div}(\mathcal{U})\mathcal{U} + (\rho+\mathcal{P})\mathrm{D}_{\mathcal{U}}\mathcal{U} + H^{\perp}(\operatorname{grad}\mathcal{P}, \bullet).$$

It is very instructive to compute this for yourself. Therefore,

$$\operatorname{div}\Big(\overline{T}(\mathcal{U}^\flat)\Big) = -\dot{\rho} - (\rho+\mathcal{P})\operatorname{div}\mathcal{U}$$

and

$$\operatorname{div}\Big(\overline{T}(X^\flat)\Big) = (\rho+\mathcal{P})X^\flat(\mathrm{D}_{\mathcal{U}}\mathcal{U}) + X^\flat(\operatorname{grad}_\perp \mathcal{P}).$$

Equating both equations to zero gives us the assertion. For homework, do the same calculation using Ricci calculus and index gymnastics. The equations of motion need to be consistent with the conservation of the energy-momentum tensor, otherwise the system would be overdetermined.

Dictionary

Let $\mathcal{A} \in \mathrm{Ten}^0_2(M)$ and $V_1, V_2 \in \mathrm{Vec}(M)$. Then $\mathcal{A}(V_1, V_2) \in \mathrm{Fun}(M)$ and evaluating it at $p \in M$ gives $\mathcal{A}(V_1, V_2)(p) \in \mathbb{R}$. However, to actually compute anything we have to resort to some coordinates. We choose a coordinate neighbourhood (a domain and coordinate functions). The vector V expressed in terms of the coordinate functions is $V = V^a e_a$ where V^a are the components of the vector in the coordinate system and e_a are the basis vectors. The evaluation of the tensor on the give two vectors is

$$\mathcal{A}(V_1, V_2) = V_1^{a_1} e_{a_1}, V_2^{a_2} e_{a_2} = V_1^{a_1}\mathcal{A}(e_{a_1}, e_{a_2})V_2^{a_2} = V_1^{a_1}\mathcal{A}_{a_1 a_2}V_2^{a_2}.$$

How does a directional derivative in the direction of X act on this tensor?

$$(\mathrm{D}_X\mathcal{A})(V_1, V_2) = \mathrm{D}_X[\mathcal{A}(V_1, V_2)] - \mathcal{A}(\mathrm{D}_X V_1, V_2) - \mathcal{A}(V_1, \mathrm{D}_X V_2).$$

Appendix for geometrical model building

Let (B, g_B) and (F, g_F) be two (pseudo-)Riemannian manifolds (a base and a fibre). We would like to construct a new manifold (M, g_M), which is the product of the two manifolds, $M = B \times F$. We can define the projectors $\pi_{B,F} : M \to B, F$. The natural way to construct the metric on the product manifold to use the pullback of the projectors:

$$g_M = \pi_B^*(g_B) + \pi_F^*(g_F).$$

The pullback of the metric is defined as $\pi^*(g)(V_1, V_2) = g(\pi(V_1), \pi(V_2))$. For homework, you can try to work this out in an abstract coordinate system. But this pullback is not the only thing we can do. We can *warp*. We are interested in homothetically warping the product metric on each fibre $p \times F$ ($p \in B$).

Definition 1.2. Let $f > 0$ be a smooth function on B. The *warped product* $M = B \times_f F$ is the product manifold with the metric

$$g_M = \pi_B^*(g_B) + (f \circ \pi_B)^2 \pi_F^*(g_F).$$

This function f modifies the (boring) Cartesian product by warping the fibre space along the base space. This is what we actually do in cosmology; F is the spatial hypersurface and B is the temporal line. Both spaces B and F will be rather easy to determine, the only thing that is a bit more tricky is the warping function f.

Example 1. Let $M = \mathbb{R}^3 - \{0\}$ with the metric $g_M = \mathrm{d}r \otimes \mathrm{d}r + r^2 g_{S^2}$. M diffeomorphic to $S^2 \times \mathbb{R}^+$ and we would write it as $M = \mathbb{R}^+ \times_r S^2$.

A 2-dimensional subspaces Π of $T_p(M)$ is called a *tangent plane* to M at p. For $v, w \in T_p(M)$, we define $Q(v,w) = \langle v,v\rangle \langle w,w\rangle - \langle v,w\rangle^2$. The modulus of this operator $|Q(v,w)|$ is the square of the area of the parallelogram spanned by v and w.

Claim 1.1. *Given any two vectors v, w in a scalar product space, there exist vectors $\overline{v}, \overline{w}$ arbitrarily close to v, w such that the plane spanned by $\overline{v}, \overline{w}$ is non-degenerate.*

Proof. This can be seen as follows. Assume that v, w are linearly independent. We are interested in a case where v, w span degenerate tangent plane. We distinguish two cases:

- If v is null choose a vector $x \in T_p(M)$ such that $\langle v, x\rangle \neq 0$.
- If v is not null, choose $x \in T_p(M)$ with opposite character (space-like, time-like).

In both cases $Q(v,x) < 0$. For $\delta > 0$ the vectors v and $w + \delta x$ span a non-degenerate tangent plane. By direct computation, we can show that

$$Q(v, w+\delta x) = \underbrace{Q(v,w)}_{0} + \underbrace{Q(v,\delta x)}_{\mathcal{O}(\delta^2)} + \underbrace{\delta b(v,w)}_{\mathcal{O}(\delta)} = \delta^2 Q(v,x) + \delta b$$

for some function b. In the case $b = 0$ then $Q(v, w+\delta x) = Q(v,x) < 0$ by the choice of x. If $b \neq 0$ then we can shrink δ so much that the second term dominates, and it is by definition not zero. □

The Riemann tensor in our notation is the object $R_{w_1 w_2}$ that eats a vector v, parallel-transports it along an infinitesimal loop in the plane spanned by w_1 and w_2 and returns the difference. Hence, $R_{w_1 w_2} v \in \mathrm{Vec}(M)$.

Definition 1.3. Let Π be a non-degenerate tangent plane to M at p and $v, w \in \Pi$. Then the number

$$K(\Pi) = \frac{\langle R_{vw} v, w\rangle}{Q(v,w)}$$

is called the *sectional curvature* of the plane Π at p.

Using the Claim 1.1, we can rest assured that the sectional curvature K is well-defined. Does K also determine R?

Claim 1.2. *If $K(\Pi) = 0$ for every tangent plane Π at p, then $R_{xy}z = 0$ for all $x, y, z \in T_p(M)$.*

Proof. First we show that $\langle R_{vw} v, w\rangle = 0$ for all $v, w \in T_p(M)$. The trivial case is when v and w span a non-degenerate tangent plane. But from the discussion above, any pair of vectors in $T_p(M)$ can be seen as the limit of a pair of vectors that span a non-degenerate tangent plane. For any $x \in T_p(M)$ it holds

$$\langle R_{v,w+x} v, w+x\rangle = \langle R_{vw} v, w\rangle + \langle R_{vw} v, x\rangle + \langle R_{wx} v, w\rangle + \langle R_{wx} v, x\rangle .$$

We apply the previous statement to the left-hand side and to the first and last term on the right-hand side. We get (by the index symmetry of the Riemann tensor)

$$\langle R_{vw}v, x\rangle = -\langle R_{vx}v, w\rangle = -\langle R_{vw}v, x\rangle .$$

Hence, $\langle R_{vw}w, x\rangle = 0$ for all $x \in T_p(X)$ and $R_{vw}w = 0$. Next, consider

$$0 = R_{v+x,w}(v + x) = R_{x,w}v + R_{v,w}x.$$

Reordering gives $R_{vw}x = R_{wx}v = -R_{xw}v$. This implies $R_{vw}x = R_{wx}v = R_{xv}w$. So $R_{vw}x$ is unchanged under cyclic permutations of $v, w, x \in T_p(M)$. The first Bianchi identity gives

$$\sum_{\text{cyc perm}} R_{vw}x = 0$$

and hence $R_{vw}x = 0$ for all $x \in T_p(M)$. □

Definition 1.4. If Riemann tensor R is identically zero at every point of M, then M is called *flat*.

M is flat if and only if the sectional curvature K vanishes everywhere. There is even more to gain from this discussion. Let $F : T_p(M)^{\times 4} \to \mathbb{R}$ be a curvature-like multilinear function that has the same symmetries as $(v, w, x, y) \to \langle R_{vw}x, y\rangle$. The preceding argument only made use of the symmetries. Therefore, if $F(v, w, v, w) = 0$ for $v, w \in T_p(M)$ spanning a non-degenerate tangent plane, then $F(v, w, x, y) = 0$. Let F be a curvature-like function on $T_p(M)$ such that $K(v, w) = F(v, w, v, w)/Q(v, w)$. Then $\langle R_{vw}x, y\rangle = F(v, w, x, y)$ for all $v, w, x, y \in T_p(M)$. Furthermore, if M has a constant sectional curvature K everywhere, then the Riemann tensor can be expressed as

$$R_{XY}Z = K(\langle Z, X\rangle Y - \langle Z, Y\rangle X).$$

1.3 Semi-Riemannian submanifolds

Let $\overline{M}$ be a (semi-Riemannian) manifold and let M be a submanifold $M \subset \overline{M}$. Let $X : M \to T(\overline{M})$ be a vector field that is defined on M but lives in the tangent space of the ambient manifold $\overline{M}$. We will denote $\overline{\mathrm{Vec}}(M)$ the set of all such vector fields. Note that if $Y \in \mathrm{Vec}(\overline{M})$, then $Y|_M \in \overline{\mathrm{Vec}}(M)$. Since the tangent space of the ambient manifold can be identified with the direct sum of the tangent space of the submanifold and the normal space

$$T_p(\overline{M}) = T_p(M) \oplus T_p(M)^{\perp}$$

every vector field $X \in T(\overline{M})$ has a unique decomposition

$$X = \tan X + \mathrm{nor}\, X$$

where

$$\tan : T(\overline{M}) \to T(M), \qquad \mathrm{nor} : T(\overline{M}) \to T(\overline{M})^{\perp}$$

are the projections onto the tangent and normal space.

Let $\overline{\mathrm{D}}$ denote the *Levi-Civita connection* on $\overline{M}$. Consider the induced connection on the submanifold M

$$\overline{\mathrm{D}} : \mathrm{Vec}(M) \times \overline{\mathrm{Vec}}(M) \to \overline{\mathrm{Vec}}(M).$$

The induced connection satisfies the Levi-Civita properties:

(1) $\overline{\mathrm{D}}_V X$ is $\mathrm{Fun}(M)$-linear in V,

(2) $\overline{\mathrm{D}}_V X$ is $\mathbb{R}$-linear in X,

(3) $\overline{\mathrm{D}}_V(fX) = V(f)X + f\overline{\mathrm{D}}_V X$ (Leibniz rule),

(4) $[V, W] = \overline{\mathrm{D}}_V W - \overline{\mathrm{D}}_W V$,

(5) $V(\langle X, Y\rangle) = \langle \overline{\mathrm{D}}_V X, Y\rangle + \langle X, \overline{\mathrm{D}}_V Y\rangle$ (metric compatibility).

Note. In general, $\overline{\mathrm{D}}_V W \notin \mathrm{Vec}(M)$ even if $V, W \in \mathrm{Vec}(M)$.

Let D be the Levi-Civita connection of M. If $V, W \in \mathrm{Vec}(M)$ then

$$\mathrm{D}_V W = \tan \overline{\mathrm{D}}_V W$$

by the Levi-Civita theorem. However, $\mathrm{nor}(\overline{\mathrm{D}}_V W)$ is new.

Definition 1.5. The function

$$\begin{aligned} \mathscr{S} : \mathrm{Vec}(M) \times \mathrm{Vec}(M) &\to \mathrm{Vec}(M)^\perp \\ V, W &\mapsto \mathrm{nor}(\overline{\mathrm{D}}_V W) \end{aligned}$$

is called the *shape tensor* of $M \subset \overline{M}$.

The shape tensor is $\mathrm{Fun}(M)$-bilinear and symmetric,

$$\mathscr{S}(V, fW) = \mathscr{S}(fW, V) = f\mathscr{S}(V, W).$$

The decomposition of an induced connection $\overline{\mathrm{D}}_V W$ is hence

$$\overline{\mathrm{D}}_V W = \mathrm{D}_V W + \mathscr{S}(V, W).$$

This decomposition relates the induced Riemann tensor $\overline{R}_{VW}$ to the Riemann tensor R_{VW} of the submanifold M

$$\langle \overline{R}_{VW} X, Y\rangle = \langle R_{VW} X, Y\rangle - \langle \mathscr{S}(V, X), \mathscr{S}(W, Y)\rangle + \langle \mathscr{S}(W, X), \mathscr{S}(V, Y)\rangle .$$

This is called the *Gauss equation*, and it relates the curvature of the ambient manifold to the curvature of the submanifold. For $v, w \in T_p(M)$ the induced sectional curvature is

$$K(v, w) = \overline{K}(v, w) + \frac{\langle \mathscr{S}(v, v), \mathscr{S}(w, w)\rangle - \langle \mathscr{S}(v, w), \mathscr{S}(v, w)\rangle}{Q(v, w)}.$$

Example 2. Let ambient manifold be $\overline{M} = \mathbb{R}^{n+1}$ and the submanifold $M = S^n(r) \subset \mathbb{R}^{n+1}$ be a sphere with radius r. The position vector field

$$P = \sum_{a=1}^{n+1} u^a \partial_a$$

is normal to $S^n(r)$. Since any plane in $\mathbb{R}^{n+1}$ has vanishing sectional curvature it must hold $\overline{K}(v, w) = 0$ for any two vectors $v, w \in T_p(\mathbb{R}^{n+1})$. The sectional curvature of the sphere is then

$$K(v, w) = \frac{\langle \mathscr{S}(v, v), \mathscr{S}(w, w)\rangle - \langle \mathscr{S}(v, w), \mathscr{S}(v, w)\rangle}{Q(v, w)}.$$

