

Faculty of physics

Differential geometry

Notes of lectures by Prof. Bernhard Leeb, Ph.D.

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Disclaimer: these notes are not official and were not endorsed by professor. The content is almost entirely based on the lectures, but occasionally the notes were modified or extended slightly to improve clarity or add extra information. In particular, the mistakes are almost surely mine. If you find any, please let me know at simon.bukovsek@tum.de or Simon.Bukovsek@campus.lmu.de.

Overview of the course

In this course, we will meet the following topics:

- differential manifolds: objects that locally look like Euclidean space,
- Lie groups: we didn't discuss them due to lack of time,
- construction of important manifolds as homogenous space,
- de Rham cohomology,
- tangent and cotangent bundles, vector fields, flows, distributions (and foliations),
- higher dimensional generalization: bundles of differential forms, calculus of differential forms, exterior derivative,
- de Rham complex.

Chapter I

Differentiable manifolds

Differentiable manifolds are geometric objects. There are two layers of structure:

- underlying topological space;
- "on top of it" a differentiable structure, which allows us to do calculus.

I.1 Topological manifolds

The essential property of topological manifolds is that they are locally parametrizable by finitely many independent coordinates. The number of coordinates is called *dimension*. In other words, an n-dimensional manifold locally looks like $\mathbb{R}^n$ (which is called a local model for n-dimensional manifolds). Intuitively, we can imagine a manifold as being "sawn" or "patched" together from "patches" of $\mathbb{R}^n$.

Definition I.1. A topological space is called n-dimensional *locally Euclidean* if it is locally homeomorphic to $\mathbb{R}^n$. That means that every point has an open neighbourhood homeomorphic to (an open subset of) $\mathbb{R}^n$ (see Fig. I.1a).

Note that n-balls are homeomorphic to $\mathbb{R}^n$. In order to avoid pathologies or strange phenomena and to allow certain constructions, we define:

Definition I.2. An n-dimensional *topological manifold* is an n-dimensional Euclidean Hausdorff space that is second countable.

Note. A topology is *first countable*, if and only if every point has a countable neighbourhood basis. A topology is *second countable* if the whole topology has a countable basis. In $\mathbb{R}^n$ under standard topology, a neighbourhood basis for a point are n-balls with radii $1/n$ for $n \in \mathbb{N}$. A basis for the whole $\mathbb{R}^n$ topology are such balls for every point with rational coordinates. Such basis is countable, so $\mathbb{R}^n$ is second countable in its standard topology. More generally, metric spaces are second countable if and only if they are *separable* (contain a dense countable subset).

The *Hausdorff property* (also called T_2) is a separation property. It means that every two points admit disjoint neighbourhoods. It implies limits of sequences and is necessary for the

existence of metric structures (for metrics that induce a given topology).

Example 1. A prime example of a non-Hausdorff space is a line with two origins. Take the set of two lines $\mathbb{R} \times \{0, 1\}$ and join them with the equivalence relation $(x, 0) \sim (x, 1)\ \forall x \in \mathbb{R} - \{0\}$.

———————:———————

The space $\mathbb{R} \times \{0, 1\}/\sim$ is a 1-dimensional locally Euclidean topological space, but the origins cannot be separated by open sets. Any sequence with a limit in one origin must also have a limit in the other origin.

Remark. Non-Hausdorff locally Euclidean spaces do occur naturally as quotient spaces, e.g. as orbit spaces of flows (1-parameter groups of transformations) or more generally as leaf spaces of foliations.

Example 2. The map $\mathbb{R} \to \mathbb{R}^2 - \{0\}$; $t \mapsto (e^t, e^{-t})$ is a non-Hausdorff orbit space (some sort of cross with four origins).

Second countability is a size restriction:

- for locally Euclidean spaces it is equivalent to the existence of cover by open topological balls (that is open subsets homeomorphic to an open ball in $\mathbb{R}^n$);
- it implies the existence of partitions of unity subordinate to open covers. This is a basic tool for local-to-global glueing constructions, i.e. for building global objects by patching together local ones.

Locally Euclidean spaces without a countable basis for their topology, at least for connected ones, are less natural. Nevertheless, they do exist.

Example 3. Let us construct one such space, called a long half-line. It is a locally 1-dimensional Euclidean path connected and contains an uncountable family of disjoint open intervals. It is constructed by a transfinite inductive process of gluing intervals.

The sketch of the construction is as follows. Take $W = \{w_1, \dots\}$ a minimal uncountable well-ordered set (unique up to order preserving bijections). Consider $X = W \times [0, 1] - \{(w_1, 0)\}$ with lexicographical order $\prec_X$: $(w, t) \prec_X (w', t') \iff w <_W w'$ or $w = w'$ and $t < t'$. This ordering creates the *order topology*. The minimal element in the set W is necessary for the path-connectedness.

Remark. The dimension of locally Euclidean space is well-defined. It is based on a deep result from algebraic topology (using homology) that dimension is a topological invariant of open subsets of Euclidean spaces in the following sense: Euclidean spaces of different dimensions are not locally homeomorphic, i.e. non-empty open subsets of Euclidean spaces of different dimensions are not homeomorphic. For example, dimension 1 is characterized topologically by the fact that removing a point locally disconnects the space. In dimension 2 removing a point locally produces non-contractible embedded topological 1-spheres.

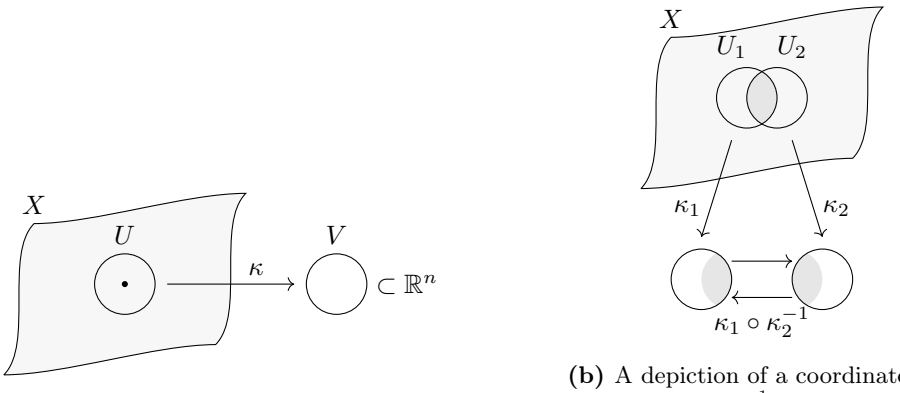
I.2 Differentiable structures

On a locally Euclidean space, one can put various kinds of additional geometric structures. The example mainly relevant for us is a differentiable structure. It is needed in order to do differential calculus on our space, starting with being able to define a notion of differentiability. Additional geometric structure on locally Euclidean space comes in the form of *atlas*.

Let X be an n-dimensional locally Euclidean space.

Definition I.3.

1. A local *chart* or *coordinate map* (U, κ) for X is a map
$$X \supset U_{\text{open}} \xrightarrow[\text{homeo}]{\kappa} V_{\text{open}} \subset \mathbb{R}^n.$$
In this case, $\mathbb{R}^n$ is called a local model (see Fig. I.1a).
2. The *coordinate change* or *transition map* from one chart (U_1, κ_1) to another chart (U_2, κ_2) is the homeomorphism of (possibly empty) open subsets of $\mathbb{R}^n$ (see Fig. I.1b):
$$\kappa_1(U_1 \cap U_2) \xrightarrow{\kappa_2 \circ \kappa_1^{-1}} \kappa_2(U_1 \cap U_2).$$
3. An *atlas* is a family of charts (U_ι, κ_ι) for $\iota \in I$, which cover X, i.e. $\bigcup_{\iota \in I} U_\iota = X$. The set I is called the index set.



(a) A depiction of a chart (U, κ).

(b) A depiction of a coordinate transform $\kappa_1 \circ \kappa_2^{-1}$.

Figure I.1: Charts on a topological space X.

The *full atlas* $\mathcal{A}_{\text{top}}(X)$, consisting of all charts, defines no additional structure on X; it just represents the topology itself. The additional structure arises when one imposes restrictions on the coordinate changes of an atlas. The kind of structure one obtains depends on the nature of the restrictions. We now introduce the type of additional structure most relevant to us, a differentiable structure.

Definition I.4.

1. Two charts are called $\mathcal{C}^n$-compatible ($1 \leq k \leq \infty$) if the coordinate changes between them are $\mathcal{C}^n$-diferentiable. Because they also have a $\mathcal{C}^n$-differentiable inverse, they are $\mathcal{C}^n$-diffeomorphisms.
2. An atlas is called $\mathcal{C}^n$-differentiable or a $\mathcal{C}^n$-atlas if any two of its charts are $\mathcal{C}^n$-diferentiable.

Note. Every $\mathcal{C}^n$-atlas is contained in a unique maximal $\mathcal{C}^n$-atlas, which is generated by adding all $\mathcal{C}^n$-compatible charts (with all charts in the starting atlas). The graphical depiction is in Fig. I.2.

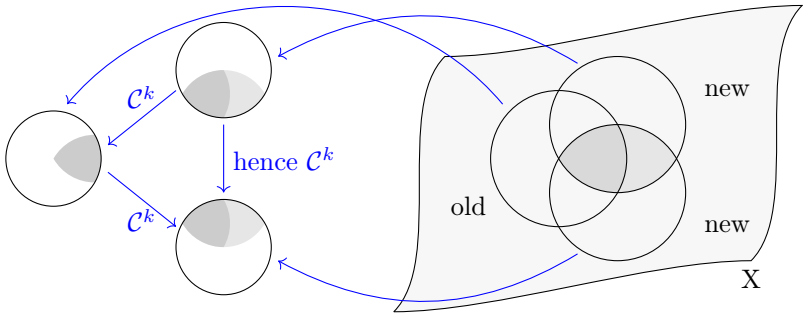


Figure I.2: From existing atlas we can create a maximal atlas $\mathcal{A}_{\text{top}}$ by adding new charts that have $\mathcal{C}^k$ coordinate changes with the existing charts. This procedure ensures that the coordinate changes between the new charts will also be $\mathcal{C}^k$.

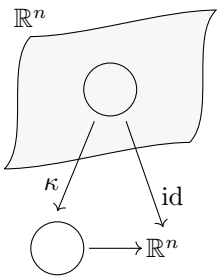
Definition I.5.
1. A $\mathcal{C}^k$-differentiable structure on an n-dimensional locally Euclidean space (not necessarily Hausdorff or second countable) is a *maximal* $\mathcal{C}^k$-atlas whose charts have values in $\mathbb{R}^n$.
2. An n-dimensional $\mathcal{C}^k$-*differentiable manifold* is an n-dimensional topological manifold together with a $\mathcal{C}^k$-differentiable structure.

Remark. For $1 \leq l \leq k \leq \infty$, a $\mathcal{C}^k$-differentiable structure can be weakened to a $\mathcal{C}^l$-differentiable structure by relaxing the restriction on the coordinate changes from $\mathcal{C}^k$ to $\mathcal{C}^l$:

$$\text{larger atlas} \quad \Longleftrightarrow \quad \text{weaker structure.}$$

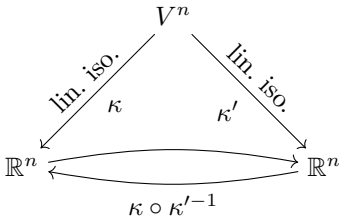
First examples of differentiable manifolds:

1. $\mathbb{R}^n$ is an n-dimensional $\mathcal{C}^\infty$-manifold. It carries a natural structure, which means that is generated by the standard topology and standard $\mathcal{C}^\infty$-structure. The latter is generated by the chart $\text{id}_{\mathbb{R}^n}$ and consists of all $\mathcal{C}^\infty$ diffeomorphisms $U \xrightarrow{\cong} V$ of open subsets of $\mathbb{R}^n$. A weaker standard $\mathcal{C}^k$-differentiable structure ($k \in \mathbb{N}$) consists of all $\mathcal{C}^k$-diffeomorphisms.



2. Finite dimensional $\mathbb{R}$-vector spaces are $\mathcal{C}^\infty$-manifolds. They are equipped by the standard topology, which is induced by any norm (in *finite* dimensional vector spaces all norms are equivalent). The standard smooth structure is generated by a $\mathcal{C}^\infty$-atlas, which consists of all linear isomorphisms to the model $\mathbb{R}^n$ of the same dimension (as a vector space).

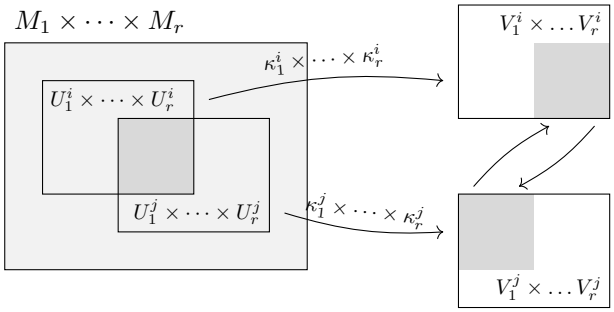
Linear isomorphisms are homeomorphisms and the coordinate change $\kappa \circ \kappa'^{-}1$ is $\mathcal{C}^\infty$-diffeomorphism.