Introduce $\hat{P} = P/\|P\|_{\overline{g}}$ (where $\overline{g}$ is the induced metric on $S^n(r)$). From previous discussion we conclude that $\hat{P}$ is also normal to $S^n(r)$. We write

$$\langle \mathscr{S}(V,W), \hat{P}\rangle = \frac{\langle \mathrm{nor}(\overline{\mathrm{D}}_V W, P)\rangle}{\|P\|_{\overline{g}}} = \frac{\langle \overline{\mathrm{D}}_V W, P\rangle}{\|P\|_{\overline{g}}}.$$

The first step was just the definition of the shape tensor, and the second step uses the fact that P is already normal and hence $\langle \tan(\overline{\mathrm{D}}_V W), P\rangle = 0$. Using the metric compatibility (5) we can rewrite this as

$$\frac{\langle \overline{\mathrm{D}}_V W, P\rangle}{\|P\|_{\overline{g}}} = \frac{1}{\|P\|_{\overline{g}}}\left(V(\langle W, P\rangle) - \langle W, \overline{\mathrm{D}}_V P\rangle\right) = -\frac{\langle W, \overline{\mathrm{D}}_V P\rangle}{\|P\|_{\overline{g}}} = -\frac{\langle W, V\rangle}{\|P\|_{\overline{g}}}.$$

The first works because P is normal and W is tangent and thus $\langle P, W\rangle = 0$. In the second step, we used that the directional derivative of the position vector field P simply points towards the direction of the derivative, i.e. $\overline{\mathrm{D}}_V P = V$. With this, we have shown

$$\langle \mathscr{S}(V,W), \hat{P}\rangle = -\frac{1}{\|P\|_{\overline{g}}}\langle V, W\rangle \hat{P}$$

and a solution for the shape tensor is

$$\mathscr{S}(V,W) = -\frac{1}{\|P\|_{\overline{g}}}\langle V, W\rangle \hat{P}.$$

Of course, it also holds $\|P\|_{\overline{g}} = r$, so

$$K(v,w) = \frac{1}{r^2}\frac{\langle v,v\rangle \hat{p}, \langle w,w\rangle \hat{p} - \langle\langle v,w\rangle \hat{p}, \langle v,w\rangle \hat{p}\rangle}{Q(v,w)} = \frac{1}{r^2}\langle p,p\rangle \frac{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2}{Q(v,w)} = \frac{1}{r^2}.$$

We have used that for normalized position vectors, the scalar product equals 1. With this, we have shown that the sectional curvature of the sphere is constant and equal to $1/r^2$ using nothing but the geometrical approach.

Chapter 2

Friedmann spacetime

We will consider a semi-Riemannian manifold

$$(M, g) = (T \times S, g = -\mathcal{U}^\flat \otimes \mathcal{U}^\flat + g_S)$$

where

1) $T \subset \mathbb{R}$ is an open time interval,

2) S is a connected 3-dimensional Riemannian manifold and

3) $\mathcal{U} = \partial/\partial t$ is the lift to $T \times S$ of the standard $\mathrm{d}/\mathrm{d}t$ on T.

The vector field $\mathcal{U}$ gives the velocity (of a galaxy). The path of an object is given by a curve

$$\begin{aligned} \gamma : T &\to T \times S, \\ t &\mapsto \gamma_p(t) \coloneqq (t, p). \end{aligned}$$

Holding $t = \text{const}$ gives hypersurfaces $S_t \coloneqq \{(t, p) : p \in S\} \subset T \times S$. Furthermore, we will be working with the following three assumptions.

(i) $g(\mathcal{U},\mathcal{U}) = -1$: each galaxy is a particle with a proper time $t : M \to T$.

(ii) $\mathcal{U} \perp_g S_t$: relative motion of galaxies is irrelevant, S_t is Riemannian and space-like per surface.

(iii) Each $(t, p) \in M = T \times S$ has a neighbourhood $\mathcal{N}$ with the following property: given a unit tangent vectors $x, y \in T_{(t,p)}(S_t)$, there is an isometry $\phi : \mathrm{id}_T \times \phi_S$ of $\mathcal{N}$ such that $\mathrm{d}\phi(x) = y$.

When we wrote $g = -\mathcal{U}^\flat \otimes \mathcal{U}^\flat + g_S$ at the beginning of this chapeter, it wasn't actually fixed yet, and even after the assumptions it is not uniquely determined. However, the choice we use is canonical.

Claim 2.1. *If the preceding three assumptions hold in $M = T \times S$ then*
(1) each hypersurface S_t has constant curvature,
(2) for any $s, t \in T$ the map $\mu : S_s \to S_t$, $(s, p) \to (t, p)$ is a spatially homogeneous dilation (homothety).

Proof sketch.

(1) Since $\dim(S) = 3$, any tangent plane Π_x to S_t has a unit normal vector x in the tangent space to S_t at p. By assumption (iii) there is a local isometry $\phi = \mathrm{id}_T \times \phi_s$ such that $\mathrm{d}\phi(x) = y$ for any other $y \in T_{(t,p)}S_t$. But y is also the unit normal of some other tangent plane Π_y at p. In other words, $\mathrm{d}\phi(\Pi_x) = \Pi_y$. Since the definition of sectional curvature of this tangent plane solely employs isometry invariance, Π_x and Π_y have the same sectional curvature. This shows that S_t has constant curvature. For a stricter argument, we would need to use Schur's lemma (that says that certain curvatures are pointwise constant then they are forced to be globally constant).

(2) We will prove this part in two steps.

- First, we will show that μ is conformal, i.e.,

$$\|\mathrm{d}\mu(x)\| = \text{const.}, \quad \forall x \in T_{(s,p)}S_s \quad \text{with } \|X\| = 1.$$

Note that differential of a map acts on vectors. Any other unit vector y can be expressed as $y = \mathrm{d}\phi(x)$. Since μ acts only on the first factor of $T \times S$ and ϕ acts only on the second factor (S), they commute. Hence, also their norms commute

$$\|\mathrm{d}\mu(\mathrm{d}\phi(x))\| = \|\mathrm{d}\phi(\mu(x))\| = \|\mathrm{d}\mu(x)\|.$$

The last step used the fact that $\mathrm{d}\phi$ is just a rotation and does not change the norm of the vector.

- Now we know that μ is conformal and by definition also that $\mu(s,p) = (t,p)$. The pullback μ^* of the metric g is defined as

$$\mu^* g(u,v) = g(\mathrm{d}\mu(u), \mathrm{d}\mu(v))$$

for all $u, v \in T_{(s,p)}(S_s)$. Using the conformal property, we can write

$$\mu^*(g_{p\in S_t}) = h^2(s,p,t) g_{p\in S_s}.$$

The factor $h(s,p,t)$ is called the *scale factor* of the homothety μ. Our goal is to prove that $h(s,p,t)$ is independent of p.

Let $\sigma : I \to S_s$ be a geodesic in S_s with initial condition $\sigma(0) = (s,p)$ and $\dot\sigma(0) = x \in T_{(s,p)}S_s$. Consider a specific isometric isotropy $\phi : \mathrm{id}_T \times \phi_S$ such that

$$\mathrm{d}\phi(\sigma'(0)) = -\sigma'(0), \qquad \mathrm{d}\phi(\sigma'(u)) = -\sigma'(-u)$$

for all $u \in I$. The pullback of the metric $\mu^* g$ along σ is

$$\begin{aligned}\mu^*(g_{S_t})(\sigma'(u),\sigma'(u)) &= g_{S_t}(\mathrm{d}\mu(\sigma'(u)), \mathrm{d}\mu(\sigma'(u))) = \|\mathrm{d}\mu(\sigma'(u))\|^2_{S_t} \\ &= h^2(\sigma(u),t)\|\sigma'(u)\|^2_{S_s}.\end{aligned}$$

Since $\sigma'(u)$ is normalized in S_s, it holds $\|\sigma'(u)\|^2_{S_s} = 1$ and

$$h(\sigma(u),t) = \|\mathrm{d}\mu(\sigma'(\mu))\|_{S_t}.$$

But since $\mathrm{d}\phi$ and $\mathrm{d}\mu$ commute, we also have

$$h(\sigma(u),t) = \|\mathrm{d}\phi(\mathrm{d}\mu(\sigma'(u)))\| = \|\mathrm{d}\mu(\mathrm{d}\phi(\sigma'(u)))\| = \|\mathrm{d}\mu(\sigma'(-u))\| = h(\sigma(-u),t)$$

for all $u \in I$. In the last step, we used the definition of the scale factor h and the normalization of the vector $\sigma'(u)$ in S_s. Since $h(\sigma(u), t) = h(\sigma(-u), t)$ it follows

$$\frac{\mathrm{d}}{\mathrm{d}u} h(\sigma(u), t) = 0$$

and h does not depend on p.

Joining the two steps, we have shown that μ is a homothety with a scale factor $h(s, t)$ that does not depend on p. This means that the spatial hypersurfaces S_s and S_t are related by a homothety.

□

Remark.

(1) In cosmology, the scale factor $h(s, t)$ is usually expressed as $h(s, t) = a(t)/a(s)$, where $a(t)$ is the scale factor we know from physics.

(2) By definition of sectional curvature, $h^2(s, t)K(t) = K(s)$. In other words, $a^2(t)C(t) = \text{const}$. This means that $C(t)$ maintains its sign during the entire evolution of the universe

$$C = \operatorname{sgn}(C)|C| = k|C|.$$

We can introduce a pivot scale a_* such that $a_*^2|C_*| = 1$ and then

$$a^2(t)C(t) = k \in \{-1, 0, 1\}.$$

Submanifold S has sectional curvature k and every injection $S \to S_t$ is spatially homogeneous dilation of the scale factor $a(t)$.

(3) The spatially constant curvature of S_t us $k/a(t)^2$.

(4) For $U, W \in TS_t$ it holds $g_M(U, V) = a^2(t) g_S(\pi_S(U), \pi_S(V))$.

(5) The three assumptions (i), (ii) and (iii) together with the claim above express $T \times S$ geometrically as a *warped product* $T \times_f S$ with base T and fibre S and warping function $f(t) = a(t)$.

Let's summarize the discussion above. Let S be a connected 3-dimensional Riemannian manifold with constant curvature $k \in \{-1, 0, 1\}$ and let $a > 0$ be a smooth function on an open interval $T \subset \mathbb{R}$. Then the warped product $M_{k,a} = T \times_a S$ is called the *Friedmann spacetime*. To elaborate the previous statement, $M_{k,a}$ is equipped with the metric $g = -\mathcal{U}^\flat \otimes \mathcal{U}^\flat + a^2(t) g_S$ where g_S denotes the metric of S lifted to $M_{k,a}$. Submanifold S is called the space of $M_{k,a}$. The three options for the curvature k are

- $k = 1$: the universe is closed, i.e. it is a 3-sphere S^3,
- $k = 0$: the universe is flat, i.e. it is $\mathbb{R}^3$,
- $k = -1$: the universe is open, i.e. it is a hyperbolic space H^3.

2.1 Connection and curvature of Friedmann spacetimes

As always, we will denote the flow vector field $\mathcal{U} = \partial/\partial t$ as future pointing unit vector field normal to the hypersurfaces S_t ($\forall t \in T$). For a generic vector field V, we have the two projections

$$\operatorname{nor} V = -\langle V, \mathcal{U}\rangle \mathcal{U}, \quad \text{and} \quad \tan V \perp_g \operatorname{nor} V.$$

In the following discussion we will assume three fields X, Y, Z tangent to S_t.

(1) $\mathrm{D}_{\mathcal{U}}\mathcal{U} = 0$. This is the geodesic equation for a comoving observer. It can be shown explicitly using the Koszul's formula (defined below) and it is one of the tasks for homework.

(2) $\mathrm{D}_{\mathcal{U}} X = \mathrm{D}_X \mathcal{U} = hX$, where $h = \dot{a}/a$ (writing it out with unit vectors gives $\mathrm{D}_{e_0} e_\alpha \sim \Gamma^a{}_{0\alpha} e_a$ or with the Ricci calculus $(\mathrm{D}_{\mathcal{U}} X)^\alpha = X^\alpha{}_{;\beta}\mathcal{U}^\beta$). The first equality follows directly from the fact that the Levi-Civita connection is torsion-free. For the second equality, we can use the *Koszul's formula*:

$$2\langle \mathrm{D}_U V, W\rangle = U\langle V, W\rangle + V\langle W, U\rangle - W\langle U, V\rangle - \langle U, [V, W]\rangle + \langle V, [W, U]\rangle + \langle W, [U, V]\rangle$$

for any three vector fields $U, V, W \in \mathrm{Vec}(M)$. Note that if a vector is written to the left of a function, it is understood as a directional derivative of the function. In our case, we have $U = \mathcal{U}$, $V = X$ and $W = Y$, so

$$2\langle \mathrm{D}_{\mathcal{U}} X, Y\rangle = \mathcal{U}\langle X, Y\rangle + X\langle Y, \mathcal{U}\rangle - Y\langle \mathcal{U}, X\rangle - \langle \mathcal{U}, [X, Y]\rangle + \langle X, [Y, \mathcal{U}]\rangle + \langle Y, [\mathcal{U}, X]\rangle .$$

We will see that five of the six terms vanish. The second and third term vanish because they contain a scalar product between a vector field in S_t and the normal vector field $\mathcal{U}$. The fifth and sixth term vanish because of the naturality of the Lie bracket, i.e. $[\mathcal{U}, X] = 0$ and $[Y, \mathcal{U}] = 0$ as $\mathcal{U}$ is in some sense "perpendicular" to X and Y. The fourth term vanishes because X and Y are tangent to S_t and hence $[X, Y]$ is also tangent to S_t. The first term is the only one that does not vanish, and it gives

$$\begin{aligned} 2\langle \mathrm{D}_{\mathcal{U}} X, Y\rangle_{(t,q)} &= \mathcal{U}\langle X, Y\rangle = \frac{\partial}{\partial t} a^2(t) g_S(X, Y)_q = 2a(t)\dot{a}(t) g_S(X, Y)_q \\ &= 2h(t)a(t)^2 g_S(X, Y)_q = 2h(t)\langle X, Y\rangle_q . \end{aligned}$$

Since this holds for all $Y \perp_g \mathcal{U}$ the assertion is proven.