3. Open subsets $O \subset M$ of $\mathcal{C}^k$-manifolds are $\mathcal{C}^k$-manifolds. The inherited topology is a *subset topology*, which consists of open subsets of M contained in O. The induced (restricted) $\mathcal{C}^k$-differentiable structure consists of charts with domains in the subset O. If $\mathcal{A}$ is a $\mathcal{C}^k$-structure on M, then the restricted $\mathcal{C}^k$-structure on O is $\mathcal{A}\big|_O = \{(U,\kappa) \in \mathcal{A} | U \subset O\}$.

4. Countable disjoint unions of n-dimensional $\mathcal{C}^k$-manifolds are n-dimensional $\mathcal{C}^k$-manifolds. The *sum topology* is generated by the union of the topologies of component manifolds. Induced $\mathcal{C}^k$-structure is generated by the union of $\mathcal{C}^k$-structures of the component manifolds.

5. Finite products of $M_1 \times M_2 \times \cdots \times M_r$ of $\mathcal{C}^k$-manifolds are $\mathcal{C}^k$-manifolds. They are equipped with *product topology* generated by open "rectangles" (products of open subsets) $O_1 \times \cdots \times O_r \subset M_1 \times \ldots M_r$. The induced $\mathcal{C}^k$-structure is generated by products of charts in the $\mathcal{C}^k$-structures of the factors. The new coordinate changes are the products of coordinate changes on the factors.

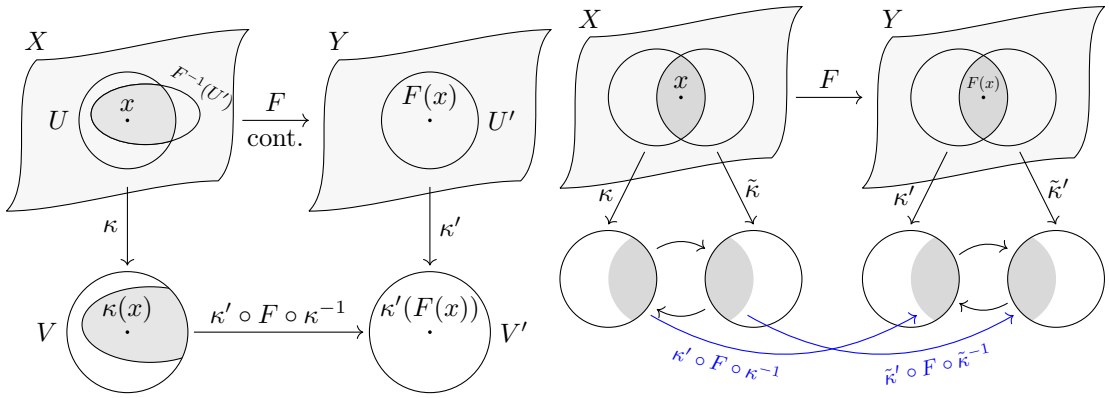


The differentiable structure can be used (is designed to) carry over the notion of differentiability from Euclidean spaces to manifolds (more generally to locally Euclidean spaces). The differentiability of maps is defined via local coordinate representations and, due to restrictions imposed on coordinate changes, *independent* of the chosen chart.

Consider a continuous map $X^n \xrightarrow{F} Y^m$ of locally Euclidean spaces, equipped with $\mathcal{C}^k$-structures. Let (U,κ) be a chart for X near the target $x \in X$ and (U',κ') be a chart for Y near the target $F(x) \in Y$ (see Fig. I.3a). Then

$$V \supset \kappa(U \cap F^{-1}(U')) \xrightarrow{\kappa' \circ F \circ \kappa^{-1}} V'$$

is a local coordinate representation for F near x (see Fig. I.3a). Note that $F^{-1}(U')$ is an open set, as well as $\kappa(F^{-1}(U'))$. The local coordinate changes are also $\mathcal{C}^k$-diffeomorphisms, since near



(a) Local coordinate representation of a map $F : X \to Y$ near $x \in X$.

(b) Coordinate change of a map $F : X \to Y$ near $x \in X$.

Figure I.3: Construction of coordinate change and local coordinate representation of a continuous map.

$\tilde{\kappa}(x)$ we can write (see Fig. I.3b)

$$\tilde{\kappa}' \circ F \circ \tilde{\kappa}^{-1} = \underbrace{(\tilde{\kappa}' \circ \kappa'^{-1})}_{\text{coo. change}} \circ (\kappa' \circ F \circ \kappa^{-1}) \circ \underbrace{(\kappa \circ \tilde{\kappa}^{-1})}_{\text{coo. change}}.$$

It, therefore, makes sense to define the following.

Definition I.6. A map F is said to be *differentiable* in $x \in X$ if some (and hence any) local coordinate representation of F near x has this property (i.e. being differentiable). The same holds if we change "differentiable" with l times differentiable or $\mathcal{C}^l$ for $1 \leq l \leq k$.

Remark.

1. More than k-fold differentiability cannot be defined using a $\mathcal{C}^k$-atlas.
2. The charts of a $\mathcal{C}^k$-structure are $\mathcal{C}^k$-diffeomorphisms.

$$\begin{array}{ccc} U & \xrightarrow{\kappa \text{ as a map}} & V \\ \kappa \text{ as a chart} \downarrow & & \downarrow \mathrm{id}_V \text{ as a chart} \\ & \xrightarrow[\text{local coo. rep. } \mathcal{C}^k]{} & \end{array}$$

3. Compositions of $\mathcal{C}^k$-maps are $\mathcal{C}^k$-maps.
4. In the special case of vector valued functions ($Y = \mathbb{R}^m$) we can use the global chart $\mathrm{id}_{\mathbb{R}^m}$ for target $\mathbb{R}^m$. $X \xrightarrow[\text{continuous}]{F} \mathbb{R}^n$ gives a local coordinate representation $F \circ \kappa^{-1}$ for charts κ on the domain.

We next describe a useful method of establishing locally Euclidean and differentiable structure on a set "at once". Take a set X and integers $n \geq 0$ and $k \geq 1$. Given a family (future atlas) of

maps (future charts)

$$X \supset U_\iota \xrightarrow[\text{bijective}]{\kappa_\iota} V_\iota \subset \mathbb{R}^n$$

such that $X = \bigcup_{\iota \in I} U_\iota$ and maps $\kappa_{\iota'} \circ \kappa_\iota^{-1}$ (future coordinate changes) defined on open subsets of V_ι and $\mathcal{C}^k$ for $\iota', \iota \in I$. Note that V_ι are open sets. Then a topology is induced on X by defining $O \subset X$ open if and only if $\kappa_\iota(O \cap U_\iota) \subset V_\iota$ is open for every $\iota \in I$. This is clearly a topology. But is it a good one (i.e. does it have enough subsets)? For a specific $\iota \in I$ and subsets $O \subset U_\iota$ it holds that O is open if and only if $\kappa_\iota(O) \subset V_\iota$ is open. The right direction (necessity) is satisfied by the definition of "open in X". But we have to verify the left direction (sufficiency). For $\iota' \in I$ we have

$$\kappa_{\iota'}(O \cap U_{\iota'}) = (\kappa_{\iota'} \circ \kappa_\iota^{-1})(\kappa_\iota(O \cap U_\iota)) = (\kappa_{\iota'} \circ \kappa_\iota^{-1})(\kappa_\iota(O) \cap \kappa_\iota(U_\iota \cap U_{\iota'}))$$

open (see Fig. I.4). Note that $O \cap U_{\iota'} = O \cap (U_\iota \cap U_{\iota'}) \subset U_\iota \cap U_{\iota'}$. Then $\kappa_\iota(O)$ is open by assumption and $\kappa_\iota(U_\iota \cap U_{\iota'})$ is open because it is the domain of $\kappa_{\iota'} \circ \kappa_\iota^{-1}$. This means that charts

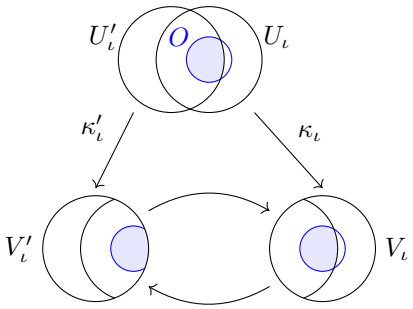


Figure I.4: The mapping of an open set O under future charts and coordinate changes.

are homeomorphisms. This topology makes X n-dimensional locally Euclidean. The (U_ι, κ_ι) become charts and form a $\mathcal{C}^k$-atlas, and we obtain a $\mathcal{C}^k$-structure on X. The Hausdorff and second countability properties need to be checked in the situation at hand.

Remark. One can assume more generally that the V_ι are n-dimensional locally Euclidean spaces equipped with $\mathcal{C}^k$-structure

Example 4. In this example, we will construct a *real projective space* as an "abstract" manifold:

$$\mathbb{R}\mathrm{P}^n := \{\text{1-dimensional linear subspaces } L \subseteq \mathbb{R}^{n+1}\}.$$

Points in the projective space are denoted as $[x] = [x_0 : \cdots : x_n]$. They are called homogenous coordinates and are unique up to scaling: $\mathbb{R}x$ for $x \in \mathbb{R}^{n+1} - \{0\}$. Now define the following affine charts:

$$U_i := \underbrace{\{[x] \in \mathbb{R}\mathrm{P}^n | x_i \neq 0\}}_{\text{"affine patch"}} \xrightarrow[\text{scaling}]{\text{bij}} \underbrace{\{x \in \mathbb{R}^{n+1} | x_i = 1\}}_{\text{affine hyperplane}} \xrightarrow[\cong]{\text{bij}} \mathbb{R}^n.$$

The components are transformed as

$$[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right) \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right).$$

In the last mapping, we just leave out the coordinate x_i/x_i. The chart is denoted by $\kappa_i : U_i \to \mathbb{R}^n$. Coordinate changes $\kappa_j \circ \kappa_j^{-1}$ are defined on the complement of a hyperplane, which is open and

$\mathcal{C}^\infty$, e.g.

$$\{y \in \mathbb{R}^n | y_1 \neq 0\} \xrightarrow{\kappa_0^{-1}} U_0 \cap U_1 \xrightarrow{\kappa_1} \{z \in \mathbb{R}^n | z_1 \neq 0\},$$
$$(y_1, \ldots, y_m) \mapsto [1 : y_1 : \cdots : y_n] = \left[\frac{1}{y_1} : 1 : \cdots : \frac{y_n}{y_1}\right] \mapsto \left(\frac{1}{y_1}, \frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1}\right).$$

In order to check that $\mathbb{R}\mathrm{P}^n$ is a manifold, we have to check that we have n-dimensional locally Euclidean topology and a $\mathcal{C}^\infty$-structure. We will next show that $\mathbb{R}\mathrm{P}^n$ is a compact manifold.

- First, prove that it is Hausdorff. Consider the subsets

  $$O_{i,j} \coloneqq \{[x] \in \mathbb{R}\mathrm{P}^n | \, |x_i| < c|x_j|\}$$

  with $0 \leq i, j \leq n$, $c > 0$ and i being distinct from j. The $<$ sign in the definition of the set is scale invariant. Sets $O_{i,j}$ are open since intersections with affine charts U_l are open. If we switch the coordinates i and j, we get $O_{i,j} \cap O_{j,i} = \varnothing$. These pairs of disjoint open subsets suffice to separate pairs of points.

- Second, prove that it is compact. There exists a cover by finitely many compacts contained in affine charts. Let's look at sets

  $$B_i \coloneqq \{[x] \in \mathbb{R}\mathrm{P}^n | \, |x_j| \leq |x_i| \, \forall j\} \subset U_i \xrightarrow[\kappa_i]{\cong} \mathbb{R}^n \supset \{y \in \mathbb{R}^n | \, |y_{j'}| \leq 1 \, \forall j'\} = [-1, 1]^n.$$

  We know that unit cubes are compact, hence $\mathbb{R}\mathrm{P}^n = \bigcup_{i=0}^n B_i$ is compact.

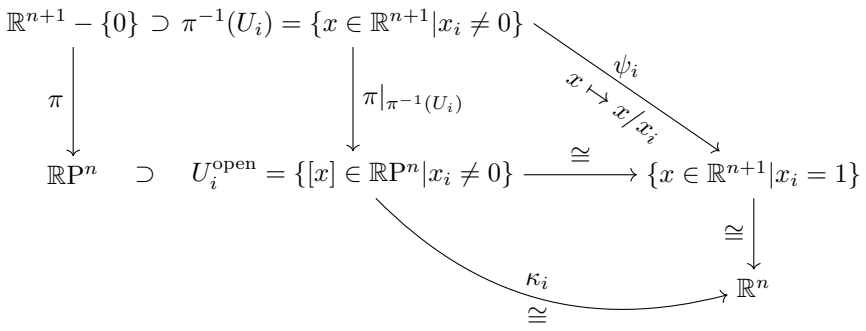
- Lastly, we should check that our space is second countable. That is trivial, as there exists a cover by finitely many charts.