(3) $\operatorname{nor}(\mathrm{D}_X Y) = \Pi(X, Y) = h\langle X, Y\rangle \mathcal{U}$. To show this, we have to remember the definition of the normal part of the connection, $\operatorname{nor}(\mathrm{D}_X Y) = -\langle \mathrm{D}_X Y, \mathcal{U}\rangle \mathcal{U}$. Since $Y \perp_g \mathcal{U}$ by assumption, it holds $X\langle Y, \mathcal{U}\rangle = 0$. So we write

$$X\langle Y, \mathcal{U}\rangle = \langle \mathrm{D}_X Y, \mathcal{U}\rangle + \langle Y, \mathrm{D}_X \mathcal{U}\rangle$$

and

$$\langle \mathrm{D}_X Y, \mathcal{U}\rangle = -\langle Y, \mathrm{D}_X \mathcal{U}\rangle = -\langle Y, hX\rangle .$$

(4) $\tan(\mathrm{D}_X Y) = 0$,

(5) The Riemann tensor (curvature operator) can be expressed as

$$R_{XY} Z = \left\{ \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right\} (\langle Z, X\rangle Y - \langle Z, Y\rangle X) .$$

To prove this, we can use the Gauss equation

$$\langle R^S_{XY} Z, W\rangle = \langle R_{XY} Z, W\rangle - \langle \mathscr{S}(X,Z), \mathscr{S}\pi(Y,W)\rangle + \langle \mathscr{S}(Y,Z), \mathscr{S}\pi(X,W)\rangle\,.$$

Since S_t has constant curvature $C = k/a^2$, by a computation we find

$$R^S_{XY} Z = C\,\{\langle Z,X\rangle\, Y - \langle Z,Y\rangle\, X\}\,.$$

Now we can use the point (3) to express the normal part of the connection in terms of the Hubble parameter $h = \dot{a}/a$:

$$\mathrm{nor}(\mathrm{D}_X Z) = \mathscr{S}(X,Z) = \langle X,Z\rangle\, \frac{\dot{a}}{a}\mathcal{U}.$$

Inserting this into the Gauss equation and using $\langle \mathcal{U},\mathcal{U}\rangle = -1$ gives the assertion.

(6) Suppose we measure the curvature tensor on a loop of one spatial and one temporal direction for a vector field pointing in the temporal direction. We get

$$R_{X\mathcal{U}}\mathcal{U} = \frac{\ddot{a}}{a} X.$$

To show this, we use the naturality of the Lie bracket $[X,\mathcal{U}] = 0$. Then we get

$$R_{X\mathcal{U}}\mathcal{U} = -\mathrm{D}_X \mathrm{D}_{\mathcal{U}}\mathcal{U} + \mathrm{D}_{\mathcal{U}}\mathrm{D}_X\mathcal{U} = \mathrm{D}_{\mathcal{U}}\mathrm{D}_X\mathcal{U} = \mathrm{D}_{\mathcal{U}}\left(\frac{\dot{a}}{a}X\right).$$

The first term vanishes because $\mathrm{D}_{\mathcal{U}}\mathcal{U} = 0$ is just the geodesic equation for a comoving observer and the last step is done by (2). Now we use the Leibniz rule for the covariant derivative (as $\mathcal{U}$ is just the time derivative (globally))

$$R_{X\mathcal{U}}\mathcal{U} = \mathrm{D}_{\mathcal{U}}((\dot{a}/a)X) = \frac{\ddot{a}a - \dot{a}^2}{a^2} X + \frac{\dot{a}}{a}\mathrm{D}_{\mathcal{U}} X = \frac{\ddot{a}}{a} X.$$

(7) Another option to find an action of the curvature tensor is to consider moving a temporal vector field $\mathcal{U}$ along a spatial loop XY. The result is $R_{XY}\mathcal{U} = 0$, which the reader can check for homework.

(8) The last non-trivial component of the curvature tensor is

$$R_{X\mathcal{U}} Y = \frac{\ddot{a}}{a}\,\langle X,Y\rangle\,\mathcal{U}.$$

To show this, we use again $[X,\mathcal{U}] = 0$ to get

$$R_{X\mathcal{U}} Y = -\mathrm{D}_X \mathrm{D}_{\mathcal{U}} Y + \mathrm{D}_{\mathcal{U}}\mathrm{D}_X Y = \left(\mathrm{D}_{\mathcal{U}} - \frac{\dot{a}}{a}\right)\mathrm{D}_X Y.$$

We split the connection into the normal and tangential part, $\mathrm{D}_X Y = \mathrm{nor}(\mathrm{D}_X Y) + \tan(\mathrm{D}_X Y)$. The tangential part vanishes, and we are left with:

$$R_{X\mathcal{U}} Y = \left(\mathrm{D}_{\mathcal{U}} - \frac{\dot{a}}{a}\right)\mathrm{nor}(\mathrm{D}_X Y) \overset{(3)}{=} \left(\mathrm{D}_{\mathcal{U}} - \frac{\dot{a}}{a}\right)\left(\langle X,Y\rangle\,\frac{\dot{a}}{a}\mathcal{U}\right).$$

The assertion follows by using the Leibniz rule for the covariant derivative. With this, we have extracted all the information from the Riemann tensor (any other combinations are either related by symmetry to the ones computed or equal zero by symmetry).

(9) In order to compute the Einstein equations, we will also need the Ricci tensor. The results are

$$\mathrm{Ric}(\mathcal{U},\mathcal{U}) = -3\frac{\ddot{a}}{a}, \qquad \mathrm{Ric}(\mathcal{U},X) = 0, \qquad \mathrm{Ric}(X,Y) = \left(2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} + \frac{\ddot{a}}{a}\right)\langle X,Y\rangle\,.$$

(10) The Ricci scalar is

$$\mathcal{S} = 6\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} + \frac{\ddot{a}}{a}\right).$$

We will show the calculation for the first component of the Ricci tensor. For that, we have to depart slightly from the strictly geometrical approach, and we introduce the *frame fields* E_i. We know that they exist, but we will never choose any specific one. Any vector field V can be expressed in terms of the frame fields

$$V = \sum_{m=0}^{3} \varepsilon_m \langle V, E_m\rangle E_m.$$

The coefficients ε_m are the signatures that convey the information about causality. In our case $\varepsilon_0 = -1$ and $\varepsilon_i = +1$ for $i = 1,2,3$. For convenience, we will use $E_0 = \mathcal{U}$. The Ricci tensor expressed in terms of the frame fields is

$$\mathrm{Ric}(V,W) = \sum_{i=0}^{3} \varepsilon_i \langle R_{V\overbrace{E_m}W,\overbrace{E_m}}\rangle\,.$$

For the particular case of $V = W = \mathcal{U}$, we can only sum over $m = 1,2,3$, because $R_{\mathcal{U}\mathcal{U}\mathcal{U}} = 0$. We get

$$\mathrm{Ric}(\mathcal{U},\mathcal{U}) = \sum_{r=1}^{3} \varepsilon_i \langle R_{\mathcal{U}E_r}\mathcal{U}, E_r\rangle \overset{(6)}{=} -\frac{\ddot{a}}{a}\sum_{r=1}^{3}\langle E_r, E_r\rangle = -\frac{\ddot{r}}{r}\dim(S_t) = -3\frac{\ddot{a}}{a}.$$

2.2 The Friedmann fluid

Claim 2.2. *If $\mathcal{U}$ is the flow vector field on a Friedmann spacetime $M_{k,a}$ then $(\mathcal{U},\rho,\mathcal{P})$ is a perfect fluid with the energy density ρ and pressure $\mathcal{P}$ given by*

$$\frac{8\pi G}{3}\rho = \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3\mathcal{P}).$$

They are called the *Friedmann equations*. The first equation is only first order, but it is non-linear. Thus, we see that even in the simplest case of the theory, the Einstein equations are still non-linear. The second equation is sometimes also called the *Raychaudhuri equation*.

Proof. To show the Friedmann equations, we use the Einstein equations:

$$T = \frac{1}{8\pi}\left(\mathrm{Ric} - \frac{1}{2}\mathcal{S}g\right).$$

By direct computation, we find

$$\rho = T(\mathcal{U},\mathcal{U}) = \frac{1}{8\pi}\left(\mathrm{Ric}(\mathcal{U},\mathcal{U}) - \frac{1}{2}g(\mathcal{U},\mathcal{U})\mathcal{S}\right) = \frac{3}{8\pi}\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right).$$

The energy density is only a function of time, hence it is constant on all spatial slices S_t. The other components of the energy-momentum tensor are

$$T(\mathcal{U}, X) = 0, \qquad T(X, Y) = \mathcal{P}\,\langle X, Y\rangle\,.$$

The spatial components of the energy-momentum tensor give us the second Friedmann equation. □

Combining both equations gives the equation of perfect Friedmann fluid:

$$3\frac{\ddot{a}}{a} = -4\pi(\rho + 3\mathcal{P}).$$

Claim 2.3. *For a perfect Friedmann fluid, it holds*

$$\dot{\rho} = -3(\rho + \mathcal{P})\frac{\dot{a}}{a}.$$

This is also called the *continuity equation* for the perfect Friedmann fluid. This equation together with the first Friedmann equation gives the Raychaudhuri equation.

Proof. We start by the divergence of the energy-momentum tensor:

$$(\mathrm{div}\,T)(\mathcal{U}) = \sum_{m=0}^{3} \varepsilon_m (\mathrm{D}_{E_m} T)(E_m,\mathcal{U}) = -(\mathrm{D}_{\mathcal{U}} T)(\mathcal{U},\mathcal{U}) + \sum_{r=1}^{3} (\mathrm{D}_{E_r} T)(E_r,\mathcal{U}).$$

Remember that $T = \rho\,\mathcal{U}^\flat \otimes \mathcal{U}^\flat + \mathcal{P}(H^\perp)^\flat$ and $(H^\perp)^\flat = g + \mathcal{U}^\flat \otimes \mathcal{U}^\flat$. We have

$$(\mathrm{D}_{\mathcal{U}} H^{\perp\flat})(\mathcal{U},\mathcal{U}) = 0, \qquad (\mathrm{D}_{E_p} H^{\perp\flat})(E_p,\mathcal{U}) = -\frac{\dot{a}}{a}\,\langle E_p, E_p\rangle\,.$$

Since ρ and $\mathcal{P}$ are only functions of time the covariant conservation implies $(\mathrm{div}\,T)(\mathcal{U}) = 0$. What is left non-zero is exactly

$$\dot{\rho} = -3(\rho + \mathcal{P})\frac{\dot{a}}{a}.$$

□

In order to solve the last equation, we need to know the relationship between the energy density and the pressure (the equation of state). If we assume that the universe is cold and particles have practically no kinetic energy then the pressure is negligible (so-called *cold matter*),

and we can directly relate the energy density and the scale factor. Choosing an arbitrary time $t_* \in T$ gives

$$\frac{\rho}{\rho_*} = \left(\frac{a}{a_*}\right)^{-3}.$$

The star in the index of any object denotes the value of that object at the time t_*, $\mathcal{O}_* = \mathcal{O}(t_*)$. This equation is called the *scaling relation* and can also be written as $\rho \sim a^{-3}$. In order to also include the pressure, we need to know the *equation of state*

$$\mathcal{P} = f(\rho, S)$$

where S is the entropy density. For the ideal fluid, the equation of state doesn't depend on the entropy, and we can write it as $\mathcal{P} = f(\rho)$.

The solution is rather simple if we assume that the pressure is proportional to the energy density, $\mathcal{P} = w\rho$ where w is a constant. An example of this is the radiation, where $w = 1/3$ or the cold matter, where $w = 0$. With this equation of state, we have

$$\dot{\rho} = -3(1+w)\rho\frac{\dot{a}}{a}.$$

If we denote $\kappa = -3(1+w)$, the solution is

$$\frac{\rho(t)}{\rho(t_*)} = \left(\frac{a(t)}{a(t_*)}\right)^{\kappa}.$$

For light, the density changes according to $\rho \sim a^{-4}$. The consequence of the fact that the radiation density changes differently to the mass density is that we can distinguish different epochs of the universe when different forms of energy dominated.

There is a non-physical singularity at $a = 0$ and there is no solution yet for this problem. However, the picture is consistent with the first law of thermodynamics, $\mathrm{d}U = 0$ (at zero pressure). We have

$$0 = \mathrm{d}(U) = \mathrm{d}(\rho \cdot V_{\text{phys}}) = \mathrm{d}(\rho \cdot a^3 V_{\text{comoving}})$$

and this gives exactly $\mathrm{d}\ln\rho = -3\mathrm{d}\ln a$. This describes a fixed amount of energy in an expanding volume.

2.3 Friedmann singularities

For this section, we will assume that the Hubble parameter at some time $t_0 \in T$ is strictly positive, $H_0 = (\dot{a}/a)_0 > 0$. This agrees well with actual physical observations.

Claim 2.4. *Let $M_{k,a} = T \times_a S$ be the Friedmann spacetime. If $\rho + 3\mathcal{P} > 0$ on the time interval T and $H_0 = (\dot{a}/a)(t_0) > 0$ for some $t_0 \in T$, then the interval T has an initial point $t^{\text{in}}_{\text{end}}$ with*

$$t_0 - H_0^{-1} < t^{\text{in}}_{\text{end}} < t_0$$

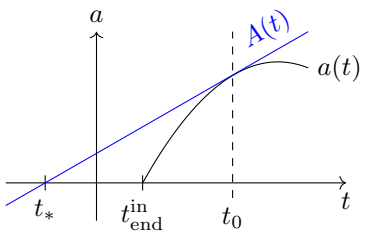
and either

1. *$\dot{a} > 0$ on T or*
2. *the scale factor a has a maximum on T after t_0, and T is a finite interval $(t^{\text{in}}_{\text{end}}, t^{\text{f}}_{\text{end}})$.*

Proof sketch. For a perfect Friedmann fluid, it holds

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3\mathcal{P}).$$

Since $a > 0$ and $\rho + 3\mathcal{P} > 0$, it follows $\ddot{a} < 0$ on the whole interval T. The curve on the graph $a(t)$ will always run below the tangent line $A(t)$ and will only touch it at t_0.



The tangent line $A(t)$ at time t_0 has the equation

$$A(t) = a(t_0) + \dot{a}(t_0)(t - t_0) = a_0 + H_0 a_0 (t - t_0) \geq a(t).$$

As t decreases, this line eventually approaches the t-axis, and crosses it at $t_* = t_0 - H_0^{-1}$. The scale factor is positive definite and ($\ddot{a} < 0$) always below (or equal) to $A(t)$. Both requirements cannot be satisfied at t_* Therefore, t_* cannot be included in T and hence we need an initial point $t_{\text{end}}^{\text{in}}$ such that $t_0 > t_{\text{end}}^{\text{in}} > t_*$. □

A Friedmann spacetime $M_{0,a}$ with Euclidean hypersurfaces corresponds today to a *critical energy density*

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G}.$$

This corresponds to 10^{-5} protons/cm^3. For any type of energy density ρ_j we introduce a dimensionless parameter called the *density parameter*

$$\Omega_{j,0} \coloneqq \frac{\rho_{j,0}}{\rho_{\text{crit},0}}$$

where the zero in the index denotes the value of the density at the present time t_0. The Friedmann equation can be expressed in terms of dimensionless density parameters as follows (for $k = 0$):

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{\text{r},0}\left(\frac{a_0}{a}\right)^4 + \Omega_{\text{m},0}\left(\frac{a_0}{a}\right)^3 + \Omega_{\Lambda,0}.$$

The density associated with the cosmological constant Λ is given by

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}.$$

We can show this by a direct computation:

$$\frac{H^2}{H_0^2} = \frac{\rho}{\rho_{\text{crit},0}} = \frac{1}{\rho_{\text{crit},0}}\left(\rho_{\text{r},0}\frac{a_0^4}{a^4} + \rho_{\text{m},0}\frac{a_0^3}{a^3} + \rho_\Lambda\right).$$

The Friedmann equation implies the following constraint:

$$\Omega_{\mathrm{r}} + \Omega_{\mathrm{m}} + \Omega_\Lambda = 1.$$

For most of its history, our universe (modelled as a Friedmann spacetime $M_{0,a}$) underwent expansion stages by a single type (r, m, Λ) of ideal fluid. This follows from the scaling relations

$$\rho_{\mathrm{r}} \propto a^{-4}, \quad \rho_{\mathrm{m}} \propto a^{-3}, \rho_\Lambda \propto a^0.$$

The scaling relations imply the following hierarchy of epochs.

$$? \longrightarrow \begin{matrix}\text{radiation}\\ \text{domination}\end{matrix} \longrightarrow \begin{matrix}\text{matter}\\ \text{domination}\end{matrix} \longrightarrow \begin{matrix}\Lambda\\ \text{domination}\end{matrix}$$

Let's first discuss the single component solutions.

Claim 2.5. *Consider a Friedmann spacetime $M_{0,a} \equiv T \times_a S$ with a single ideal fluid with constant equation of state parameter w. For $t_1, t_2 \in T$ the solution is*

$$\frac{a(t_2)}{a(t_1)} = \begin{cases} \left(\frac{t_2}{t_1}\right)^{\frac{2}{3(1+w)}}, & w \neq -1, \\ \exp(\sqrt{\Omega_\Lambda} H_0 (t_2 - t_1)), & w = -1. \end{cases}$$

Proof. Let Ω denote the dimensionless density parameter associated with the single ideal fluid. Introduce

$$\kappa_0 = \sqrt{\Omega} H_0 a_0^\mu, \quad \mu = \frac{3}{2}(1+w).$$

The Friedmann equation becomes $H = \kappa_0 a^{-\mu}$. For $w \neq -1$ we find

$$\frac{a(t_2)}{a(t_1)} = \left(\frac{t_2}{t_1}\right)^{1/\mu}.$$

In the case $w = -1$ we know that $\Omega = \Omega_\Lambda$. Then $\mathrm{d} \ln a / \mathrm{d}t = \kappa_0$ with the solution

$$a(t_2) = a(t_1) \exp(\sqrt{\Omega_\Lambda} H_0 (t_2 - t_1)).$$

□

Fluid	$a(t)$	$H(t)$
radiation	$t^{1/2}$	$1/(2t)$
matter	$t^{2/3}$	$2/(3t)$
dark energy	$\exp(\sqrt{\Omega_\Lambda} H_0 t)$	$\sqrt{\Omega_\Lambda} H_0$

For the matter dominated universe, we can find the age by $t_0 = 2/(3H_0)$, where the measurements show $H_0 \sim 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. The result is $t_0 \sim 9 \cdot 10^9$ years. This is less than the age of the oldest stars, hence the universe cannot be matter dominated.

Definition 2.1. For a Friedmann spacetime $M_{0,a} \equiv T \times_a S$, define

$$\mathrm{d}\eta \coloneqq \frac{\mathrm{d}t}{a(t)}.$$

It is called the *conformal time*.

Remark. We can define the Hubble constant that depends on the conformal time:

$$\mathcal{H}(\eta) = \frac{\mathrm{d}}{\mathrm{d}\eta} \ln a(\eta).$$

Fluid	$a(\eta)$	$\mathcal{H}(\eta)$
radiation	η	η^{-1}
matter	η^2	$2\eta^{-1}$
dark energy	η^{-1}	$-\eta^{-1}$

For the homework, we will show that for $w \neq -1/3$ the relation between the conformal time and scale factor is

$$a(\eta) = \eta^{2/(1+3w)}.$$

As we can see, the conformal time simplifies many of the results.

This finishes the discussion about the single-component fluids.

The obvious next step is to consider the case of two fluids. In conformal time, the Friedmann equation and the Raychaudhuri equation for $k = 0$ become

$$(a')^2 = \frac{8\pi G}{3}\rho a^4,$$

$$a'' = \frac{4\pi G}{3}(\rho - 3\mathcal{P})a^3.$$

The derivative with respect to the conformal time is denoted by a prime. Note that the minus sign in front of the pressure term is not a mistake. The two equations above can be shown by a direct computation using

$$\dot{a} = \frac{a'}{a} = \mathcal{H}, \quad \ddot{a} = \frac{a''}{a^2} - \frac{(a')^2}{a^2}.$$

The Raychaudhuri equation in conformal time doesn't have a source term supporting a radiation fluid.

Claim 2.6. *Consider a Friedmann spacetime $M_{0,a}$ whose expansion is sourced by an ideal radiation fluid and an ideal matter fluid. Then there is a conformal timescale $\eta_* \sim \eta_{\mathrm{eq}}$ so that the scale factor is given by*

$$\frac{a(\eta)}{a(\eta_{\mathrm{eq}})} = \left(\frac{\eta}{\eta_*}\right)^2 + 2\frac{\eta}{\eta_*}.$$

Proof. Write

$$\rho = \rho_{\mathrm{m}} + \rho_{\mathrm{r}} = \frac{1}{2}\rho_{\mathrm{eq}}\left[\left(\frac{a_{\mathrm{eq}}}{a}\right)^3 + \left(\frac{a_{\mathrm{eq}}}{a}\right)^4\right]$$

where a_{eq} is the scale factor where both densities were equal ($\rho_{\mathrm{eq}}/2$). The energy term in the conformal Raychaudhuri equation is

$$\rho - 3\mathcal{P} = \rho_{\mathrm{m}} + \rho_{\mathrm{r}} - 3\mathcal{P} = \rho_{\mathrm{m}}.$$

Hence, the Raychaudhuri equation becomes

$$a'' = \frac{2\pi G}{3}\rho_{\mathrm{eq}} a_{\mathrm{eq}}^3.$$

The solution is

$$a(\eta) = \kappa_{\mathrm{eq}}\eta^2 + C_1\eta + C_0$$

where $\kappa_{\mathrm{eq}} = \pi G\rho_{\mathrm{eq}} a_{\mathrm{eq}}^3/3$. The boundary conditions are $a(0) = 0$ which gives $C_0 = 0$ and $(a'(0))^2 = C_1^2 = 4\pi G\rho_{\mathrm{eq}} a_{\mathrm{eq}}^4/3$ (by Friedmann equation). The full solution is

$$a(\eta) = \frac{\pi}{3}G\rho_{\mathrm{eq}} a_{\mathrm{eq}}^3\eta^2 + \sqrt{4\pi G\rho_{\mathrm{eq}} a_{\mathrm{eq}}^4/3}\eta = \kappa_{\mathrm{eq}}\eta^2 + \sqrt{4\kappa_{\mathrm{eq}} a_{\mathrm{eq}}}\eta.$$

Next, we introduce $\eta_* \coloneqq (\kappa_{\mathrm{eq}}/a_{\mathrm{eq}})^{-1/2}$. We get

$$\frac{a(\eta)}{a_{\mathrm{eq}}} = \left(\frac{\eta}{\eta_*}\right)^2 + 2\frac{\eta}{\eta_*}.$$

It remains to show that $\eta_{\mathrm{eq}} \sim \eta_*$. By direct computation (solving the preceding solution at η_{eq} for η_*) we find

$$\eta_* = \frac{1}{\sqrt{2}-1}\eta_{\mathrm{eq}}.$$

□

The other two component fluid that is analytically solvable is the combination of matter and dark energy. The derivation of the solution is given as a homework.

Chapter 3

Hot Big Bang

In this chapter, we will try to probe what shortly after the singularity at $t_{\text{end}}^{\text{in}}$ happened. Around that time, the value of a was small and the value of $\dot{a}$ was large, hence, the energy density was enormous. This leads us to the term *hot Big Bang* (HBB).

In order to discuss the particles in the Friedmann fluid, we first have to talk about the distribution functions. In principle, the distribution function f maps from a cotangent space of an equal-time hypersurface (the space of all possible momenta), $f : T^*S_t \to \mathbb{R}$. Due to the isometries of Friedmann, f does not depend on particle positions. In (thermal) equilibrium, the time dependence is replaced by a temperature dependence. The important examples are

$$f_\pm(p; T, \mu) = \left[\exp\left(\beta(E(p) - \mu)\right) \pm 1\right]^{-1}$$

where $+$ corresponds to *Fermi-Dirac distribution* and $-$ corresponds to *Bose-Einstein distribution* and $\beta = 1/T$.

Next, we want to determine observables. For $\mathcal{O} \in \mathcal{C}^\infty(T^*S_t, \mathbb{R})$ that is q-independent it holds (in equilibrium)

$$\langle \mathcal{O} \rangle_f \coloneqq g \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f(p; T, \mu) \mathcal{O}(p).$$

The letter g represents the internal degrees of freedom of the particle, e.g. the spin.

Example 3.

a) The average number density: $n(T, \mu) = \langle 1 \rangle_f (T, \mu)$.

b) The average energy density: $\rho(T, \mu) = \langle E(p; m) \rangle_f (T, \mu; m)$.

c) The average pressure: $\mathcal{P}(T, \mu) = \langle \|p\|^2 / (3E(p)) \rangle_f (T, \mu; m)$.

Remark.

1. Each species j has its own temperature T_j and chemical potential μ_j.
2. At sufficiently early times, $\mu_j / T_j \ll 1$ for all species j. (This is more of an assumption than a provable statement.)

Next, we introduce three new dimensionless parameters: $x = m/T$, $y = \mu/T$ and $k = \|p\|/T$. With all suitable constants, we get

$$x = \frac{mc^2}{k_\mathrm{B}T}, \quad y = \frac{\mu}{k_\mathrm{B}T}, \quad k = \frac{\|p\|c}{k_\mathrm{B}T}.$$

With these parameters, we can express the distribution function as

$$f_\pm(k; x, y) = \left[\exp(\sqrt{k^2 + x^2} - y) \pm 1\right]^{-1}.$$

Furthermore, for an operator $\mathcal{O}$, we define

$$\langle\!\langle \mathcal{O} \rangle\!\rangle_{f_\pm}(x, y) = \int_{\mathbb{R}^+} \mathrm{d}k\, k^2 f_\pm(k; x, y)\mathcal{O}.$$

It will be convenient in the future to introduce two more integrals:

$$I_\pm(x, y) \coloneqq \langle\!\langle 1 \rangle\!\rangle_{f_\pm}(x, y) = \int_{\mathbb{R}^+} \mathrm{d}k\, k^2 f_\pm(k; x, y),$$
$$J_\pm(x, y) \coloneqq \langle\!\langle \sqrt{k^2 + x^2} \rangle\!\rangle_{f_\pm}(x, y) = \int_{\mathbb{R}^+} \mathrm{d}k\, k^2 f_\pm(k; x, y)\sqrt{k^2 + x^2}.$$

In the language of the average densities, we have

$$n = \frac{g}{2\pi^2}T^3 I_\pm(x, y), \qquad \rho = \frac{g}{2\pi^2}T^4 J_\pm(x, y).$$

Why is this a smart reparametrization? It is very convenient in the (ultra)relativistic limit where $m = 0$ and $\mu \approx 0$, so $(x, y) = (0, 0)$. In this limit, we have

$$I_-(0, 0) = 2\zeta(3), \quad I_+(0, 0) = \frac{3}{4}I_-(0, 0).$$

To see this, we write

$$f_\pm(k; 0, 0) = \frac{1}{e^k \pm 1} = \frac{e^{-k}}{1 \pm e^{-k}}.$$

We expand this in the geometric series:

$$f_\pm(k; 0, 0) = \sum_{l=1}^{\infty} (\mp 1)^{l-1} e^{-lk}$$

and thus

$$I_\pm = \sum_{l=1}^{\infty} (\mp 1)^{l-1} \int_0^\infty \mathrm{d}k\, k^2 e^{-lk} = \sum_{l=1}^{\infty} (\mp 1)^{l-1} \frac{2}{l^3}.$$

So $I_-(0, 0) = 2\zeta(3)$ and $I_+(0, 0) = 3\zeta(3)/2$. As a result we get

$$n_\pm(T) = \frac{\zeta(3)}{\pi^2} g T^3 \cdot \begin{cases} 1 & \text{for bosons,} \\ \frac{3}{4} & \text{for fermions} \end{cases}$$

and

$$\rho_\pm(T) = \frac{\pi^2}{30} g T^4 \cdot \begin{cases} 1 & \text{for bosons,} \\ \frac{7}{8} & \text{for fermions.} \end{cases}$$

For a massless species with vanishing chemical potential, the pressure is given by $\mathcal{P} = \rho/3$.