Hence, $\mathbb{R}\mathrm{P}^n$ is a compact smooth n-dimensional manifold. Since $\mathbb{R}\mathrm{P}^n$ is constructed as a quotient, namely as the orbit space $\mathbb{R}\mathrm{P}^n \coloneqq (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^*$ of the action $\mathbb{R}^* \to \mathbb{R}^{n+1} - \{0\}$ by scalar multiplication, we wonder whether the topology constructed by us coincides with the quotient topology. That is indeed the case.

Claim I.1. *The canonical projection (quotient map)* $\mathbb{R}^{n+1} - \{0\} \xrightarrow{\pi} \mathbb{R}\mathrm{P}^n$, $x \mapsto [x]$
1. *is open, and*
2. *the topology on* $\mathbb{R}\mathrm{P}^n$ *constructed above coincides with the quotient topology with respect to* π.

Beware that quotient projections $Z \to Z/\sim$ are, in general, not open (examples are abundant: e.g. a disjoint union of a point and a line with a projection of the point on the plane).

Proof. It suffices to verify the claim "chart by chart" due to the following consideration



Then:

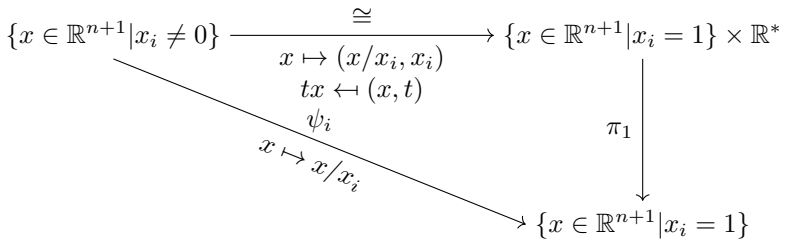
1. If ψ_i is open for all i then π is open. This holds because if $W \subset \mathbb{R}^{n+1} - \{0\}$ is open then

$$\pi(W) = \bigcup_i (\pi(W) \cap U_i) = \bigcup_i \pi|_{\pi^{-1}(U_i)}(W \cap \pi^{-1}(U_i)),$$

   where W and $\pi^{-1}(U_i)$ are both open. Lines not continued in $x_i = 0$ intersecting W.

2. If $\{x \in \mathbb{R}^{n+1} | x_i = 1\} \cong \mathbb{R}^n$ carries quotient topology with respect to ψ_i then $\mathbb{R}\mathrm{P}^n$ carries quotient topology with respect to π. For $\overline{W} \subset \mathbb{R}\mathrm{P}^n$ it holds that: $\overline{W}$ is open in $\mathbb{R}\mathrm{P}^n$ if and only if $\overline{W} \cap U_i$ is open in U_i for all i. Moreover, $\pi^{-1}(\overline{W})$ is open in $\mathbb{R}^{n-1} - \{0\}$ if and only if $\pi^{-1}(\overline{W}) \cap \pi^{-1}(U_i) = (\pi|_{\pi^{-1}(U_i)})^{-1}(\overline{W} \cap U_i)$ is open in $\pi^{-1}(U_i)$ for all i.

To check the claim for ψ_i, consider



Since the claim holds for projections from a product to their factors (here π_1), it carries over to ψ_i. □

Many other sorts of geometric structures of different flavours can be defined this way i.e. by specifying a local model and the admissible coordinate changes (transition maps) between open subsets of the local model. This is called the Thurston point of view. Strengthening the restrictions on the coordinate changes would produce a structure that is more rigid ("stiff"). Relaxing the restrictions would produce more flexible ("flabby") structures.

Example 5.

- Bilipschitz structures: local model $\mathbb{R}^n$ with bilipschitz coordinate changes is more flexible than $\mathcal{C}^1$.
- Real-analytic structures: local model $\mathbb{R}^n$ with real-analytic coordinate changes is more rigid than $\mathcal{C}^\infty$.

- Complex structures: local model $\mathbb{C}^n$ with holomorphic coordinate changes is more rigid than $\mathcal{C}^\infty$.
- Oriented differentiable structures: local model $\mathbb{R}^n$ with orientation preserving coordinate changes.
- Differentiable structures with boundary: local model is no longer $\mathbb{R}^n$ but a closed half-space.
- Foliations (more on that later).

I.3 Differentiable submanifolds behaviour of generic differential maps

Differentiable manifolds occur naturally as

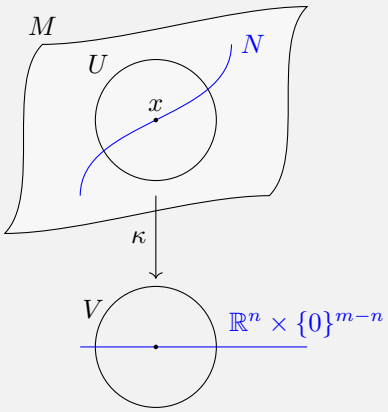
- *embedded* objects: motivating examples are differentiable submanifolds of Euclidean spaces, e.g. solutions sets of "nice" (systems) equations or, equivalently, level sets of "nice" functions. What is nice is determined by the implicit function theorem. This is due to the fact that "generic" differentiable maps have "simple" local behaviour.
- *Abstract* manifolds (non-embedded): e.g. moduli spaces or, more precisely, configuration spaces (Grassmannians $\mathrm{Gr}(n,k)$ and projective spaces $\mathrm{P}^n = \mathrm{Gr}(n+1,1)$, would be an example).

Submanifolds are subsets of manifolds which locally look like affine subspaces of Euclidean spaces.

Definition I.7. Let M^m be an m-dimensional $\mathcal{C}^k$-differentiable manifold and $n \leq m$. A subset $N \subset M$ is called an n-dimensional $\mathcal{C}^k$-*submanifold* of M if for every $x \in N$ exists a $\mathcal{C}^k$-differentiable chart for M near x:

$$M \supset U^{\text{open}} \xrightarrow[\mathcal{C}^k\text{-diffeo}]{\kappa} V^{\text{open}} \subset \mathbb{R}^m$$

such that $\kappa(N \cap U) = (\mathbb{R}^n \times \{0_{\mathbb{R}^{m-n}}\}) \cap V$. We would call this an *extrinsic* (local) chart for submanifold N near x. The expression $(\mathbb{R}^n \times \{0_{\mathbb{R}^{m-n}}\})$ is the coordinate subspace of $\mathbb{R}^n$.



One refers to submanifolds as defined above also as *embedded* submanifolds in order to distinguish them from more general *immersed* submanifolds, which will be discussed later (an example

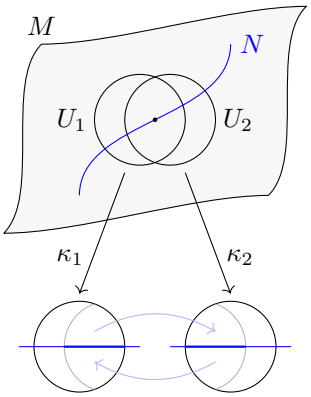
would be a helix densely coiled around a torus).

Differentiable submanifolds are a source of examples for "abstract" manifolds since they inherit structures as differentiable manifolds from the ambient manifolds they sit in. Firstly, it is clear from the definition that n-dimensional $\mathcal{C}^k$-manifolds are, equipped with the relative topology, n-dimensional topological manifolds. As for their induced differentiable structure, we note that extrinsic submanifold charts restrict to intrinsic manifold charts:

$$\begin{array}{ccccccc} M & \supset & U^{\text{open}} & \xrightarrow[\mathcal{C}^k\text{-diffeo}]{\kappa} & V^{\text{open}} & \subset & \mathbb{R}^m \\ \cup & & \cup & & \cup & & \cup \\ N & \supset & \underset{\text{(rel) open}}{U \cap N} & \xrightarrow[\text{homeo}]{\kappa|_{N\cap U}} & \underset{\text{(rel) open}}{V \cap (\mathbb{R}^n \times \{0\})} & \subset & \mathbb{R}^n \times \{0\} \supset \mathbb{R}^n \end{array}$$

The map $\kappa|_{N\cap U}$ is called *intrinsic* local chart for N near x. In this way, the $\mathcal{C}^k$-structure on M includes on N an atlas of $\mathbb{R}^n$ valued charts. It is a $\mathcal{C}^k$-atlas because the intrinsic coordinate changes are restrictions of the extrinsic ones to coordinate changes of Euclidean spaces, and the extrinsic coordinate changes are $\mathcal{C}^k$.

$$\begin{array}{ccccccc} \mathbb{R}^m & \supset & \kappa_1(U_1 \cap U_2)^{\text{open}} & \xrightarrow[\mathcal{C}^k]{\kappa_2 \circ \kappa_1^{-1}} & \kappa_2(U_1 \cap U_2)^{\text{open}} & \subset & \mathbb{R}^m \\ \cup & & \cup & & \cup & & \cup \\ \mathbb{R}^n \times \{0\} & \supset & \underset{\text{(rel) open}}{\kappa_1(N \cap U_1 \cap U_2)} & \xrightarrow[\text{hence } \mathcal{C}^k]{\kappa_2|_{N\cap U_2} \circ (\kappa_1|_{N\cap U_1})^{-1}} & \underset{\text{(rel) open}}{\kappa_2(N \cap U_1 \cap U_2)} & \subset & \mathbb{R}^n \times \{0\} \end{array}$$



The map $\kappa_2|_{N\cap U_2} \circ (\kappa_1|_{N\cap U_1})^{-1}$ is $\mathcal{C}^k$ since locally the restriction of a $\mathcal{C}^k$-map to a coordinate subspace. The induced $\mathcal{C}^k$ atlas on N generates its induced structure as an n-dimensional $\mathcal{C}^k$-manifold. In this sense, submanifolds are manifolds.

I.3.1 Local behaviour of differentiable maps

We will make precise in what sense "generic" differentiable maps have "simple" local behaviour or normal forms, which then will imply that their level sets and, locally, their images are submanifolds. We will formulate genericity assumptions in terms of the *rank of the differential*. However, since we still postpone the discussion of the tangent bundle (of differential manifolds) and the differential (of differentiable maps), we will define its rank by resorting to local coordinates. To understand why this leads to a well-defined quantity, let us look at the differentials of the local

coordinate representations (these are already defined) and the effect of coordinate changes.

Consider a map of $\mathcal{C}^k$-manifolds $M^m \xrightarrow{F} N^n$ which is continuous everywhere and moreover differentiable in a point $x \in M$ (see Fig. I.5).

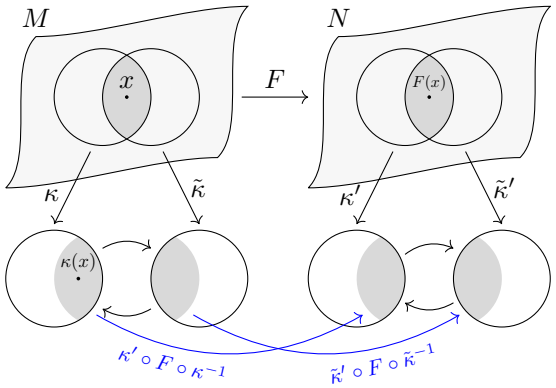


Figure I.5: Local coordinate representation and coordinate change of a map $F : M \to N$ near $x \in M$.

We have:

$$\tilde{\kappa}' \circ F \circ \tilde{\kappa}^{-1} = (\tilde{\kappa}' \circ \kappa'^{-1}) \circ (\kappa' \circ F \circ \kappa^{-1}) \circ (\kappa \circ \tilde{\kappa}^{-1}).$$

The left term is the local coordinate representation, differentiable in $\tilde{\kappa}(x)$, the first term on the right side is coordinate change $\mathcal{C}^k$-diffeomorphism, the middle term is the local coordinate representation, differentiable in $\kappa(x)$ and the right term is also a coordinate change $\mathcal{C}^k$-diffeomorphism. With the chain rule, we get

$$\mathrm{d}(\tilde{\kappa}' \circ F \circ \tilde{\kappa}^{-1})_{\tilde{\kappa}(x)} = \mathrm{d}(\tilde{\kappa}' \circ \kappa'^{-1})_{\kappa'(F(x))} \circ \mathrm{d}(\kappa' \circ F \circ \kappa^{-1})_{\kappa(x)} \circ \mathrm{d}(\kappa \circ \tilde{\kappa}^{-1})_{\tilde{\kappa}(x)}$$

and

$$\operatorname{rank} \mathrm{d}(\tilde{\kappa}' \circ F \circ \tilde{\kappa}^{-1})_{\tilde{\kappa}(x)} = \operatorname{rank} \mathrm{d}(\kappa' \circ F \circ \kappa^{-1})_{\kappa(x)}$$

since both coordinate changes are $\mathcal{C}^k$-diffeomorphisms and their differentials are invertible.