Another limit that we can study is the non-relativistic limit, where $x \gg 1$ and $x \gg y$. The distribution function can be simplified to

$$f_{\pm}(k; x, y) \approx \exp(-\sqrt{k^2 + x^2}).$$

In this case, we observe that the dominant contribution is from the region where $k^2 \ll x^2$. In a not particularly rigorous fashion, we can write

$$I_{\pm}(x, 0) \approx \int_0^{\infty} \mathrm{d}k\, k^2 \exp(-\sqrt{k^2 + x^2}).$$

We introduce a new variable $z = k/x = \|p\|/m \ll 1$:

$$I_{\pm}(x, 0) \approx x^3 \left(\int_0^{\varepsilon} + \int_{\varepsilon}^{\infty} \right) \mathrm{d}z\, z^2 \exp(-x\sqrt{1 + z^2}).$$

We expand the first integral's integration domain (which gives a small error) and the second part gives a sub-leading contribution:

$$I_{\pm}(x, 0) \approx x^3 e^{-x} \int_0^{\infty} \mathrm{d}z z^2 \exp(-xz^2/2) + R_{\varepsilon}(x) = \sqrt{\frac{\pi}{2}} x^{3/2} e^{-x} + R_{\varepsilon}(x).$$

The particle density in this limit is

$$n(T) \approx g \left(\frac{mT}{2\pi} \right)^{3/2} \exp\left(-\frac{m}{T} \right)$$

and the energy density is

$$\rho(T) \approx mn(T) \left(1 + \frac{3}{2} \frac{T}{m} + \dots \right).$$

The pressure obeys the ideal gas law $\mathcal{P} = n(T)T$. Hence, the energy density can also be expressed as

$$\rho(T) \approx mn(T) + \frac{3}{2} \mathcal{P}(T).$$

3.1 Counting degrees of freedom

Suppose we have a mixture of ideal fluids. Let all species be relativistic with temperatures T_s (this is a tuple of temperatures) and chemical potentials $\mu_s = 0$.

Claim 3.1. *The total energy density is given by*

$$\rho(T) = \frac{\pi^2}{30} g_*(T, T_s) T^4$$

where T is the temperature of the universe and $g_*(T, T_s)$ is the effective number of relativistic degrees of freedom at temperature T and species temperatures T_s.

Proof. For each relativistic boson, we write

$$\rho_{\mathrm{b}}(T_{\mathrm{b}}) = \frac{g_{\mathrm{b}}}{2\pi} T_{\mathrm{b}}^4 J_-(x_{\mathrm{b}}, 0)$$

and for each relativistic fermion

$$\rho_{\mathrm{f}}(T_{\mathrm{f}}) = \frac{g_{\mathrm{f}}}{2\pi} T_{\mathrm{f}}^4 J_+(x_{\mathrm{b}}, 0).$$

We rewrite the bosonic density as

$$\rho_{\mathrm{b}} = \frac{1}{2\pi^2} J_-(0,0) T^4 g_{\mathrm{b}} \left(\frac{T_{\mathrm{b}}}{T}\right)^4 \frac{J_-(x_{\mathrm{b}}, 0)}{J_-(0,0)}.$$

The total energy density is

$$\rho(T) = \frac{\pi^2}{30} \underbrace{\left(\sum_{\mathrm{b}} g_{\mathrm{b}} \left(\frac{T_{\mathrm{b}}}{T}\right)^4 \frac{J_-(x_{\mathrm{b}}, 0)}{J_-(0,0)} + \sum_{\mathrm{f}} g_{\mathrm{f}} \left(\frac{T_{\mathrm{f}}}{T}\right)^4 \frac{J_+(x_{\mathrm{f}}, 0)}{J_-(0,0)} \right)}_{g_*(T, T_s)} \cdot T^4.$$

□

For $x_{\mathrm{b,f}} \approx 0$, the effective number of relativistic degrees of freedom simplifies to

$$g_*(T) = \sum_{\mathrm{b}} g_{\mathrm{b}} \left(\frac{T_{\mathrm{b}}}{T}\right)^4 + \frac{7}{8} \sum_{f} g_{\mathrm{f}} \left(\frac{T_{\mathrm{f}}}{T}\right)^4.$$

If the species are in the thermal equilibrium, then $T_{\mathrm{b}} = T_{\mathrm{f}} = T$ and the task of determining g_* is a counting exercise:

$$g_* \approx \sum_{\mathrm{b}} g_{\mathrm{b}} + \frac{7}{8} \sum_{\mathrm{f}} g_{\mathrm{f}} \overset{\mathrm{SM}}{\approx} 106.75.$$

3.2 Entropy

In a Friedmann spacetime, the entropy S of any species in equilibrium at temperature T with vanishing chemical potential is conserved. Instead of a microscopic formulation,

$$T \mathrm{d}S = \mathrm{d}U + \mathcal{P} \mathrm{d}V$$

(for vanishing chemical potential) we use instead the extensive variables $s = S/V$ and $\rho = U/V$. The relation we get is

$$\left(\frac{\mathrm{d}s}{\mathrm{d}T} - \frac{1}{T} \frac{\mathrm{d}\rho}{\mathrm{d}T} \right) \mathrm{d}T = \frac{1}{T} \left(\rho + \mathcal{P} - Ts \right) \frac{\mathrm{d}V}{V}.$$

Since temperature and volume can vary independently, we get

$$s = \frac{\rho + \mathcal{P}}{T}, \qquad \frac{\mathrm{d}s}{\mathrm{d}T} = \frac{1}{T} \frac{\mathrm{d}\rho}{\mathrm{d}T}.$$

This implies that the scaling of the entropy density can only be trivial,

$$\frac{\dot{s}}{s} = -3 \frac{\dot{a}}{a}$$

and $sa^3 = \text{const}$. If we add the chemical potential, the situation gets more spicy.

Claim 3.2. *The entropy density in a Friedmann universe is given by*

$$s(T) = \frac{\rho(T)}{T} + \int_0^T \mathrm{d}\tilde{T}\, \frac{\rho(\tilde{T})}{\tilde{T}^2}.$$

Proof. This is obtained by partial integration of the equation $\mathrm{d}s/\mathrm{d}T = (1/T)\mathrm{d}\rho/\mathrm{d}T$. The integral term is exactly equal to $\mathcal{P}(T)/T$. □

Claim 3.3. *Under the conditions we are working with (zero chemical potential, masses of particles much smaller than temperatures), the parameter w in the equation of state $\mathcal{P} = w\rho$ depends on the temperature T and is given by*

$$w(T) = \int_0^1 \mathrm{d}y\, y^2 \frac{g_*(Ty)}{g_*(T)}.$$

Proof. To show this, we write

$$\frac{\mathcal{P}(T)}{T} = \int_0^T \mathrm{d}\tilde{T}\, \frac{\rho(\tilde{T})}{\tilde{T}^2}.$$

Consequently,

$$w(T) = \frac{\mathcal{P}(T)}{\rho(T)} = \int_0^T \frac{\mathrm{d}\tilde{T}}{T} \frac{T^2}{\tilde{T}^2} \frac{\rho(\tilde{T})}{\rho(T)}.$$

After expanding ρ and renaming variables we get the desired result. □

Claim 3.4. *There is a function $g_{*s}(T)$ such that*

$$s(T) = \frac{2\pi^2}{45} g_{*s}(T) T^3.$$

Proof. This can be seen as follows. For each species j we have

$$s_j(T_j) = \frac{\rho_j(T_j) + \mathcal{P}_j(T_j)}{T_j} = \frac{4}{3}\frac{\rho_j(T_j)}{T_j}.$$

Assuming $T_j \gg m_j$, it holds

$$s_j(T_j) = \frac{2\pi^2}{45}\left[g_j \left(\frac{T_j}{T}\right)^3 \left(\delta_{i\mathrm{b}} + \frac{7}{8}\delta_{j\mathrm{f}}\right)\right] T^3.$$

Since entropy is extensive quantity, the assertion follows:

$$g_{*s} = \sum_{\mathrm{b}} g_{\mathrm{b}} \left(\frac{T_{\mathrm{b}}}{T}\right)^3 + \frac{7}{8}\sum_{\mathrm{f}} g_{\mathrm{f}} \left(\frac{T_{\mathrm{f}}}{T}\right)^3.$$

□

Knowing the dependence of g_{*s} on the temperature is highly non-trivial as it overcomes a phase transition. Namely, the QCD bound states have different degrees of freedom than the free quarks and gluons (primordial plasma).

3.3 First look at the CMB

The earliest photo of the universe we can take is from the time when all charged particles were bound into bound states and photons were free to propagate unobstructed. At the temperature around 1 eV our universe was still opaque for photons, since there was a lot of scattering. There was Thomson scattering of photons on electrons and also Coulomb scattering of protons and electrons (hence, indirectly, protons also interacted with photons). At temperature around 0.3 eV electrons form bound states with protons to produce atoms. The density of free charges drops and the mean free path of photons rapidly increases and exceeds the Hubble length J^{-1}. At a temperature 0.25 eV our universe becomes transparent for photons.

Consider a scattering process between four species $1 + 2 \to 3 + 4$. This implies that the chemical potentials of the species are related by $\mu_1 + \mu_2 = \mu_3 + \mu_4$.

- Photons have vanishing chemical potential $\mu_\gamma = 0$, hence, the number of photons is not conserved.
- The chemical potentials of a particle and antiparticle differ by a sign, $\mu_{\bar{\jmath}} = -\mu_j$. This can be seen by the annihilation process $j+\bar{\jmath} \to \gamma+\gamma$. The chemical balance gives $\mu_j+\mu_{\bar{\jmath}} = 2\mu_\gamma = 0$.

Consider the process

$$e^- + p^+ \to \mathrm{H} + \gamma.$$

This is called the *hydrogen recombination*. Let the collection of particles $\{e^-, p^+, H\}$ be in (local thermal) equilibrium at temperature T below any of the masses in the collection. The ratio

$$\left(\frac{n_\mathrm{H}}{n_{e^-} n_{p^+}}\right)_\mathrm{eq}$$

is independent of the chemical potentials of the particles. Since $T < m_{e^-}$ by assumption, the equilibrium abundances are

$$n_j(T_j, \mu_j) = g_j \left(\frac{n_j T}{2\pi}\right)^{3/2} \exp\left(-\frac{m_j - \mu_i}{T}\right).$$

Thus, the ratio of the number densities is

$$\left(\frac{n_\mathrm{H}}{n_{e^-} n_{p^+}}\right)_\mathrm{eq} \propto \exp\left(\frac{\mu_\mathrm{H} - \mu_{e^-} - \mu_{p^+}}{T}\right) = 1.$$

No observational evidence suggests that the universe is charged. Thus, we can assume $n_{e^-} = n_{p^+}$. We introduce two dimensionless parameters:

$$X_e = \frac{n_e}{n_e + n_\mathrm{H}} \in [0, 1]$$

is the *abundance ratio* of electrons and

$$\eta = \frac{n_B}{n_\gamma} = \frac{\pi^4}{45\zeta(3)} g_{*s} \frac{n_B}{s} \approx 1.8\, g_{*s}(T) \frac{n_B}{s}(T)$$

is the *baryon-to-photon ratio*. After the electron-positron annihilation, the baryon-to-photon ratio is approximately constant, $\eta \approx 7 n_B/s$.

Claim 3.5. *Under the above assumptions, the free electron fraction satisfies the Saha equation*

$$\left(\frac{1-X_e}{X_e^2}\right)_{\text{eq}} = \frac{n_{\text{H}}}{n_e^2}\eta n_\gamma.$$

Proof. By definition, $X_e = n_e/(n_e + n_{\text{H}})$ and $n_B = \eta n_\gamma$ and $n_e = n_p$. Then the result follows by direct computation. □

For homework, we will solve the Saha equation for $X_e = f(T, \eta)$, sketch the solution and calculate the temperature T_{rec} of the recombination.

To discuss the behaviour of particle interactions beyond equilibrium, we have to talk about the *Boltzmann equation*. In the absence of particle interactions, the abundance of particles in a given volume is conserved

$$\dot{n} = -3Hn, \quad \Longrightarrow \quad na^3 = \text{const.}$$

The abundance of particles in a Friedmann spacetime follows the *integrated Boltzmann equation*:

$$\frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}(n_j a^3) = C_j(\{n_i\})$$

where $C_j(\{n_i\})$ is called the *collision term* and is a function of the particle abundances n_i of all species i.

Example 4. Consider the process $1 + 2 \underset{\text{production}}{\overset{\text{destruction}}{\rightleftharpoons}} 3 + 4$ We can decompose the collision term as

$$C_1(n_1, n_2, n_3, n_4) = C_1^{\text{dest}}(n_1, n_2) + C_2^{\text{prod}}(n_3, n_4).$$

A basic exercise in kinetic theory gives

$$C_1^{\text{dest}}(n_1, n_2) = -\alpha n_1 n_2$$
$$C_2^{\text{prod}}(n_3, n_4) = +\beta n_3 n_4$$

where $\alpha = \langle \sigma^{\text{dest}} v \rangle$ is the averaged cross-section ($\langle \cdot \rangle$ denotes averaging over the relativistic velocities of the incoming particle). To get β, we think about the equilibrium condition:

$$C(n_1, n_2, n_3, n_4) = \sigma \quad \Longrightarrow \quad \beta = \left(\frac{n_1 n_2}{n_3 n_4}\right)_{\text{eq}} \alpha.$$

This is called the *detailed balance condition*.

Claim 3.6. *Assuming that $1 + 2 \leftrightarrow 3 + 4$ is the only reaction open for the species 1 in the Friedmann spacetime, its total number N_1 varies with time according to*

$$\frac{\mathrm{d}\ln N_1}{\mathrm{d}\ln a} = -\frac{\Gamma_1}{H}\left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\text{eq}} \frac{N_3 N_4}{N_1 N_2}\right],$$

where $\Gamma_1 = \langle \sigma v \rangle n_2$ is the interaction rate of species 1 with 2 and H is the Hubble constant. The ratio Γ_1/H is called the interaction efficiency of the species 1.