We thus define the *rank* of F in x as the rank of the differential of some (any) of its local coordinate representations near x:

$$\operatorname{rank}(F, x) := \operatorname{rank} \mathrm{d}F_x := \operatorname{rank} \mathrm{d}(\kappa' \circ F \circ \kappa^{-1})_{\kappa(x)}$$

with charts κ, κ' as before. In particular, even though we did not yet define the differential of F in x, we can nevertheless already make sense of saying that it is injective/surjective/invertible/has a certain rank. For a $\mathcal{C}^k$-map

$$M^m \xrightarrow{F \in \mathcal{C}^k} N^n$$

we thus have a well-defined rank function

$$M \xrightarrow{\operatorname{rank}(F_i)} \{0, 1, \ldots, \min(m, n)\},$$

$$x \mapsto \operatorname{rank}(F, x) =: \operatorname{rank} \mathrm{d}F_x.$$

We ask ourselves the following question: what are the continuity properties of $\operatorname{rank}(F,\bullet)$? We will understand them by locally expressing $\operatorname{rank}(F,\bullet)$ as the rank of the differential (Jacobian) of a local coordinate representation (see Fig. I.3a). Let (U,κ) be a chart on M and (U',κ') be a chart on N. Then F can be written in these coordinates as $\kappa'\circ F\circ\kappa^{-1}$ on the open subset $U\cap F^{-1}(U')$. There we can express $\operatorname{rank}(F,\cdot)$ as the composition

$$U\cup F^{-1}(U')\to \operatorname{Hom}(\mathbb{R}^m,\mathbb{R}^n)\xrightarrow{\text{rank}}\{0,1,\dots,\min(m,n)\},$$

$$x\mapsto \mathrm{d}(\kappa'\circ F\circ\kappa^{-1})_{\kappa(x)},$$

which is $\mathcal{C}^k$-map between open subsets of Euclidean spaces and its Jacobian depends on $\mathcal{C}^{k-1}$ in $\kappa(x)$. We see that the first step of the composition is continuous. In order to address the above question, we need to understand the semicontinuity properties of the rank function on linear maps/matrices.

What even is semicontinuity? With continuity, we require that the value of the function very close to the point doesn't increase or decrease too much. With lower/upper semicontinuity, we need just one of these two conditions. On functions valued in discrete sets (such as function rank is valued in natural numbers), semicontinuity essentially means boundedness in one direction or another.

$$\begin{array}{ccc} M & \xrightarrow{\operatorname{rank}(F,\cdot)} & \{0,1,\dots,\min(m,n)\} \\ \cup & \nearrow & \uparrow\text{rank} \\ U\cap F^{-1}(U') & \xrightarrow[\text{cont}]{} & \operatorname{Hom}(\mathbb{R}^m,\mathbb{R}^n)\ \cong\ \mathbb{R}^{n\times m} \end{array}$$

Consider $\operatorname{Hom}(\mathbb{R}^m,\mathbb{R}^n)\xrightarrow{\text{rank}}\{0,1,\dots,\min(m,n)\}$.

Basic observation. Small perturbations of a linear map can only (weakly) increase the rank, i.e. the rank function is lower semicontinuous.

More precisely

$$\{A\in\operatorname{Hom}(\mathbb{R}^m,\mathbb{R}^n)|\operatorname{rank}A\geq r\}\subset\operatorname{Hom}(\mathbb{R}^m,\mathbb{R}^n)$$

is open for every $r\in\mathbb{N}_0$. The geometric reason for this is that linear independence is an open condition for r-tuples of vectors in $\mathbb{R}^n$. For more technically precise justification, this is a consequence of the continuity of the r-fold *$\wedge$-products*:

$$v_1\wedge\cdots\wedge v_r\mapsto Av_1\wedge\cdots\wedge Av_r.$$

If $v_1,\dots,v_r$ are linearly dependent, this wedge product is zero, and this also holds for $Av_1,\dots,Av_r$. If, for a certain combination of r_i, the images Ar_i are linearly independent, then by continuity, any small enough change in r_i would not make $Av_1\wedge\cdots\wedge Av_r$ equal to zero. In more basic linear algebra terms, it is the consequence of the continuity of the determinant and, hence, the continuity of the function

$$A\mapsto\text{max of absolute values of determinants of } r\times r\text{-submatrices.}$$

Equivalent (complementary) to the above is the following basic observation.

Basic observation. When passing to the limits of linear maps, the rank can only (weakly) decrease.

More precisely

$$\{A \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m) | \operatorname{rank} A \leq r\} \subset \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$$

is closed for all $r \in \mathbb{N}_0$. Consequently,

$$\overline{\{A \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m) | \operatorname{rank} A = r\}} \subset \{A \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m) | \operatorname{rank} A \leq r\}$$

where the over-line denotes the topological closure. We strengthen this last assertion by adding another observation about adding approximability, namely that every linear map of non-maximal rank can be arbitrarily well approximated by linear maps with a strictly larger rank. Therefore,

$$\overline{\{A \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m) | \operatorname{rank} A = r\}} = \{A \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m) | \operatorname{rank} A \leq r\}.$$

In particular, the set of all homomorphisms of maximal rank

$$\{A \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m) | \operatorname{rank} A = \min(m, n)\} \subset \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$$

is open and dense. In this sense, "maximal" is a "generic" property for linear maps. And more generally, so is the property "rank $= r$" among linear maps of rank $\leq r$ for any $r \leq \min(m, n)$.

Now is the time to return to $\mathcal{C}^k$-differentiable maps $M^n \xrightarrow{F} N^n$ and our question about the semicontinuity property of the rank function. The rank function F is the composition of a continuous function and a lower semicontinuous function, hence it is semicontinuous itself. We conclude that

$$\{x \in M | \operatorname{rank}(F, x) = \max \operatorname{rank}(F, \cdot)\} \subset M$$

is open and not empty (well, if the manifold is not empty) and in particular

$$\{x \in M | \operatorname{rank}(F, x) = \min(m, n)\} \subset M$$

is open but possibly empty. In the first case, F is a constant rank; in the second example, F is a submersion or immersion and is the maximal possible rank. Thus, the generic infinitesimal behaviour of the map F (in the sense of the max rank of its differential) takes place in open regions, and it is in these regions where we study the local behaviour of F.

We therefore restrict our attention to the case of $\mathcal{C}^k$-maps of constant rank. Regarding the special case of $\mathcal{C}^k$-maps of everywhere maximal possible rank one has the two important (overlapping) notions.

Definition I.8. A $\mathcal{C}^k$-differentiable map $F : M^m \to N^n$ is a $\mathcal{C}^k$-*submersion* ($\mathcal{C}^k$-*immersion*) if its differential is everywhere surjective (injective). They can happen when $m \geq n$ ($m \leq n$).

Examples for $\mathcal{C}^k$-maps with constant rank are affine-linear maps $\mathbb{R}^m \to \mathbb{R}^n$. We will see later that these are indeed the local models for general constant-rank $\mathcal{C}^k$-maps.

The case $m = n = r$ is covered by the Inverse function theorem, $m \geq n = r$ is covered by Submersion theorem, $m \geq n = r$ is covered by Immersion theorem and $m, n > r$ is covered by Constant rank theorem. Both Immersion and Submersion theorem are parts of the implicit function theorem.

The basic result, from which the other (apparently) more general ones can be deduced rather directly, concerns the case of equal dimensions and max rank (and has been proven in Analysis

II with rather hard work). Note that the local results discussed here are local in nature and immediately reduce from the manifold case to the Euclidean case.

Theorem I.2 (*Inverse function theorem*). *Let*

$$F : M^m \xrightarrow{\mathcal{C}^k} N^{n=m}.$$

Then F has invertible differential in $x_0 \in M$ and near x_0, and F is local $\mathcal{C}^k$-diffeomorphism near x_0, i.e. there exist open neighbourhoods $U \subset M$ of x_0 and $U' \subset N$ of $y_0 = F(x_0)$ such that

$$F|_U : U \xrightarrow{\mathcal{C}^k\text{-}diffeo} U'$$

(see Fig. I.6)

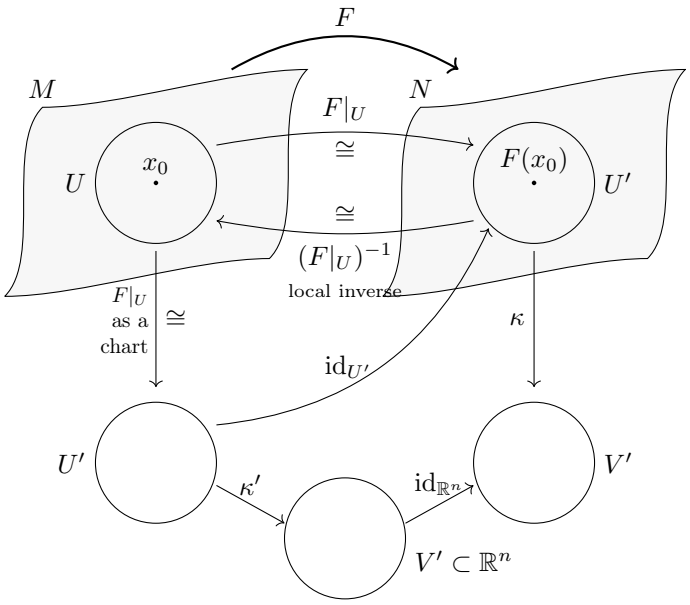


Figure I.6: Local $\mathcal{C}^k$-diffeomorphism of F near x_0.

As a consequence F is locally $\mathcal{C}^k$-equivalent near x_0 to $\mathrm{id}_{\mathbb{R}^{m=n}}$, i.e. in suitable coordinates, F is near x_0 given by $\mathrm{id}_{\mathbb{R}^{m=n}}$, which is called local normal form.

Remark. This is a result that maps infinitesimal to local.

Next, we deduce from this the submersion case, which is covered by the Implicit function theorem:

Theorem I.3 (*Submersion theorem*). *Let*

$$F : M^m \xrightarrow{\mathcal{C}^k} N^{n \leq m}$$

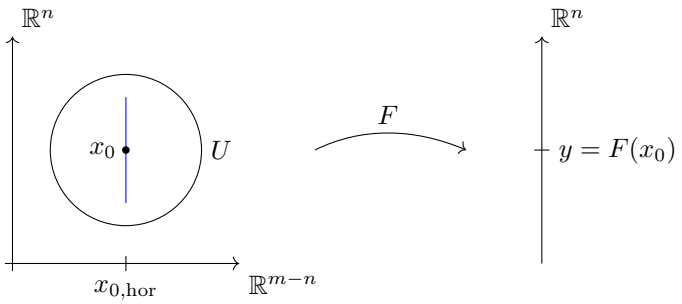
be a $\mathcal{C}^k$-submersion. Then the following holds: if F has a surjective differential in some point $x_0 \in M$ (and hence also near x_0), then F is locally (near x_0) $\mathcal{C}^k$-equivalent to a surjective linear map $\mathbb{R}^m \to \mathbb{R}^n$.

We could strengthen the notion of $\mathcal{C}^k$-equivalence to linear $\mathcal{C}^k$-equivalence since surjective linear maps are all equivalent to each other modulo linear coordinate change.

Idea of proof. It is enough to work locally. Let's consider a map

$$\mathbb{R}^m \supset U^{\text{open}} \xrightarrow[\mathcal{C}^k\text{-submersion}]{F} \mathbb{R}^{n \leq m}$$

and $x_0 \in U$.



Without loss of generality (after permuting coordinates in the domain), we force F-level sets $F^{-1}(F(x_0)) = F^{-1}(y_0)$ through x_0 to be "transversal" to the vertical direction. We make our map locally invertible by adding $m - n$ dimensions to the image, i.e. pass to $\hat{F} = (\pi_{\mathbb{R}^{m-n}}, F)$, where π is the vertical projection $\mathbb{R}^m \to \mathbb{R}^{m-n}$ to the first factor. The map $\hat{F}(x) = (x_{\text{hor}}, F(x))$ preserves vertical fibration. Note that $\mathrm{d}\hat{F}_{x_0}$ is invertible, because the Jacobian is

$$J\hat{F}_{x_0} = \begin{pmatrix} \mathrm{id}_{\mathbb{R}^{m-n}} & 0 \\ \mathrm{d}F_{x_0}|_{\mathbb{R}^{m-n} \times \{0\}} & \mathrm{d}F_{x_0}|_{\{0\} \times \mathbb{R}^n} \end{pmatrix}.$$

The Inverse function theorem applies to $\hat{F}$ in x_0. The F-level sets are identified by $\hat{F}^{-1}$-preimages of horizontal layers. Hence, $\hat{F}$ serves as local chart "flattening" the F-level sets and F-level set $F^{-1}(F(x_0))$ is $(m-n)$-dimensional $\mathcal{C}^k$-submanifold near x_0.

Regarding the linear local normal form, use $\hat{F}$ as a local coordinate change near x_0. Near x_0 F is $\mathcal{C}^k$-equivalent to $F \circ (\hat{F}|_V)^{-1}$ near $\hat{F}(x_0)$, where V is a neighbourhood of $\hat{F}(x_0)$. It is equal to the projection to the vertical factor

$$(F \circ (\hat{F}|_V)^{-1})(\hat{F}(x)) = F(x) = \pi_{\mathbb{R}^n}(\hat{F}(x)),$$

where $\pi_{\mathbb{R}^n}$ is the projection to the second factor. □

Remark. $\mathcal{C}^k$ submersions are open.

Now, we further expand this to the general constant rank case.

Theorem I.4 (*Constant rank theorem*). *Let*

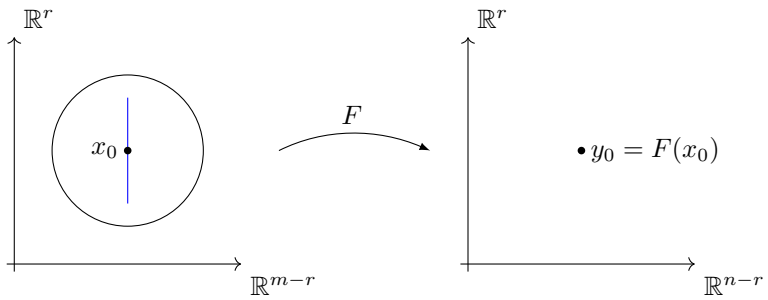
$$F : M^m \xrightarrow{\mathcal{C}^k} N^n, \quad r \leq \min(m, n).$$

The differential of F has rank $\equiv r$ *near $x_0 \in M$. Then F is locally $\mathcal{C}^k$-equivalent, near x_0, to a linear map $\mathbb{R}^m \to \mathbb{R}^n$ of rank r. These linear maps are all linearly equivalent to each other.*

Sketch of proof. We will deduce the proof from the Submersion theorem. Again, we will work locally, that is

$$\mathbb{R}^m \supset U^{\text{open}} \xrightarrow[\mathcal{C}^k, \text{rank}=r]{F} \mathbb{R}^{n \leq m}.$$

The domain is split into $\mathbb{R}^m \cong \mathbb{R}^{m-r} \times \mathbb{R}^r$ and the co-domain as well $\mathbb{R}^n \cong \mathbb{R}^{n-r} \times \mathbb{R}^r$. However, the splitting of the domain is not as important as the splitting of the co-domain.



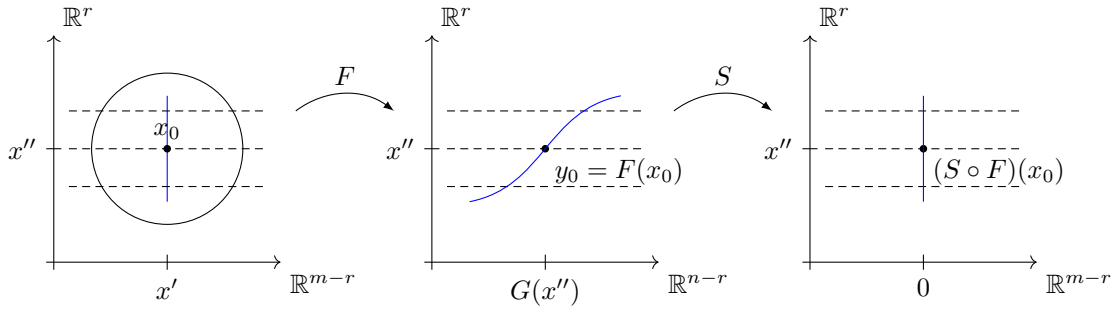
Without loss of generality, we can assume that $\mathrm{d}(\pi_{\mathbb{R}^r} \circ F)_{x_0}$ is surjective, where π is a horizontal projection to the second factor. This forces the image of F to "spread" transversely in the horizontal direction. The map $\pi_{\mathbb{R}^r} \circ F$ is $\mathcal{C}^k$-submersion near x_0 and the Submersion theorem applies there. After suitable coordinate change (in the domain and second factor of the target), $\pi_{\mathbb{R}^r} \circ F$ becomes near x_0 a surjective linear map $\mathbb{R}^m \to \mathbb{R}^r$. Without loss of generality $\pi_{\mathbb{R}^r} \circ F = \pi_{\mathbb{R}^r}$, i.e. F assumes near $x = (x', x'')$ the form $F(x', x'') = (G(x', x''), x'')$, where $x' \in \mathbb{R}^{m-r}$ and $x'' \in \mathbb{R}^r$. We know little about $G(x', x'')$ except that it is $\mathcal{C}^k$-differentiable. The differential $\mathrm{d}F_x$ has the triangular block form. The Jacobian is

$$JF_x = \begin{pmatrix} \mathrm{d}G_x|_{\mathbb{R}^{m-r} \times \{0\}} & \mathrm{d}G_x|_{\{0\} \times \mathbb{R}^r} \\ 0 & \mathrm{id}_{\mathbb{R}^r} \end{pmatrix}.$$

The rank of $\mathrm{d}F_x$ is r and thus

$$r = \operatorname{rank} \mathrm{d}F_x = \operatorname{rank} \mathrm{d}G_x|_{\mathbb{R}^{m-r} \times \{0\}} + \operatorname{rank} \mathrm{id}_{\mathbb{R}^r} = \operatorname{rank} \mathrm{d}G_x|_{\mathbb{R}^{m-r} \times \{0\}} + r.$$

Therefore, $\operatorname{rank} \mathrm{d}G_x|_{\mathbb{R}^{m-r} \times \{0\}} = 0$ and G does not depend on x': $G(x', x'') = G(x'')$. Near x_0 F has the form $F(x', x'') = (G(x''), x'')$. One more small step is needed to achieve the desired linear normal form, namely a horizontal *shear* in the image.



The transformation is $S : (y, x'') \mapsto (y - G(x''), x'')$ and the inverse is $s^{-1} : (z, x'') \mapsto (z + G(x''), x'')$. Lastly, we have

$$(S \circ F)(x', x'') = (0, x'') = \pi_{\mathbb{R}^n}(x', x'').$$

□

As a special case of the Constant rank theorem, we also obtain

Theorem I.5 (*Immersion theorem*). *Let $M^m \xrightarrow[\mathcal{C}^k]{F} N^{n \geq m}$ be a $\mathcal{C}^k$-immersion. If F has injective differential in $x_0 \in M$ (and hence near it), then F is locally (near x_0) $\mathcal{C}^k$-equivalent to an injective linear map $\mathbb{R}^m \to \mathbb{R}^n$.*

Let's look at the consequences of the Constant rank theorem. For $M^m \xrightarrow[\mathcal{C}^k, \text{rank}=r]{F} N^n$ it holds that

1. the level sets $F^{-1}(y) \subset M$ for $y \in N$ are embedded $\mathcal{C}^k$-submanifolds of co-dimension r,
2. the image of F is everywhere locally an embedded $\mathcal{C}^k$-submanifold of dimension r, i.e. for every $x \in M$ has an open neighbourhood U such that the image $F(U) \subset N$ is an embedded $\mathcal{C}^k$-submanifold of dimension r.

This motivates the notion of a differentiable (sub)manifold. Consider $M^m \xrightarrow[\mathcal{C}^k]{F} N^n$.

Definition I.9.
1. A point $x \in M$ is called a *regular point* of F if F has surjective differential in x, and a *critical point* otherwise.
2. A point $y \in N$ is called a *regular value* for F if all points in the level set $F^{-1}(y) \subset M$ are regular points, and a *critical value* otherwise.

For submersions, all points and values are regular. In general, the subset of regular points is open, and the map restricts to a submersion. The Submersion theorem applies:

Corollary I.6. *If $m \geq n$, then for every regular value $y \in N$ of F, the level set $F^{-1}(y) \subset M$ is an embedded $\mathcal{C}^k$-submanifold of codimension n.*

Remark. If $m < n$, then the regular values of F are precisely the points in $N - F(M)$, and the level sets are empty.

The critical values are precisely the images of the critical points (which form a closed subset). However, the images of the regular points (which form an open subset of M) are not necessarily regular values. Generally, the subsets of the regular (and, respectively, critical) values have no good topological properties. However, they are certainly measurable (which comes from the fact that the set of critical points is closed and, hence, a countable union of compacts). Regarding the existence of regular values, there is a basic result asserting that values of sufficiently smooth maps are generally regular (see Milnor's "Topology from the differentiable viewpoint"). We state the smooth case:

Theorem I.7 (*Sard*)**.** *For a smooth map, the set of critical values has measure zero, i.e. measure zero in any local coordinates. (wiki)*

Remark. In general, one needs $\mathcal{C}^{k \geq \max(m-n+1,1)}$ (e.g. $\mathcal{C}^1$) is enough for $m \leq n$.

Corollary I.8 (*Brown*)**.** *The set of regular values of a smooth map is dense.*

In particular, the image of $M^m \xrightarrow[\mathcal{C}^1]{F} N^{n>m}$ has measure zero (empty interior) in N.

I.3.2 Embeddings versus immersions

We will state topological and differentiable definitions of embeddings.

Definition I.10. A continuous map (of topological spaces) is an *embedding* if it is a homeomorphism onto its image (equipped with a relative topology).

Definition I.11. A $\mathcal{C}^k$-map $M^m \xrightarrow{F} N^{n \geq m}$ is a $\mathcal{C}^k$-embedding if $F(M) \subset N$ is an embedded n-dimensional $\mathcal{C}^k$-submanifold and $M \xrightarrow{F} F(M) \subset N$ is a $\mathcal{C}^k$-diffeomorphism.

So, $\mathcal{C}^k$-embeddings "are" inclusion maps of $\mathcal{C}^k$-submanifolds. Clearly, $\mathcal{C}^k$-embeddings are $\mathcal{C}^k$-immersions. The converse is true locally:

Proposition I.9. *$\mathcal{C}^k$-immersions are locally $\mathcal{C}^k$-embeddings, i.e.*

$$M^m \xrightarrow[\mathcal{C}^k\text{-immersion}]{F} N^{n \geq m} \implies \exists x \in W^{open} \subset M : \ F|_W \text{ is a } \mathcal{C}^k\text{-embedding.}$$

Proof. Reduces by Immersion theorem to the case of linear embeddings. □

Globally, we have:

Proposition I.10. *For a $\mathcal{C}^k$-map $M^m \xrightarrow[\mathcal{C}^k]{F} N^{n\geq m}$ it holds that*

$$\mathcal{C}^k\text{-embedding} \iff \left\{ \begin{array}{c} \text{topological embedding} \\ \text{(injective) } \mathcal{C}^k\text{-immersion} \end{array} \right\}.$$

Note that the word "injective" is redundant.

Proof.

$\Longrightarrow$ This is clear.

$\Longleftarrow$ Suppose $x \in M$ and

$$M^m \xrightarrow[\mathcal{C}^k\text{-immersion, topological embedding}]{F} N^{n\geq m}.$$

Since $\mathcal{C}^k$-immersions are locally $\mathcal{C}^k$-embeddings

$$\exists x \in W^{\text{open}} \subset M : \ F|_W \text{ is } \mathcal{C}^k\text{-embedding}$$

it follows $F(W) \subset N$ is an n-dimensional embedded $\mathcal{C}^k$-submanifold. Since F is also a topological embedding, $F(W)$ "stays away" from the rest of the image of F in the sense that $F(W)$ is relatively open in $F(M) \subset N$, i.e. $\exists U^{\text{open}} \subset N$, such that $F(W) = F(M) \cap U$ and $F(M) \subset N$ is an m-dimensional embedded $\mathcal{C}^k$-submanifold near $F(x)$. Moreover, $M \xrightarrow{F} F(M)$ is everywhere locally $\mathcal{C}^k$-diffeomorphic and also injective, so it is a $\mathcal{C}^k$-diffeomorphism.

□

Remark. Injective $\mathcal{C}^k$-immersion does not imply topological embedding.



However, if the source (domain) is compact, topological "accidents" cannot happen.

Corollary I.11. *Injective $\mathcal{C}^k$-immersions from compact sources are embeddings.*

It follows from a useful topological lemma:

Lemma I.12. *Let X be a compact topological space and Y be a Hausdorff space. Then continuous injections $X \xrightarrow{inj} Y$ are topological embeddings.*

The corollary holds more generally for proper immersions.

Let's generalize slightly the notion of differentiability for maps between Euclidean spaces. Consider maps

$$\mathbb{R}^n \supset A \xrightarrow{F} \mathbb{R}^m,$$

where A is an arbitrary set. Cases of interests are

- A is open.
- A is an intersection of open subsets with a closed half-space.
- A is an intersection of open subsets with finitely many closed half-spaces, which are sublevels of linearly independent linear functionals.