Proof. In the previous example, we have modelled the detailed balance condition. Now, we can use the Saha equation to get

$$\frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}(n_1 a^3) = -\langle\sigma v\rangle\left[n_1 n_2 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{\mathrm{eq}} n_3 n_4\right].$$

This can be manipulated as follows:

$$\frac{1}{n_1 a^3}\frac{\mathrm{d}}{\mathrm{d}t}(n_1 a^3) = -\Gamma_1\left[1 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{\mathrm{eq}} \frac{n_3 n_4}{n_1 n_2}\right].$$

Next, we introduce $n_j = N_j s$ where s is the entropy density and N_j is some parameter. We get

$$\frac{1}{N_1}\frac{1}{sa^3}\frac{\mathrm{d}}{\mathrm{d}t}(N_1 s a^3) = -\Gamma_1\left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\mathrm{eq}} \frac{N_3 N_4}{N_1 N_2}\right].$$

Since sa^3 is conserved, we can write

$$\frac{1}{N_1 sa^3}\frac{\mathrm{d}}{\mathrm{d}t}(N_1 s a^3) = \frac{1}{N_1}\frac{\mathrm{d}N_1}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\ln N_1 = Ha\frac{\mathrm{d}}{\mathrm{d}a}\ln N_1 = H\frac{\mathrm{d}\ln N_1}{\mathrm{d}\ln a}.$$

□

An application of this result is the kinetic and chemical decoupling of *WIMPs* (weakly interacting massive particles). In this model, we consider a massive particle X and its antiparticle $\overline{X}$ that annihilate into some lighter particles ℓ and $\overline{\ell}$, $X + \overline{X} \leftrightarrow \ell + \overline{\ell}$. We further assume $m_X \gg m_\ell$, $n_X = n_{\overline{X}}$ and $n_e = n_e^{\mathrm{eq}}$. The integrated Boltzmann equation for the number of WIMPs is

$$\frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}(n_X a^3) = -\langle\sigma v\rangle\,(n_X^2 - (n_X^{\mathrm{eq}})^2).$$

This follows immediately by substituting $1 = X$, $2 = \overline{X}$, $3 = \ell$ and $4 = \overline{\ell}$ into the integrated Boltzmann equation. We introduce two new variables $Y_X = n_X/T^3$ and $x = m_X/T$ (a new dimensionless measure of time). Since the temperature scales as $T \propto a^{-1}$, we can write

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Hx.$$

Claim 3.7. *The variable Y_X satisfied the Ricatti equation:*

$$\frac{\mathrm{d}Y_X}{\mathrm{d}x} = -\frac{\lambda}{x^2}[Y_X^2 - (Y_X^{\mathrm{eq}})^2]$$

where λ is a dimensionless parameter.

Proof. Starting from the left side of the integrated Boltzmann equation, we can write

$$\frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}\left(n_X a^3\right) = \frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}\left(Y_X T^3 a^3\right) = T^3\frac{\mathrm{d}}{\mathrm{d}t}Y_X = T^3\dot{x}\frac{\mathrm{d}}{\mathrm{d}x}Y_x = T^3 Hx\frac{\mathrm{d}Y_X}{\mathrm{d}x}.$$

For the number density of WIMPs, we have $n^2 = Y_X^2 T^6$. The integrated Boltzmann equation becomes

$$T^3 H x \frac{\mathrm{d}Y_X}{\mathrm{d}x} = -T^6 \langle\sigma v\rangle (Y_X^2 - (Y_X^{\mathrm{eq}})^2).$$

After rewriting and using the definition of x, we get

$$\frac{\mathrm{d}Y_X}{\mathrm{d}x} = -\frac{1}{x^4} \underbrace{\frac{M_X^3 \langle\sigma v\rangle (T)}{H(T)}}_{\tilde{\lambda}(T)} (Y_X^2 - (Y_X^{\mathrm{eq}})^2).$$

Since $H(T)/T^2 = H(M_X)/M_X^2$ and assuming $\langle\sigma v\rangle (T) = \langle\sigma v\rangle_0 + \mathcal{O}(T/M_X)$, we have

$$\tilde{\lambda}(T) = x^2 \frac{M_X^3 \langle\sigma v\rangle (T)}{H(M_X)} = x^2(\lambda + \mathcal{O}(x^{-1})), \qquad \lambda = \frac{M_X^3 \langle\sigma v\rangle_0}{H(M_x)}.$$

□

The *freeze-out density* of species X is defined by

$$Y_X^\infty = \lim_{x\to\infty} Y_X(x).$$

Can we make a statement about the freeze-out density? Yes, we can.

Claim 3.8. *Assuming that the number density of X starts to deviate from the equilibrium at around $x \approx x_f$, then its freeze-out density is approximately given by*

$$Y_X^\infty \approx \frac{x_f}{\lambda}.$$

Proof. For $x \gg x_f$, we expect $Y_X \gg Y_X^{\mathrm{eq}}$ and thus

$$\frac{\mathrm{d}Y_X}{\mathrm{d}x} = -\frac{1}{x^2}\lambda Y_X^2.$$

Integrating this equation from x_f to ∞ gives

$$\frac{1}{Y_X^\infty} - \frac{1}{Y_X(x_f)} \approx \frac{\lambda}{x_f}.$$

Assuming that $Y_X(x_f) \ll Y_X^\infty$, we can neglect the second term and get the desired result. □

Using this equation, we could relate the freeze-out abundance to its present-day value. The density parameter Ω_X of the species X can be expressed as

$$\Omega_X \approx \frac{8\pi G}{3H_0^2} M_X T_0^3 \frac{g_{*s}(T_0)}{g_{*s}(M_X)} Y_X^\infty.$$

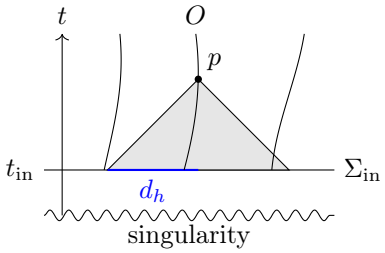
For the homework, we will prove this relation and inform ourselves about the WIMP miracle.

Chapter 4

Cosmological inflation

4.1 The horizon problems

We consider the Friedmann spacetime $M_{0,a} = T \times \Sigma$. Let Σ_{in} be an (initial) Euclidean hypersurface in the vicinity of the hot big bang singularity in the domain of general relativity. Consider a family of particles whose histories are time-like geodesics originating from Σ_{in}. Let p denote an event on the world-line of an observer (or particle) O in this family.



The past light-cone of p is the set of events in $M_{0,a}$ which can be observed by O at $t(p)$. The division of particles into these sea by O at the event p and the others is called the *particle horizon* d_h for G at p.

Remark. The particle horizon d_h is (using conformal time) greatest comoving distance from which O at time $t(p)$ will be able to receive a signal:

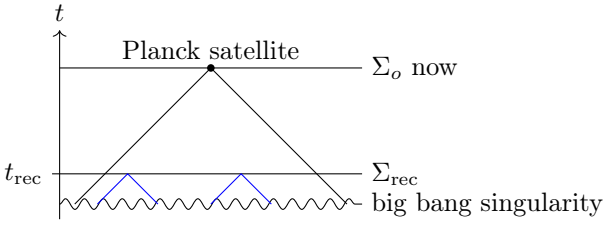
$$d_h(\eta) = \eta(t) - \eta(t_{\text{in}}) = \int_{t_{\text{in}}}^{t} \mathrm{d}\tilde{t}\, a^{-1}(\tilde{t})$$

Claim 4.1. *The particle horizon for an observer O at an event p is given by*

$$d_h(\eta) = \int_{\ln a_{\text{in}}}^{\ln a} \mathrm{d}\ln\tilde{a}\, \mathscr{H}^{-1}.$$

The statement can be proven by the definition and direct computation.

Consider a Friedmann universe $M_{0,a}$ filled with ideal radiation and matter fluids. Then any two events on the so-called hypersurface of recombination Σ_{rec} separated by an angle $\theta > 2°$ (observed at the present time) have non-intersecting past light-cones. This is called the (first) *horizon problem*.



Setting $a_0 = 1$, the Hubble rate is given by

$$H^2 = H_0^2(\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4})$$

with $\Omega_r = \Omega_m$ at a_{eq}. Of course, in the early universe, we can neglect the effects of the dark energy. Expressing the Hubble rate with a_{eq} gives

$$H^2 = H_0^2\Omega_{m,0}(a + a_{\text{eq}})a^{-4}.$$

The conformal Hubble rate is

$$\mathscr{H}^{-1} = (aH)^{-1} = \frac{1}{H_0\sqrt{\Omega_{m,0}}}\frac{a}{\sqrt{a + a_{\text{eq}}}}.$$

We can get the conformal time by integrating the inverse conformal Hubble rate:

$$\eta = \int_{a_{\text{in}}}^{a} \frac{\mathrm{d}\ln\tilde{a}}{\mathcal{H}} = \frac{2}{H_0\sqrt{\Omega_{m,0}}}\left(\sqrt{a + a_{\text{eq}}} - \sqrt{a_{\text{in}} + a_{\text{eq}}}\right) \approx \frac{2}{H_0\sqrt{\Omega_{m,0}}}\left(\sqrt{a + a_{\text{eq}}} - \sqrt{a_{\text{eq}}}\right).$$

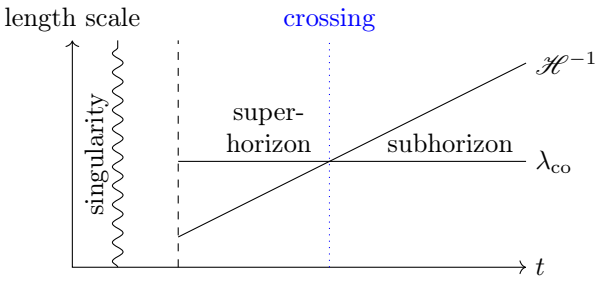
During the radiation domination $a \ll a_{\text{eq}}$ the conformal time is proportional to a, during the matter domination ($a \gg a_{\text{eq}}$), it is proportional to $\sqrt{a}$. The conformal time today is

$$\eta_0 = \frac{2}{\sqrt{\Omega_{0,m}}}\frac{1}{H_0}$$

and at recombination $a_{\text{rec}} = 1/1100$ it was $\eta_{\text{rec}} = 0.0175\,\eta_0$. The Hubble radius at the time of recombination was $d_H(\text{rec}) = 265\,\text{Mpc}$, while the comoving distance to the hypersphere of the last scattering is $d_{\text{ls}}(\eta_{\text{rec}}) = \eta_0 - \eta_{\text{rec}} \approx \eta_0 \approx 15.1\,\text{Gpc}$. The angle on the sky where we expect the particles on the recombination sphere to be causally connected is

$$\theta = \frac{2d_H(\text{rec})}{d_{\text{ls}}(\eta_{\text{rec}})} \approx 2°.$$

The horizon problem can also be stated in terms of fluctuations. Under the previous assumptions, in the radiation fluid with a finite comoving wavelength λ originated on super-horizon scales (scales that were bigger than the comoving Hubble radius – so it should be more appropriately named super-Hubble). This can be best seen on the following sketch, where you can see the comparison of comoving distances. The y-axis represents the beginning of time and the wavy line represents the time when the universe could first be described by the General relativity. The transition must have happened in a way that is not detectable by today's observations.



4.2 Inflation

The main idea of inflation is to postulate an epoch prior to the recombination, during which the cosmic expansion history is characterized by a decreasing comoving Hubble radius:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}^{-1} < 0.$$

Since $\mathscr{H} = \dot{a}$, this statement is equivalent to

$$-\frac{\dot{\mathscr{H}}}{\mathscr{H}^2} = \frac{\ddot{a}}{\mathscr{H}^2} < 0, \qquad \Longrightarrow \qquad \ddot{a} > 0.$$

This epoch is referred to as the *cosmic inflation*.

Claim 4.2. *Let $M_{0,a}$ be a Friedmann spacetime filled with a single component ideal fluid with equation of state parameter that violates the strong energy condition. Then the hot big bang singularity is located at negative conformal time and $\eta_{\mathrm{in}} \to -\infty$ for $a_{\mathrm{in}} \to 0$.*

Proof. Under these assumptions, $H^2 = H_0^2 a^{-3(1+w)}$ and the density parameter is constant $\Omega = 1$. Thus, $\mathscr{H}^{-1} = H_0^{-1} a^{(1+3w)/2}$. We can now directly compute the Hubble radius

$$d_{\mathrm{H}}(a) = \frac{2}{H_0(1+3w)} \left[\tilde{a}^{(1+3w)/2}\right]_{a_{\mathrm{in}}}^{a} = \eta - \eta_{\mathrm{in}}.$$

By assumption, $1 + 3w < 0$, the particle horizon is shrinking and

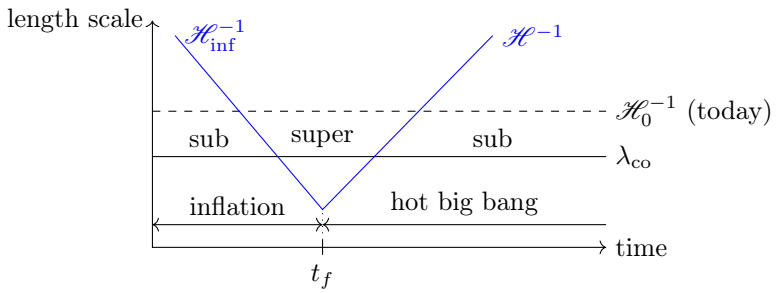
$$\lim_{a_{\mathrm{in}}\to 0} \eta_{\mathrm{in}} = \lim_{a_{\mathrm{in}}\to 0} \frac{-2}{H_0|1+3w|} a_{\mathrm{in}}^{(1+3w)/2} = -\infty.$$

□

For homework, one can show that for a sufficiently large duration of inflation, the past light-cones of any two events had enough conformal time to interact.

Claim 4.3. *Consider the Friedmann spacetime $M_{0,a}$, predominantly filled with an ideal fluid characterized by a constant equation of state parameter that violates strong energy condition. Then inflation allows for a solution of the Horizon problems within the domain of general relativity.*

Proof. The proof is by sketch, but it is possible to make it more precise.