Differentiability in a point of A is defined as a linear approximability:

Definition I.12. A map F is *differentiable* in $x_0 \in A$ if and only if there exists a linear map $\mathbb{R}^n \xrightarrow[\text{linear}]{L} \mathbb{R}^m$ such that $F(x) = F(x_0) + L(x - x_0) + r(x)$. The first two terms are affine-linear transformation and $r(x)$ is the error term, which must be *sublinear* $r(x) = o(\|x - x_0\|)$, i.e. $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\|r(x)\| \leq \varepsilon \|x - x_0\|$ for all $x \in A \cap B_\delta(x_0)$.

Let's look at the relation of linear approximation to partial derivatives. Suppose that F is differentiable in $x_0 \in A$ (as defined above) and that A contains a segment initiating in x_0 of the form $x_0 + tv \in A$ for $0 \leq t < \tau$, $\tau > 0$ and $v \in \mathbb{R}^n$. Denote ∂_{v^+} a one-sided "forward" partial derivative in the direction of v. Then

$$(\partial_{v^+} F)(x_0) \coloneqq \frac{\mathrm{d}}{\mathrm{d}t} F(x_0 + tv)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \left(F(x_0) + tL(v) + o(t)\right)\bigg|_{t=0} = Lv.$$

The values of L on directions represented by segments in A are partial derivatives of F in x_0. Suppose now more generally that A contains a differentiable curve starting in x_0 in the direction of $v \in \mathbb{R}^n$,

$$[0, \tau) \xrightarrow[\text{diff in } 0]{c} A \subset \mathbb{R}^n$$

such that $c(0) = x_0$ and $\dot{c}(0) = v$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} F(c(t))\bigg|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} F(x + tv + o(t))\bigg|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \left(F(x_0) + L(tv + o(t)) + o(|tv + o(t)|)\right)\bigg|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \left(F(x_0) + tL(v) + o(t)\right)\bigg|_{t=0} \\
&= L(\dot{c}(0)).
\end{aligned}$$

Regarding the uniqueness of the linear approximation of F in x_0. We will not discuss the optimal conditions; rather, we will just give one sufficient condition.

Sufficient condition for uniqueness: If A contains segments (or more generally differentiable curves as above) initiating in x_0 whose directions (initial velocities) span $\mathbb{R}^n$ as a vector space, then the linear map L is unique.

This is certainly the case if A is open or relatively open in a closed half-space, quadrant, octant or n-dimensional orthant. If L is unique, we call it the *differential* of F in x_0 and denote it by

$$\mathrm{d}F_{x_0} \coloneqq L \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m).$$

Remark. If there exists a function $U \xrightarrow{\hat{F}} \mathbb{R}^m$ defined on a neighbourhood U of x_0 in $\mathbb{R}^n$ and differentiable in x_0 such that $\hat{F}|_{A\cap U} = F|_{A\cap U}$, then F is differentiable in x_0. When the above-mentioned sufficient condition for uniqueness holds, then $\mathrm{d}F_x = \mathrm{d}\hat{F}_x$.

If A satisfies the sufficient condition for uniqueness in all points and if F is everywhere differentiable, then its *differential* is the map $\mathrm{d}F$

$$\begin{aligned} A &\to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{m\times n} \cong \mathbb{R}^{mn}, \\ x_0 &\mapsto \mathrm{d}F_{x_0}. \end{aligned}$$

One can then speak of the continuity and differentiability of the differential and define the notion of higher differentiability (e.g. $\mathcal{C}^k$).

I.4 Manifolds with boundary

We extend the notion of manifold by using a more general inhomogeneous local model:

$$\overline{H}^n \coloneqq \{x_1 \leq 0\} \subset \mathbb{R}^n,$$

which is an n-dimensional closed half-space for $n \geq 1$. Such space has a natural decomposition into the

- *interior*: open half-space $H^n \coloneqq \{x_1 < 0\}$,
- *boundary*: coordinate hyperplane $\partial H^n \coloneqq \{x_1 = 0\} \cong \mathbb{R}^{n-1}$.

On the topological level, we define:

Definition I.13.
(i) A topological space is called *n-dimensional locally Euclidean with boundary* if it is locally homeomorphic to $\overline{H}^n$.
(ii) It is called an *n-dimensional topological manifold with boundary* if it is in addition Hausdorff and second countable.

Let X be an n-dimensional locally Euclidean space with boundary. A local *chart* (U, k) for X now takes values in the local model $\overline{H}^n$, i.e. it is the map

$$X \supset U^{\text{open}} \xrightarrow[\text{homeo}]{\kappa} V^{\text{rel. open}} \subset \overline{H}^n.$$

As before, we define coordinate changes, atlases, $\mathcal{C}^k$-structures with boundary and $\mathcal{C}^k$-manifolds with boundary. Note that we need a generalized notion of differentiability in order to define $\mathcal{C}^k$-structures with boundary. These are a special type of geometric structure with charts in $\overline{H}^n$ and transition maps that are $\mathcal{C}^k$-diffeomorphisms.

We expect that the interior-boundary decomposition of the local model carries over to manifolds with boundaries. This is indeed the case, but it requires profound results in the topological case, whereas it follows from basic calculus in the differentiable case.

The coordinate changes are now homeomorphisms between open subsets of the more general local model $\overline{H}^n$, and we need them to preserve its interior-boundary decomposition, that is

$$\overline{H}^n \supset V_1^{\text{rel. open}} \xrightarrow[\text{homeo}]{\phi} V_2^{\text{rel. open}} \subset \overline{H}^n$$

implies

$$\phi(V_1 \cap H^n) = V_2 \cap H^n \quad \text{and} \quad \phi(V_1 \cap \partial H^n) = V_2 \cap \partial H^n.$$

If ϕ is $\mathcal{C}^1$, this is an immediate consequence of the Inverse function theorem. It implies that ϕ and ϕ^{-1} must map interior points to interior points. Note that the differentials of $\mathcal{C}^1$-diffeomorphisms are well-defined also in boundary points and must be invertible there.

In the topological case, the decomposition follows a deep theorem in algebraic topology.

Theorem I.13 (Invariance of domain theorem (Brouwer))**.** *Let $\varphi : U \to \mathbb{R}^n$ be a continuous injective map from an open subset $U \subset \mathbb{R}^n$. Then $\varphi(U) \subset \mathbb{R}^n$ is open. (wiki)*

This directly implies that φ is a homeomorphism onto an open subset.

Based on the invariance of the local-boundary decomposition of the local model, we can define

Definition I.14. The interior $\mathring{X}$ and the boundary ∂X of an n-dimensional locally Euclidean space X with boundary (equipped with a $\mathcal{C}^k$-structure) is the subset of points which are mapped by some and (hence only) chart to H^n and ∂H^n, respectively.

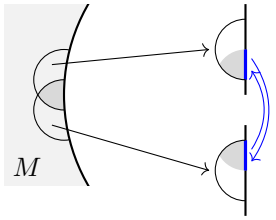
Note that in the presence of a $\mathcal{C}^k$-structure, one can only use $\mathcal{C}^k$-charts and, therefore, only needs the Inverse function theorem for well-definedness.

The interior is an open subset, and the boundary is a closed subset. Both are again locally Euclidean. Indeed, charts for X restrict to charts for the interior $\mathring{X}$ and the boundary ∂X:

$$X \supset U^{\text{open}} \xrightarrow[\cong]{\kappa} V \subset \overline{H}^n$$

more specifically

$$\begin{cases} \mathring{X} \supset \overset{\text{(rel.) open}}{U \cap \mathring{X}} \xrightarrow[\cong]{\kappa|_{U\cap\mathring{X}}} V \cap H^n \subset H^n, \\ \partial X \supset \overset{\text{(rel.) open}}{U \cap \partial X} \xrightarrow[\cong]{\kappa|_{U\cap\partial X}} V \cap \partial H^n \subset \partial H^n. \end{cases}.$$



Coordinate changes between the restricted charts are restrictions of coordinate changes for X to open subsets respectively relatively open subsets of the coordinate hyperplane $\{x_1 = 0\}$. Hence, they have the same regularity $\mathcal{C}^k$ as before their restrictions.

Consequently, for an n-dimensional locally Euclidean space with boundary equipped with a $\mathcal{C}^k$-structure it holds that

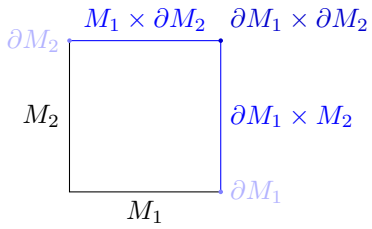
(i) $\mathring{X}$ is an n-dimensional locally Euclidean space and ∂X is an $(n-1)$-dimensional locally Euclidean ;

(ii) $\mathring{X}$ inherits a $\mathcal{C}^k$-structure of $\mathbb{R}^n$-valued charts and ∂X inherits a $\mathcal{C}^k$-structure of $\mathbb{R}^{n-1}$-valued charts.

Clearly, the Hausdorff and second countability properties are inherited as well. Thus, M being an n-dimensional $\mathcal{C}^k$-manifold with boundary implies

$$\begin{cases} \mathring{M} \text{ is an } n\text{-dimensional } \mathcal{C}^k\text{-manifold,} \\ \partial M \text{ is an } (n-1)\text{-dimensional } \mathcal{C}^k\text{-manifold.} \end{cases}$$

The deficit of the notion of the differentiable manifold with a boundary is that it is not closed under taking products.



I.4.1 Differentiable manifolds with corners

This is a further extension of the notion of a differentiable manifold. The discussion is analogous. The local model is now an n-dimensional closed *orthant*:

$$\overline{O}^n := \{x_1, \dots, x_n \leq 0\} \subset \mathbb{R}^n.$$

The d-dimensional *stratum* consists of the points X exactly d of whose coordinates are non-zero. This is a slight generalization of an orthant. The *stratification* assigns each point to the respective stratum.

It is left as an exercise to show that $\mathcal{C}^k$-coordinate changes preserve the stratification. The $\mathcal{C}^k$-manifolds with corners inherit the stratification and from the local model and the strata are $\mathcal{C}^k$-manifolds (without boundary).

I.5 Compact exhaustions and partitions of unity

Partitions of unity are a basic tool for local-to-global constructions on manifolds. Let M^m be a $\mathcal{C}^k$-manifold (possibly with boundaries and corners).

Definition I.15. A $\mathcal{C}^k$ *partition of unity* on M is a family $(\varphi_\iota)_{\iota\in I}$ of $\mathcal{C}^k$-functions $M \xrightarrow{\varphi_\iota} [0,1]$ such that

(i) The family $(\operatorname{supp}\varphi_\iota)_{\iota\in I}$ of support sets is locally finite, i.e. every compact subset of M intersects only finitely many subsets $\operatorname{supp}\varphi_\iota$.

(ii) The sum
$$\sum_{\iota\in I}\phi_\iota \equiv 1$$
is everywhere well-defined due to local finiteness.

Recall that $\operatorname{supp}\varphi = \overline{\{\varphi \neq 0\}}$.

Definition I.16. The partition of unity $(\varphi_\iota)_{\iota\in I}$ is called *subordinate* to cover $(U_\alpha)_{\alpha\in A}$ of M if there exists a map $I \xrightarrow{\alpha} A$ such that $\operatorname{supp}\varphi_\iota \subset U_{\alpha(\iota)}$ for all $\iota \in I$.

Theorem I.14. *For every open cover $(U_\alpha)_{\alpha\in A}$ of M there exists a $\mathcal{C}^k$-differentiable partition of unity subordinate to it with compact support sets $\operatorname{supp}\varphi_\iota$ and countable index set I.*

The existence of partitions of unity is a topological property. It is a consequence of second countability, and it is derived from the existence of a compact exhaustion which we will show first.

Definition I.17. A *compact exhaustion* of a topological space X is a nested sequence of relatively compact open sets
$$G_1 \subset G_2 \subset \ldots G_n \subset \cdots \subset X$$
such that $\overline{G}_n \subset G_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_n G_n = X$.

Proposition I.15. *Every second countable, locally compact Hausdorff space admits a compact exhaustion.*

Locally compact means that every neighbourhood contains a compact neighbourhood.

Proof. There exists a countable basis $(B_i)_{i\in\mathbb{N}}$ for the topology consisting of relatively compact (open) subsets. We inductively construct an increasing sequence of sequences $(i_n)_{n\in\mathbb{N}}$ of indices as follows

(i) $i_1 \coloneqq 1$,

(ii) $i_{n+1} : G_n = \bigcup_{i=1}^{i_n} B_i$ relatively compact. Then we choose $i_{n+1} \geq i_n + 1$ so that $\overline{G}_n \subset \bigcup_{i=1}^{i_{n+1}} B_i$. The sequence of G_n has the desired properties.