□

Consider the manifold $M_{0,a} \simeq T \times S$, where $T = [t_{\text{in}}, t_{\text{f}}]$, filled with a single component fluid such that $1 + 2w < 0$. The number

$$N \coloneqq \ln\left(\frac{a_{\text{f}}}{a_{\text{in}}}\right)$$

is called the *number of e-folds*.

Claim 4.4. *During inflation $\dot{H} < H^2$, consider inflation epoch $[t_{\text{in}}, t_{\text{f}}]$ during which the Hubble radius is approximately constant. Then the amount by which the comoving Hubble radius decreases is given by the amount by which the scale factor increases. For such history, there are no Horizon problems, provided the number of e-folds during the inflation satisfies*

$$N > \ln\left(\frac{T_{\text{R}}}{T_0}\right) + \frac{1}{4}\ln\left(\frac{g_{*\text{r}}}{g_{*0}}\right),$$

where index R refers to the reheating (the reheating temperature is the highest temperature plasma has reached during hot big bang).

Proof. The absence of Horizon problems happens when

$$\mathscr{H}_{\text{in}}^{-1} > \mathscr{H}_0^{-1} = \frac{\mathscr{H}_{\text{R}}}{\mathscr{H}_0}\frac{1}{\mathscr{H}_{\text{R}}}.$$

Moreover, we have

$$\mathscr{H}_{\text{R}} = a_{\text{R}} H_{\text{R}} \approx a_{\text{f}} H_{\text{f}} \approx a_{\text{f}} H_{\text{in}}.$$

Hence,

$$\frac{a_{\text{f}}}{a_{\text{in}}} > \frac{\mathscr{H}_{\text{R}}}{\mathscr{H}_0}.$$

We assume radiation domination right after the inflation, so

$$\left(\frac{\mathscr{H}_{\text{R}}}{\mathscr{H}_0}\right)^2 = \left(\frac{a_{\text{R}}}{a_0}\right)^2 \Omega_{\text{rad}} \left(\frac{a_0}{a_{\text{R}}}\right)^4 = \left(\frac{\rho_{\text{R}}}{\rho_0}\right)^{1/2} = \left(\frac{g_{*\text{R}}}{g_{*0}}\right)^{1/2} \left(\frac{T_{\text{R}}}{T_0}\right)^2.$$

The assertion follows by inserting the definition of e-folds. □

Remark. It is convenient to insert a reference scale $T_* = 10^{15}\,\mathrm{GeV}$. Then the condition for e-folds is

$$N > \ln\left(\frac{T_*}{T_0}\right) + \ln\left(\frac{T_\mathrm{R}}{T_*}\right) + \frac{1}{4}\ln\left(\frac{g_{*\mathrm{R}}}{g_{*0}}\right) \approx 64 + \underbrace{\ln(T_\mathrm{r}/10^{15}\,\mathrm{GeV})}_{2-5} + \text{small}.$$

4.3 Slow-roll approximation

One of the simplest models of inflation is the *slow-roll approximation*. We define two dimensionless functions $\varepsilon, \kappa : T \to \mathbb{R}$ given by

$$\varepsilon := -\frac{\mathrm{d}}{\mathrm{d}N}\ln H, \quad \kappa := +\frac{\mathrm{d}}{\mathrm{d}N}\ln\varepsilon$$

where $\mathrm{d}N = \mathrm{d}\ln a = H\mathrm{d}t$. These are called the firs and second *slow-roll parameters*.

Claim 4.5. *An inflationary epoch in cosmic expansion history requires $\varepsilon < 1$.*

Proof. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}^{-1} = \frac{\mathrm{d}}{\mathrm{d}N}\frac{1}{a} - \frac{1}{aH}\frac{\mathrm{d}H}{\mathrm{d}N} = -\frac{1}{a}\left(1 + \frac{\mathrm{d}\ln H}{\mathrm{d}N}\right) = -\frac{1}{a}(1-\varepsilon),$$

from which the assertion follows. □

Claim 4.6. *A vanishing first slow-roll parameter corresponds to de Sitter spacetime.*

Proof. If $\varepsilon = 0$, then $H = \mathrm{const}$ and

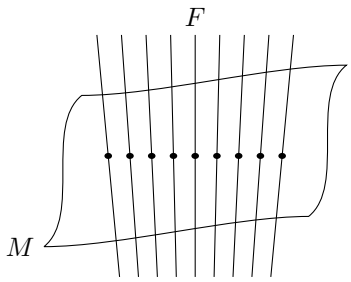
$$a(t) = a(t_\mathrm{in})\exp(H(t - t_\mathrm{in})),$$

which is the de Sitter solution. □

4.4 Sources of inflation

Since we have to agree on some language of classical field theory, we will recap the basics of classical field theory briefly.

Definition 4.1. A *configuration bundle* is a tuple $\mathcal{C} = (C, \pi, M, F)$, where M is the space-time manifold, F is the fibre (a vector space) over M, $C = M \times F$ and $\pi : C \to M$ is the projection map.



Definition 4.2. Let $\Gamma_p(\mathcal{C})$ be the set of all (local) sections of $\mathcal{C}$ on a neighbourhood of $p \in M$.

For any positive integer k, we define the following equivalent relations in $\Gamma_p(\mathcal{C})$:

$$\varphi_1 \sim_p^k \varphi$$

if for any curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$ and for any (smooth) function $f : C \to \mathbb{R}$ the compositions $f \circ \varphi_1 \circ \gamma$ and $f \circ \varphi_2 \circ \gamma$ have the same Taylor expansion at p up to and including the k-th order term. The equivalence class of φ is denoted by $j_p^k \varphi$. The quotient space of Γ is denoted by $J_p^k := \Gamma_p(\mathcal{C})/\sim_p^k$. The disjoint union of these quotients over all points p in the manifold is

$$J^k(\mathcal{C}) = \bigsqcup_{p \in M} J_p^k(\mathcal{C}).$$

Definition 4.3. A *Lagrangian* of order k is a bundle morphism

$$L : J^k(\mathcal{C}) \to A_{\dim(M)}(M),$$

where $A_m(M)$ is m-form bundle over M.

Sometimes, the bundles are named by their projection maps to the base manifold, $\pi : J^k(\mathcal{C}) \to M$ and $\tau_M : A_m(M) \to M$.

Let $\mathcal{D}$ be a compact submanifold with boundary $\partial\mathcal{D}$ and $\varphi : M \to \mathcal{C}$ is a configuration. We are interested in an m-form $L \circ j^k\varphi$ where $j^k\varphi : M \to J^k(\mathcal{C})$, $p \mapsto j_p^k\varphi$.

Definition 4.4. The *action* of the section φ over $\mathcal{D}$ is defined as

$$\mathcal{A}_\mathcal{D}(\varphi) = \int_\mathcal{D} L \circ j^k\varphi.$$

We equip $\mathcal{C}$ with fibred coordinates (x, y). A first order Lagrangian is locally expressed as

$$\begin{aligned} L : J^k(\mathcal{C}) &\to A_m(M) \\ (x, y, \partial_\mu y) &\mapsto \mathcal{L}(x, y, \partial_\mu y)\mathrm{d}\,\mathrm{vol}_g, \end{aligned}$$

where vol_g is a volume form on (M, g). But which field configurations can be qualified as physical? Consider a vertical (only pointing along the fibre) vector field V with $\operatorname{supp} V \subseteq \mathcal{D}$

with $j^{k-1}V(\partial D) = 0$. Such a vector field is called a *deformation*. The flow of V is $\mathcal{F}_s : \mathcal{C} \to \mathcal{C}$. The variation of the action

$$\mathcal{A}_{\mathcal{D}}(\varphi_s) = \int_{\mathcal{D}} L \circ j^k \varphi_s$$

is

$$\delta_V \mathcal{A}_{\mathcal{D}}(\varphi) \coloneqq \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{A}_{\mathcal{D}}(\varphi_s)\big|_{s=0},$$

where φ_s is defined as flow composed with the original section φ. The Hamilton principle states that the variation of the action for the physical field configurations vanishes,

$$\delta_V \mathcal{A}_{\mathcal{D}}(\varphi) = 0.$$

In general relativity, the action is defined as

$$L : (g^{-1}, \mathrm{Ric}) \mapsto \mathcal{L}_G(g^{-1}, \mathrm{Ric}) \mathrm{d}\,\mathrm{vol}_g$$

with Lagrange density

$$\mathcal{L}_G(g^{-1}, \mathrm{Ric}) = \mathcal{S}$$

where $\mathcal{S}$ is the Ricci scalar. The *Klein-Gordon Lagrangian* is

$$\mathcal{L}_{\mathrm{KG}} = -\left(\frac{1}{2} g^{-1}(\mathrm{d}\varphi, \mathrm{d}\varphi) + V(\varphi)\right).$$

The total Lagrange density is give by

$$\mathcal{L} = \mathcal{L}_G + \alpha_{\mathrm{KG}} \mathcal{L}_{\mathrm{KG}}$$

where $\alpha_{\mathrm{KG}} = 16\pi G$. The Hamilton principle gives

$$\mathrm{Ric} - \frac{1}{2}\mathcal{S} g = 8\pi G T_{\mathrm{KG}}, \quad \Box_g \varphi = V'(\varphi)$$

where $V'(\varphi) = \partial V / \partial \varphi$ and the energy-momentum tensor is given by

$$T_{\mathrm{KG}} = \mathrm{d}\varphi \otimes \mathrm{d}\varphi - \left(\frac{1}{2} g^{-1}(\mathrm{d}\varphi, \mathrm{d}\varphi) + V(\varphi)\right) g.$$

If we assume that the geometry respects the Friedmann isometries, then the field can only be a function of time, $\varphi = f(t)$. This gives the following equation of motion:

$$\ddot{\varphi} + 3H\dot{\varphi} = -V'(\varphi).$$

These equations describe the dynamics of an (almost) ideal Friedmann fluid with density and pressure

$$\rho_{\mathrm{KG}} = \frac{1}{2}\dot{\varphi}^2 + V(\varphi),$$

$$\mathcal{P}_{\mathrm{KG}} = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

and the energy-momentum tensor

$$T_{\mathrm{KG}} = \rho_{\mathrm{KG}} \mathcal{U}^\flat \otimes \mathcal{U}^\flat + \mathcal{P}_{\mathrm{KG}} H^\perp$$

with $\mathcal{U} = (1, \mathbf{0})$. We will drop $_{\text{KG}}$ index from now on as everything will refer to the Klein-Gordon field.

Claim 4.7. *The slow-roll parameter can be expressed in terms of the potential V and its derivatives:*

$$\varepsilon = \frac{(3/2)\dot{\varphi}^2}{\rho(\varphi)}.$$

Proof. We start by writing the Friedmann and Klein-Gordon equations

$$H^2 = \frac{8\pi}{3} G\rho(\varphi), \quad \ddot{\varphi} + 3H\dot{\varphi} = -V'(\varphi).$$

Using $\dot{H} = -4\pi G\dot{\varphi}^2$, the direct computation yields

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{(3/2)\dot{\varphi}^2}{(1/2)\dot{\varphi}^2 + V(\varphi)}.$$

□

When $\varepsilon \ll 1$, it is called the *slow-roll regime*. It is useful to introduce another parameter

$$\delta(\varphi) \coloneqq -\frac{\ddot{\varphi}}{H\dot{\varphi}}.$$

Claim 4.8. *The slow-roll parameters are related by*

$$\frac{\dot{\varepsilon}}{H\varepsilon} = \kappa = 2(\varepsilon - \delta).$$

Inflation continues as long as $\varepsilon \ll 1$ and $|\delta| \ll 1$ (show this for homework). This is exactly what we will assume now. Assume $\varepsilon \ll 1$ and $\delta \ll 1$. The system obeys the equations

$$H^2(\varphi) = \frac{8\pi}{3} GV(\varphi) + \mathcal{O}(\varepsilon),$$
$$3H\dot{\varphi} = -V'(\varphi) + \mathcal{O}(\delta).$$

This is called the *slow-roll approximation*.

Remark. The exact dependence of the equations of motion on the slow-roll parameters is given by

$$H^2(\varphi) = \frac{8\pi G}{3} V(\varphi) \frac{3}{3-\varepsilon}$$
$$3H\dot{\varphi} = V'(\varphi) \frac{3}{3-\delta}.$$

It is left as a homework to show that the slow-roll parameters are fully determined by the potential to leading order in the slow-roll parameters

$$\varepsilon \approx \frac{1}{2}\left(\frac{V'}{V}\right)^2, \quad \delta \approx \frac{V''}{V}.$$

4.5 Fluctuations

We want to add fluctuations to the picture. The main issue with them is that they disrespect Friedmann isometries. Consequently, also the spacetime is not a Friedmann spacetime any more. Let $\{g^{(\lambda)}\}_{\lambda\in\mathbb{R}}$ be a one-parameter family of metric tensors on M and $\mathrm{D}^{(\lambda)}$ the corresponding Levi-Civita connection. The difference

$$C := \mathrm{D}^{(\lambda_1)} - \mathrm{D}^{(\lambda_2)}$$

applied to one-form ω is an element of $T^1_2(M)$. In particular,

$$C(\omega, V, X) = ((\mathrm{D}^{(\lambda_1)}_V - \mathrm{D}^{(\lambda_2)}_V)\omega)(X).$$

In an abstract chart, it holds

$$(\mathrm{D}^{(\lambda_1)}_a - \mathrm{D}^{(\lambda_2)}_b)\omega_b = C^c_{ab}\omega_c.$$

4.6 Quantum theory of inflation

Consider

$$\mathscr{A}(\Phi, g) = \int_M \mathrm{d}\,\mathrm{vol}_g\ \mathcal{S} - \int_M \mathrm{d}\,\mathrm{vol}_g\ \left(-\frac{1}{2}g^{-1}(\mathrm{d}\Phi, \mathrm{d}\Phi) - V(\Phi)\right),$$

where Φ is a scalar field, $\mathcal{S}$ is the Ricci scalar, and vol_g a covariant volume form on (M, g). We could in principle also include higher order derivatives, however, it is then very difficult to get rid of ghosts (it can be done using infinitely many derivatives). There are two ways to approach the quantization of the theory. We can either use the classical background and quantize the fluctuations or we can (at least try to) quantize the whole theory — we will use the first approach. The field and the metric will be split into a background and a fluctuation part:

$$g = g^{(0)} + h, \qquad \Phi = \Phi^{(0)} + \varphi.$$

We will consider only so-called "scalar fluctuations",

$$\begin{aligned} h(\mathcal{U}, U) = h_{00}(\eta, x) &= -2a^2(\eta)\mathsf{A}(\eta, x) \\ h_{0i}(\eta, x) &= 2a^2(\eta)\partial_i\mathsf{B}(\eta, x) \\ h(X, Y) &= \langle X, Y\rangle_{\mathrm{E}} \end{aligned}$$

where A and B are fluctuations and η is the conformal time. We have chosen the spatially flat gauge, where η is the conformal time and $x \in \mathbb{R}^3$. The derivative with respect to conformal time is denoted by a prime (tick) and the derivative with respect to cosmological time is denoted by a dot. The background metric is

$$g^{(0)}(\eta) = -a^2(\eta)\mathrm{d}\eta \otimes \mathrm{d}\eta + a^2(\eta)\delta_{ij}\mathrm{d}x^i \otimes \mathrm{d}y^j.$$

It is a homework to show (or just look the notes from last semester's GR lecture) that in the first order perturbation theory we get

$$\begin{aligned} g^{00}(\eta, x) &= -a^{-2}(\eta)(1 - 2A(\eta, x)), \\ g^{0i}(\eta, x) &= a^{-2}(\eta)\partial_i B(\eta, x), \\ g^{ij}(\eta, x) &= a^{-2}(\eta)\delta^{ij}. \end{aligned}$$

The Hamilton's principle gives the equations of motion for the fluctuations:

$$\Box_g \Phi = V_{,\Phi}(\Phi).$$

Expanding this to the first order in fluctuations yields

$$\varphi'' + 2\mathscr{H}\varphi' - \Delta\varphi = (\mathsf{A}' + \Delta\mathsf{B})\Phi'^{(0)} - 2a^2 V_{,\Phi}(\Phi^{(0)})\mathsf{A} - a^2 V_{,\Phi,\Phi}(\Phi^{(0)})\varphi.$$

This is a horrible equation. However, we can simplify it by using the Einstein's equations for the perturbed fields. The idea is to eliminate A and B in favour of φ. Let's introduce a new notation:

$$\delta \operatorname{Ten} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \operatorname{Ten}(g^{(0)} + \lambda h).$$

In particular,

$$\delta G^0{}_i = -2a^{-2}(\eta)\mathscr{H}(\eta)\partial_i \mathsf{A}(\eta, x)$$

and

$$8\pi G \delta T^0{}_i = -8\pi G a^{-2}(\eta)\Phi'^{(0)}(\eta)\partial_i\varphi(\eta, x).$$

In words of the professor: "If you think about this, it is almost intuitive." This implies

$$\partial_i \mathsf{A} = 4\pi G \frac{\Phi'^{(0)}}{\mathscr{H}} \partial_i \varphi.$$

We choose

$$\mathsf{A}(\eta, x) = 4\pi G \frac{\Phi'^{(0)}}{\mathscr{H}} \varphi(\eta, x).$$

It can be more neatly expressed using the slow-roll parameter

$$\varepsilon = -\frac{\dot{H}}{H^2} = 4\pi G \left(\frac{\Phi'^{(0)}}{\mathscr{H}}\right)^2$$

which gives

$$A(\eta.x) = \varepsilon(\eta)\frac{\mathscr{H}}{\Phi'^{(0)}}(\eta)\varphi(\eta, x).$$

Doing the same steps, we can calculate $\delta G^0{}_0$ and $\delta T^0{}_0$ and get the expression for B:

$$\mathsf{B}(\eta, x) = -\varepsilon(\eta)\frac{\mathscr{H}(\eta)}{\Phi'^{(0)}(\eta)}\left(\varphi'(\eta, x) + (\delta - \varepsilon)(\eta)\mathscr{H}(\eta)\varphi(\eta, x)\right)$$

where

$$\delta = \frac{\ddot{\Phi}^{(0)}}{H\dot{\Phi}^{(0)}} = 1 - \frac{\Phi''^{(0)}}{\mathscr{H}\Phi'^{(0)}}.$$

The expression for B is only first order in perturbation theory, but exact in slow-roll parameters. Combining the expressions for A and B, we get the following expression for the field perturbation:

$$\varphi''(\eta, x) + 2\mathscr{H}(\eta)\varphi'(\eta, x) - \Delta\varphi(\eta, x) = \left((3 + 2\varepsilon - \delta)(\varepsilon - \delta)(\eta) - \frac{\delta'}{\mathscr{H}}(\eta)\right)\mathscr{H}^2(\eta)\varphi(\eta, x).$$

We want to transform this equation into its adjoint form, expressed with eigenfunctions of the Laplace operator. However, we have to be extermely careful to check that eigenfunctions actually form a complete basis. Before we do that, we will simplify the equation a bit. Using

$$\varphi'' + 2\mathscr{H}\varphi' = \frac{1}{a}(a\varphi)'' - \frac{a''}{a}\varphi$$

and introducing $f(\eta, x) = a(\eta)\varphi(\eta, x)$ we get

$$f''(\eta, x) - \left(\Delta + \frac{z''}{z}\right)\varphi(\eta, x) = 0$$

with $z(\eta) = a\Phi'^{(0)}/\mathcal{H}$. This is called the *Mukhanov-Sasaki equation* (the very same Vyacheslav Mukhanov that taught the GR course last semester). Writing the equation in the dynamical form, we get

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\begin{pmatrix} f \\ \pi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Delta - \frac{z''}{z} & 0 \end{pmatrix}\begin{pmatrix} f \\ \pi \end{pmatrix},$$

where $\pi = f'$ is the momentum conjugate to f. The Fourier modes of the fluctuations $f_k(\eta)$ (k being a continuous index) are

$$f_k''(\eta) + \omega_k^2(\eta) f_k(\eta) = 0$$

with $\omega_k^2(\eta) = \|k\|^2 - z''/z$. Approximately, we can write $z''/z \approx a''/a = 2\mathscr{H}^2$ and hence $\omega_k^2(\eta) \approx k^2 - 2\mathscr{H}^2$. There are two regimes:

1. $k^2 \gg \mathscr{H}^2$: sub-Hubble modes. The equation of motion is approximately $f_k''(\eta) + k^2 f_k(\eta) \approx 0$, which gives simple harmonic motion. Setting

$$f_k^{>}(\eta) = a_k^{>}(\eta_*)e^{-ik(\eta-\eta_*)}, \quad \text{positive energy solution,}$$

$$f_k^{<}(\eta) = a_k^{<}(\eta_*)e^{ik(\eta-\eta_*)}, \quad \text{positive energy solution,}$$

   the equation of motion gives

$$\begin{pmatrix} a_k^{>}(\eta_*) \\ a_k^{<}(\eta_*) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{i}{2k} \\ \frac{1}{2} & -\frac{i}{2k} \end{pmatrix}\begin{pmatrix} f_k^{>}(\eta_*) \\ f_k^{<}(\eta_*) \end{pmatrix}.$$

2. $k^2 \ll \mathscr{H}^2$: super-Hubble modes. The equation of motion ($z''/z = -2\mathscr{H}^2$)

$$''_k(\eta) + 2\mathscr{H}^2 f_k(\eta) \approx 0$$

   does not admit oscillatory solutions. Instead, we get two types of solutions: a growing solution

$$f_k^{\mathrm{grow}}(\eta) = C_k^{\mathrm{grow}}(\eta_*)\frac{z(\eta)}{z(\eta_*)}$$

   and a decaying solution

$$f_k^{\mathrm{decay}}(\eta) = C_k^{\mathrm{decay}}(\eta_*)\left(\frac{z(\eta_*)}{z(\eta)}\right)^2.$$

   In the spatially flat gauge, the comoving curvature preturbation is

$$R = \frac{\mathscr{H}}{\Phi'^{(0)}}\varphi(\eta, x) = \frac{1}{z(\eta)}f(\eta, x)$$

   and this is a gauge-invariant quantity.

4.6.1 Inflationary vacuum fluctuations

We will study the toy model of de Sitter inflation in the slow-roll approximation. As a reminder, we have $a(\eta) = -1/(H\eta)$ and $\mathscr{H}(\eta) = -1/\eta$. The Mukhanov-Sasaki parameters simplify to $z''/z \approx 2/\eta^2$ and the equation is

$$f_k^2(\eta) - (k^2 - 2/\eta^2) f_k(\eta) = 0.$$

The canonical quantization works by adding hats to the fields and promoting them to operators:

$$\hat{f}(f) = \hat{a}(f) + \hat{a}^\dagger(f)$$

where

$$\hat{a}(f) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \hat{a}_k f_k(\eta) e^{i\langle k,x\rangle}$$

subject to normalization condition

$$f_k \overleftrightarrow{\partial_\eta} f_k^*(\eta) = i$$

with $\overleftrightarrow{\partial_\eta} = \overrightarrow{\partial}_\eta - \overleftarrow{\partial}_\eta$. With this we assure

$$[\hat{a}_k, \hat{a}_{\tilde{k}}^\dagger] = (2\pi)^3 \delta^{(3)}(k - \tilde{k}).$$

The vacuum state (called the *Bunch-Davis vacuum*) is defined such that

$$\hat{a}_k(\eta_*) \, |0\rangle_{\mathrm{BD}} = 0$$

. For $k^2 \ll \mathscr{H}^2$, the Mukhanov-Sasaki equation becomes

$$f_k^{\mathrm{bd}}(\eta) + k^2 f_k^{\mathrm{bd}}(\eta) = 0 + \mathcal{O}(\mathscr{H}^2/k^2)$$

where $f_k^{\mathrm{bd}}(\eta) = 1/\sqrt{2k} e^{ik\eta}$ (the reader can derive the normalization constant for homework). Solving this equation with the above boundary condition gives

$$f_k(\eta) = \frac{1}{2k}\left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}.$$

This is called the *Bunch-Davies mode function*. Using the Segal field,

$${}_{\mathrm{BD}}\langle 0|\hat{f}(f)|0\rangle_{\mathrm{BD}}$$

(indicative but not yet conclusive for Gaussian statistics). The fluctuactions in the vacuum state (expected quadratic deviations from the mean value) are given by

$$\begin{aligned}
{}_{\mathrm{BD}}\langle 0|\{\hat{f}(f) - {}_{\mathrm{BD}}\langle 0|\hat{f}(f)\rangle \, |0_{\mathrm{BD}}\,\mathrm{id}\} \circ \{\hat{f}(f) - {}_{\mathrm{BD}}\langle 0|\hat{f}(f)\rangle \, |0_{\mathrm{BD}}\,\mathrm{id}\}|0\rangle_{\mathrm{BD}} &= \\
&= {}_{\mathrm{BD}}\langle 0|\hat{f}(f) \circ \hat{f}(f)|0\rangle_{\mathrm{BD}} \\
&= {}_{\mathrm{BD}}\langle 0|\hat{a}(f)\hat{a}^\dagger(f)|0\rangle_{\mathrm{BD}} \\
&= {}_{\mathrm{BD}}\langle 0|\hat{a}^\dagger(f)\hat{a}(f) + [\hat{a}(f), \hat{a}^\dagger(f)]|0\rangle_{\mathrm{BD}} \\
&= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} |f_k(\eta)|^2_{\mathbb{C}} \\
&= \int \mathrm{d}\ln k \, \frac{k^3}{2\pi^2} |f_k(\eta)|^2_{\mathbb{C}} \\
&= \int \mathrm{d}\ln k \Delta_f^2(k, \eta).
\end{aligned}$$

The variance of the quantum fluctuations is determined by the dimensionless power spectrum of the *classical* solution:

$$\Delta_f^2(k,\eta) \coloneqq \frac{k^3}{2\pi^2}|f_k(\eta)|^2_{\mathbb{C}},$$

$$\Delta_\varphi^2(k,\eta) = \left(\frac{H}{2\pi}\right)^2 \left[1+(k\eta)^2\right].$$

In the supper-Hubble limit, we have $k\eta \ll 1$ and the power spectrum becomes independent of k: $\Delta_\varphi = H/2\pi$. We want to relate Δ_φ^2 to observable fluctuations after inflation:

$$\Delta_R^2 = \left(\frac{H}{\Phi'(0)}\right)^2 \Delta_\varphi^2|_{k=aH} = \frac{1}{\pi}\frac{1}{\varepsilon}GH^2(t)|_{k=aH.}$$

The *scalar spectral index* is defined as

$$n_\mathrm{s} - 1 = \frac{\mathrm{d}}{\mathrm{d}\ln k}\ln\Delta_R^2(k) = 2\frac{\dot{H}}{H^2} - \frac{\dot{\varepsilon}}{H\varepsilon} = -2\varepsilon - \kappa.$$

Then $z = a\sqrt{\varepsilon}/\sqrt{4\pi G}$ and

$$\frac{z''}{z} = \mathscr{H}^2\left(2 - \varepsilon + \frac{3}{2}\kappa\right) + \dots$$

where $\mathscr{H} = -(1+\varepsilon)/\eta$. Solving the differential equation for z gives

$$\frac{z(\eta)}{z(\eta_*)} = \left(\frac{\eta}{\eta_*}\right)^{-1+\varepsilon+\kappa/2+\dots}$$

where we chose η_* such that $k_* = \mathscr{H}_*$ and $\eta_* = -1/k_*$. Then

$$\Delta_R^2 = \mathcal{A}_\mathrm{s}\cdot\left(\frac{k}{k_*}\right)^{n_\mathrm{s}-1}$$

with

$$\mathcal{A}_\mathrm{s} = \frac{1}{\pi}\frac{1}{\varepsilon_*}GH_*^2, \quad n_\mathrm{s} - 1 = -2\varepsilon_* - \kappa_*.$$

The CMB measurements give $n_\mathrm{s} = 0.9652 \pm 0.0042$, which is a statistically significant deviation from the scale-invariant spectrum $n_\mathrm{s} = 1$.

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