□

Proof of Theorem I.14. By the Proposition I.15, M admits a compact exhaustion (G_n). For every point p in the compact "annulus" $\overline{G}_n - G_{n-1}$ we choose an open neighbourhood $V_p \subset G_{n+1} - \overline{G}_{n-2}$ and a $\mathcal{C}^k$-function $\psi_p : M \to [0, \infty)$ such that $\operatorname{supp} \psi_p \subset V_p$ and $\psi_p(p) > 0$. Because of the compactness of the annuli $\overline{G}_n - G_{n-1}$ is covered by finitely many open subsets $\{\psi_p > 0\} \subset V_p$. Hence, there exists a countable closed discrete subset $I \subset M$ such that $(\{\psi_p > 0\})_{p \in I}$ is an open cover of M. The family of subsets $(V_p)_{p \in I}$ is locally finite and

$$\psi \coloneqq \sum_{p \in I} \psi_p$$

is a well-defined and strictly positive $\mathcal{C}^k$ function. Now define $(\varphi_p)_{p \in I}$ with

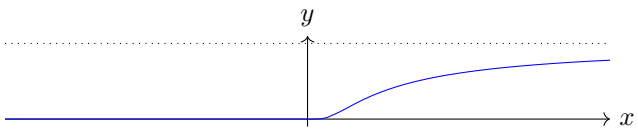
$$\phi_p \coloneqq \frac{\psi_p}{\psi},$$

which is $\mathcal{C}^k$ partition of unity with compact support sets. In order to make the partition of unity subordinate to a given cover $(U_\alpha)_{\alpha \in A}$ we choose the neighbourhoods V_p so that they are contained in the corresponding subsets $U_{\alpha(p)}$. □

Remark (Construction bump functions). The function $f : \mathbb{R} \to [0, 1)$ defined by

$$f(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

is everywhere $\mathcal{C}^\infty$. Regarding the smoothness in origin, note that the derivatives for $x > 0$ are of the form $e^{-1/x} p(1/x)$ for some polynomial p, and hence they vanish at $x = 0$.



Using it, we can for every $\delta > 0$ build a smooth function $\varphi : \mathbb{R} \to [0, 1)$ supported on $[-\delta, \delta] \subset \mathbb{R}$ and strictly positive on $(-\delta, \delta)$ by

$$\varphi(x) = f(x - \delta) f(\delta + x) = \begin{cases} e^{\frac{2\delta}{\delta^2 - x^2}}, & x \in (\delta, \delta), \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, we can construct a smooth function $\varphi : \mathbb{R}^n \to [0, 1)$ supported on $[-\delta, \delta]^n \subset \mathbb{R}^n$ and strictly positive on $(-\delta, \delta)^n$ by

$$\varphi_n(x) \coloneqq \varphi(x_1) \cdot \dots \cdot \varphi(x_n).$$

Example 6 (Applications of partitions of unity)**.**

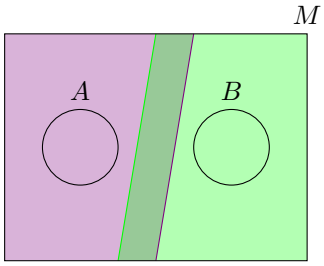
1. Differentiable *Urysohn's lemma* (separating closed subsets by functions, wiki): let $A, B \subset M$ be closed subsets of a $\mathcal{C}^k$-manifold. Consider an open cover of M by $M - B$ and $M - A$. By the partition of unity we obtain $\mathcal{C}^k$-functions $(\varphi_\iota)_{\iota \in I}$ subordinate to this cover. This means that the support of each φ_ι avoids either A or B (or both). The sum of the partitions of unity is one, so we can write

$$\sum_{\operatorname{supp} \varphi_\iota \subset M - B} \varphi_\iota + \sum_{\operatorname{supp} \varphi_\iota \subset M - A} \varphi_\iota \equiv 1.$$

Denote the first sum as f_A and the second as f_B. By the local finiteness of the family of support sets,

$$\bigcup_{\operatorname{supp}\varphi_\iota \subset M-B} \operatorname{supp}\varphi_\iota \subset M - B$$

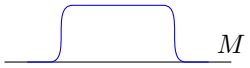
is closed, and hence $\operatorname{supp} f_A \subset M - B$ (coloured violet) and $\operatorname{supp} f_B \subset M - A$ (coloured green), which is stronger than $f_A|_B = 0$.



Hence,

$$f_A \equiv \begin{cases} 1, & \text{on a neighbourhood of } A, \\ 0, & \text{on a neighbourhood of } B. \end{cases}$$

This can be applied to constant *cut-off functions* (smooth functions that are constant (non-zero) on a closed set A and non-zero on an open set U) $A^{\text{closed}} \subset U^{\text{open}} \subset M$ and $B \coloneqq M - U$.



As above, we construct a $\mathcal{C}^k$ function $f_A : M \to [0,1]$ with $f_A \equiv 1$ on neighbourhood of A and $\operatorname{supp} f_A \subset U$.

2. Differentiable approximation of continuous functions.

Claim I.16. *If $g_1 : M \to \mathbb{R}$ and $g_2 : M \to \mathbb{R}$ are continuous functions on a $\mathcal{C}^k$-manifold with $g_1 < g_2$, then there exists a $\mathcal{C}^k$-function $f : M \to \mathbb{R}$ such that $g_1 < f < g_2$.*

Proof. We write the proof with the notation of construction of partition of unity. Choose $V_p \subset M$ such that there exists $c_p \in \mathbb{R}$ with $g_1 < c_p < g_2$ on V_p. Choose $\varphi_p \geq 0$ with $\operatorname{supp}\varphi_p \subset V_p$. Then $\varphi_p g_1 \leq c_p \varphi_p \leq \varphi_p g_2$ on M. The inequality is strict, where $\varphi_p > 0$. Lastly, we sum over $p \in I$ and use $\sum_{p\in I} \varphi_p \equiv 1$ to get

$$g_1 < \sum_{p\in I} c_p \varphi_p < g_2.$$

The middle term is a $\mathcal{C}^k$-function, exactly what we were looking for. □

Chapter II

Tangent bundle

The prerequisite for this chapter is familiarity with the local existence and uniqueness of solutions of (1st order) ordinary differential equations (ODEs) and differential dependence on initial conditions. These topics can be found in any introductory book on ODEs. If the book doesn't contain these notions, you can throw it away.

II.1 Tangent bundle to open subsets of the local model

Consider

$$U^{\text{rel. open}} \subset \overline{O}^n \subset \mathbb{R}^n.$$

Tangent vectors to U are arbitrary vectors (x, v) with footpoint in U. Here, the letter $x \in U$ denotes the *footpoint* of the tangent vector and $v \in \mathbb{R}^n$ its components.

Note. If $x \in U \cap \partial\overline{O}^n$ is a boundary point, then tangent vectors in x may "point away" from $\overline{O}^n$.

Tangent space to U in $x \in U$ is defined as

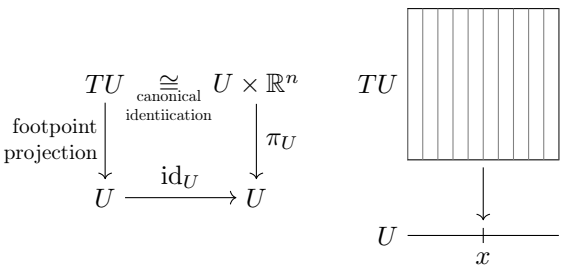
$$T_xU \coloneqq \{\text{tangent vectors to } U \text{ with footprint in } x\} \cong \{x\} \times \mathbb{R}^n \cong \mathbb{R}^n.$$

It is a "copy" of $\mathbb{R}^n$ attached to x.

The *tangent bundle* of U is a disjoint union of tangent spaces in all points:

$$TU \coloneqq \bigsqcup_{x \in U} T_xU \cong U \times \mathbb{R}^n.$$

Sometimes, we think of tangent spaces as



The set U is also called a *base* of the footpoint projection. There are two components of the structure:

- *algebraic* structure on fibres ("vertically") an n-dimensional vector spaces,
- *geometric* structure (also) "transversally" to the fibres as $2n$-dimensional $\mathcal{C}^\infty$-manifolds with corners.

It is important to know different useful ways of viewing tangent vectors to $\overline{O}^n$:

- *geometrically* as velocities: for a curve, i.e. a differentiable function c from an open interval $I \in \mathbb{R}$ to $\overline{O}^n$, $c : I \to \overline{O}^n$, regard velocity $\dot{c}(t)$ as tangent vector at $c(t)$: $\dot{c}(t) \in T_{c(t)}\overline{O}^n \cong \mathbb{R}^n$. We can represent tangential vectors, at least in the interior O^n, by differentiable curves and view them as equivalence classes of such. Define an equivalence relation on curves $(-\varepsilon_c, \varepsilon_c) \xrightarrow[\text{diff in } 0]{c} \overline{O}^n$ by $c \equiv_1 \tilde{c}$ equal in 0 up to 1st order, i.e. $c(0) = \tilde{c}(0)$ and $\dot{c}(0) = \dot{\tilde{c}}(0)$. The equivalence classes $[x]$ are called *1-jets* of curves in $\overline{O}^n$ with footprint (a 0-jet) in $c(0)$.

  There is a natural correspondence

  $$T\overline{O}^n \supset TO^n \overset{1:1}{\longleftrightarrow} \{\text{1-jets of curves } (-\varepsilon_c, \varepsilon_c) \xrightarrow{\text{diff in } 0} O^n\}$$

  with $(c(0), \dot{c}(0)) \leftrightarrow [c]$. Deficits of this interpretation of tangent vectors as 1-jets are

  - linear structure of tangent spaces is hidden (think of manifold case),
  - correspondence breaks down at the boundary;

- *analytically* as directional derivatives: let $(x, v) \in T_x\overline{O}^n$. We usually canonically identify $T_x\overline{O}^n \cong \mathbb{R}^n \ni v$. Let f be a function defined in $\overline{O}^n$ in the neighbourhood x and differentiable in x. Denote the derivative of f in the direction of v at point x by four equivalent notations

  $$\partial_{(x,v)} f = \partial_v|_x f = (\partial_v f)(x) \coloneqq \mathrm{d}f_x(v).$$

  Then, using the linearity of $\mathrm{d}f_x$, we can write

  $$(\partial_v f) = \mathrm{d}f_x(v) = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}(x) = \sum_{i=1}^n v_i (\partial_{e_i} f)(x) = \sum_{i=1}^n v_i \mathrm{d}f_x(e_i),$$

  where (e_i) is the standard basis of $\mathbb{R}^n$. Another useful notation for the standard basis of $T_x\overline{O}^n$ is

  $$\frac{\partial f}{\partial x_i}(x) \coloneqq \left.\frac{\partial}{\partial x_i}\right|_x f.$$

  If we are in the interior O^n (and sometimes also on the boundary), we can also write

  $$(x, e_i) \leftrightarrow [t \to x + te_i] \leftrightarrow \partial_{e_i}|_x = \left.\frac{\partial}{\partial x_i}\right|_x.$$

  There is a correspondence between geometric and analytic descriptions of tangent vectors: for 1-jets $[c]$, we have (cf. earlier computation in the discussion of differentiability), using the chain rule,

  $$\partial_{[c]} f = \mathrm{d}f_{[c]}(\dot{c}(0)) = (f \mathbin{\dot{\circ}} c)(0).$$

  What other properties of a derivative in the direction of the tangent vector do we need to say that it is indeed the directional derivative? The 1st order differential operator on functions which are defined near x and differentiable in x should satisfy the following properties:

– it is $\mathbb{R}$-linear,

– it satisfies the *Leibniz rule*:

$$(\partial_v(fg))(x) = (\partial_v f)(x) \cdot g(x) + f(x) \cdot (\partial_v g)(x)$$

An operator that satisfies these two properties is called *derivation*. These properties can be used to algebraically characterize directional derivatives among operators on $\mathcal{C}^\infty$-functions defined near x. This analytic viewpoint leads to the algebraic definition of tangent vectors, which generalizes to polynomial/rational functions over arbitrary ground fields. This is useful in algebraic geometry. For details, see Frank Warner's book on Foundations of Differentiable Manifolds and Lie Groups.

The advantages of tangential vectors as differential operators are

– linear structure of tangent spaces becomes apparent,

– works up to the boundary.

The deficit is that the algebraic characterization of directional derivatives becomes messy in the $\mathcal{C}^{k<\infty}$ case.

Next, we will study the effect of differentiable maps on tangent vectors. Let F be a $\mathcal{C}^k$-map between open subsets of the local models:

$$\overline{O}^n \supset \overset{\text{rel. open}}{U} \xrightarrow[\mathcal{C}^k]{F} \overset{\text{rel. open}}{V} \subset \overline{O}^m.$$

Regard the differentials in points $x \in U$ as linear maps of tangent spaces:

$$\begin{array}{ccc} T_xU & \xrightarrow[\mathrm{d}F_x]{} & T_{F(x)}V \\ \| & & \| \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \end{array}$$

with respect to the standard basis given by the Jacobi matrices

$$\mathrm{d}F_x \frac{\partial}{\partial x_j}\bigg|_x = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_j}(x) \frac{\partial}{\partial y_i}\bigg|_{F(x)}$$

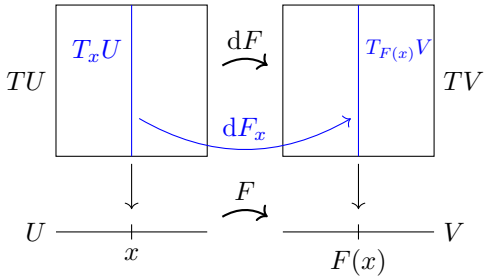
for $x \in \mathring{U}$. The differentials can be combined in all $x \in U$ to the whole differential of F:

$$\begin{aligned} U \times \mathbb{R}^n \cong TU &\xrightarrow{\mathrm{d}F} TV \cong V \times \mathbb{R}^m \\ (x, v) &\mapsto (F(x), \mathrm{d}F_x(v)). \end{aligned}$$

Remark.

- The differential of F preserves bundle structure:

$$\pi_{TV} \circ \mathrm{d}F = F \circ \pi_{TU}.$$



- It is linear on fibres.

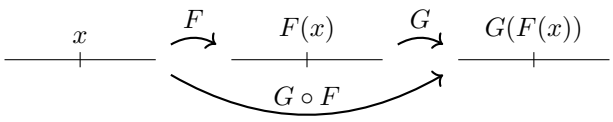
Note. If F is a $\mathcal{C}^k$-function, the differential is one order less smooth, i.e. $\mathrm{d}F$ is $\mathcal{C}^{k-1}$.

The coordinate representation of $\mathrm{d}F$ involves partial derivatives:

$$(x_1,,,x_n,v_1,,,v_n) \mapsto \left(F_1(x_1,,,x_n),,,F_m(x_1,,,x_n), \sum_{j=1}^{n} \frac{\partial F_1}{\partial x_j}(x_1,,,x_n),,, \sum_{j=1}^{n} \frac{\partial F_m}{\partial x_j}(x_1,,,x_n) \right).$$

Lastly, we mention the chain rule for differentials:

$$\mathrm{d}(G \circ F) = \mathrm{d}G \circ \mathrm{d}F.$$



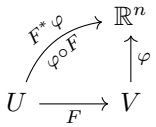
Representation of differential in terms of 1-jets is

$$\mathrm{d}F_{c(0)}(\dot{c}(0)) = (F \dot{\circ} c)(0)$$

or more compactly as $\mathrm{d}F([c]) = [F \circ c]$. In terms of directional derivatives, the chain rule reads

$$\partial_{\mathrm{d}F_x(v)}|_{F(x)}\varphi = \mathrm{d}\varphi_{F(x)}(\mathrm{d}F_x(v)) = (\mathrm{d}\varphi_{F(x)} \circ \mathrm{d}F_x)(v) = \mathrm{d}(\varphi \circ F)_x(v) = \partial_v|_x(\varphi \circ F) = \partial_v|_x F^*\varphi.$$

Compactly we write $\partial_{\mathrm{d}F_x(v)}|_{F(x)} = \partial_v|_x \circ F^*$, where F^* is the *pullback* of functions by F. A pullback is just a handy construction that allows us to simplify the notation: $\varphi \circ F = F^*\varphi$.



For a deeper explanation of the existence and uniqueness of solutions of ODEs, see Vladimir Arnold's book on Mathematical Methods of Classical Mechanics. The professor allegedly met Arnold in person but strenuously avoided answering whether he was a nice guy.

II.2 Tangent bundle of differential manifolds

Just as a manifold can be recovered from gluing the images of its charts (open subsets of (rel.) open model) using the coordinate changes as gluing maps, we will now build in the differentiable

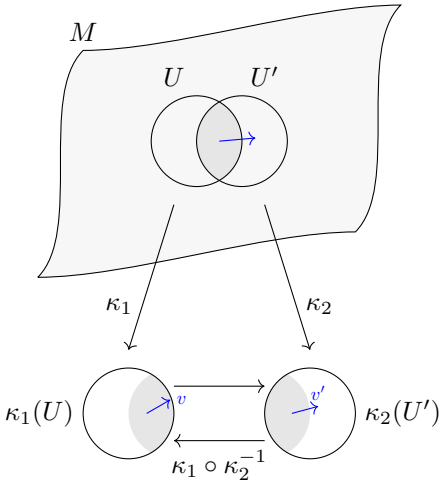


Figure II.1: Coordinate change of tangent vectors

case its tangent bundle by gluing the tangent bundles of the images of the charts (open *saturated subsets* of the tangent bundle of the local model) using as gluing maps the differentials of the coordinate changes. The local model for tangent bundles of manifolds with corners is

$$T\overline{O}^n \cong \overline{O}^n \times \mathbb{R}^n \subset \mathbb{R}^{2n}.$$

Let's start with a $\mathcal{C}^k$-manifold M^m with corners (see Fig. II.1):

$$TM := \bigsqcup_{(U,\kappa)\in\mathcal{A}} T\kappa(U)/\sim$$

with the equivalence relation

$$(x,v)\sim(x',v') \iff \begin{cases} x\in\kappa(U\cap U'), \quad x'\in\kappa'(U\cap U'),\\ (\kappa'\circ\kappa^{-1})(x)=x',\\ \mathrm{d}(\kappa'\circ\kappa^{-1})_x(v)=v'. \end{cases}$$

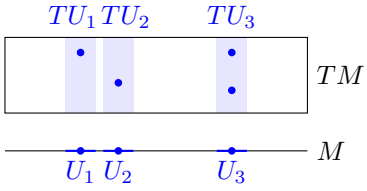
Some call this the "physicist point of view," but the professor doesn't like that name. Note that $T\kappa(U)\cong\kappa(U)\times\mathbb{R}^n$ and $\kappa(U)$ is relatively open in $\overline{O}^n$. Next, we will put geometric and algebraic structures on TM. For any chart (U,κ) of M define $TU\subset TM$ as the image of the canonical injection $T\kappa(U)\hookrightarrow TM$.

$$\begin{array}{ccccccc} T\overline{O}^n & \supset & \overset{\text{(rel.) open}}{T\kappa(U)} & \xrightarrow{1:1} & TU & \subset & TM \\ \wr\| & & \wr\| & \underset{:=T\kappa}{\overset{\text{inverse}}{\leftarrow}} & & & \\ \overline{O}^n\times\mathbb{R}^n & \supset & \kappa(U)\times\mathbb{R}^n & & & & \end{array}$$

The coordinate changes of tangent vectors are

$$T\kappa(U)\supset \overset{\text{(rel.) open}}{T\kappa(U\cap U')} \xrightarrow{T\kappa'\circ(T\kappa)^{-1}} \overset{\text{(rel.) open}}{T\kappa'(U\cap U')} \subset T\kappa'(U'),$$

where, by construction, $T\kappa' \circ (T\kappa)^{-1} = \mathrm{d}(\kappa' \circ \kappa^{-1}) \in \mathcal{C}^{k-1}$. This establishes the geometric structure on TM as a $2n$-dimensional $\mathcal{C}^{k-1}$-manifold with corners. We should check the second countability and the Hausdorff property of TM. The second countability is clear, and the Hausdorff property is easy to check (see figure below).



Remark. The induced bundle charts become $\mathcal{C}^{k-1}$-diffeomorphisms.

Next, we will put an algebraic structure on TM.

- The **footpoint projection** of the actual tangent bundle is

$$\begin{aligned} TM &\xrightarrow{\pi_{TM}} M \\ (T\kappa)^{-1}(x,v) &\mapsto \kappa^{-1}(x) \end{aligned}$$

  where $(x,v) \in T\kappa(U) \cong \kappa(U) \times \mathbb{R}^n$, $x \in \kappa(U)$, $(T\kappa)^{-1}(x,v) \in TU$ and $\kappa^{-1}(x) \in U$, is well-defined. In the local coordinates, we can write the following diagram:

$$\begin{array}{ccccc} TM \supset TU & & \xrightarrow{\pi_{TM}|_{TU}} & & U \subset M \\ T\kappa \downarrow & & & & \downarrow \kappa \\ T\kappa(U) & & \xrightarrow[\mathcal{C}^{k-1}\text{-submersion}]{\pi|_{T\kappa(U)}} & & \kappa(U) \\ \| & & & & \| \\ \kappa(U) \times \mathbb{R}^n & & \longrightarrow & & \kappa(U) \subset \overline{O}^n \end{array}$$

  Hence, the projection π_{TM} is a $\mathcal{C}^{k-1}$-submserion.

- **Tangent spaces**: $T_pM := \pi_{TM}^{-1}(p)$ for $p \in M$ arises from identifying the tangent spaces $T_{\kappa(p)}\kappa(U) \cong \mathbb{R}^n$ for all charts around p by the differentials of the coordinate changes:

$$\begin{array}{ccc} & T_pM & \\ {\scriptstyle T\kappa|_{T_pM}} \swarrow & & \searrow {\scriptstyle T\kappa'|_{T_pM}} \\ T_{\kappa'(p)}\kappa(U) & \xrightarrow[\mathrm{d}(\kappa' \circ \kappa^{-1})_{\kappa(p)}]{\cong} & T_{\kappa'(p)}\kappa'(U') \end{array}$$

  and this is consistent with the chain rule. The tangent spaces inherit a linear structure, making the $T\kappa|_{T_pM}$ linear isomorphisms.

This is the algebraic part of the structure. Next, we will look at the differentials of maps between manifolds. Consider a differentiable map $M^m \xrightarrow[\mathcal{C}^k]{F} N^n$. Its differential can be written as $TM \xrightarrow{\mathrm{d}F} TN$ and is defined locally as (see Fig. II.2):

$$\begin{array}{ccccc} TM & \supset & T(U\cap F^{-1}(\tilde{U})) & \xrightarrow{\mathrm{d}F|_{U\cap F^{-1}(\tilde{U})}} & T\tilde{U} \subset TN \\ & & \downarrow T\kappa & & \downarrow T\tilde{\kappa} \\ T\overline{O}^m & \supset & T\kappa(U\cap F^{-1}(\tilde{U})) & \xrightarrow{\mathrm{d}(\tilde{\kappa}\circ F\circ\kappa^{-1})} & T\tilde{\kappa}(\tilde{U}) \subset T\overline{O}^n \\ & & \wr\| & & \wr\| \\ & & \kappa(U\cap F^{-1}(\tilde{U}))\times\mathbb{R}^m & \longrightarrow & \tilde{\kappa}(\tilde{U})\times\mathbb{R}^n \end{array}$$

$$(x,v)\longmapsto((\tilde{\kappa}\circ F\circ\kappa^{-1})(x),\mathrm{d}(\tilde{\kappa}\circ F\circ\kappa^{-1})_x(v))$$

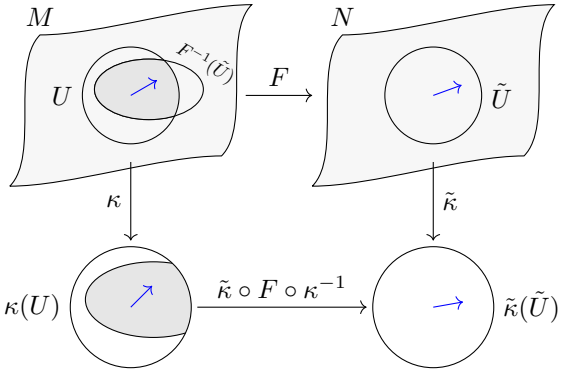


Figure II.2: Local coordinate representation of the differential of a map F.

The chain rule is well-defined since

$$\tilde{\kappa}'\circ F\circ\kappa'^{-1}=(\tilde{\kappa}'\circ\tilde{\kappa}^{-1})\circ(\tilde{\kappa}\circ F\circ\kappa^{-1})\circ(\kappa\circ\kappa'^{-1}).$$

The differential $\mathrm{d}F$ is a $\mathcal{C}^{k-1}$-map and is linear on tangent spaces:

$$T_pM\xrightarrow{\mathrm{d}F_p}T_{F(p)}N$$

is the differential of F in p. It also preserves the bundle structure (no dependence on the footpoint):

$$\begin{array}{ccc} TM & \xrightarrow{\mathrm{d}F} & TN \\ \downarrow{\scriptstyle\pi_{TM}} & & \downarrow{\scriptstyle\pi_{TN}} \\ M & \xrightarrow{F} & N \end{array}$$

Note. By construction, it holds $T\kappa=\mathrm{d}\kappa$ for charts (U,κ) of M.

More explicitly in local coordinates, the chart (U,x) for M is associated with the *bundle chart* (TU,Tx) for TM:

$$\begin{aligned} TU &\xrightarrow{Tx} x(U)\times\mathbb{R}^n \\ \sum_{i=1}^n\xi_i\frac{\partial}{\partial x_i}\Big|_p &\mapsto(x(p),(\xi_1,\ldots,\xi_n)). \end{aligned}$$

The set of partial derivatives $\frac{\partial}{\partial x_i}\big|_p$ is the *standard basis* of T_pM with respect to the chart x. Next, we introduce a more suggestive notation for coordinate changes on M (see Fig. II.3):

$$\tilde{x}(x) = \tilde{x} \circ x^{-1}.$$

The change of associated standard basis on the intersection between two different charts $U \cap \tilde{U}$ is

$$\frac{\partial}{\partial x_j}\bigg|_p = \sum_{i=1}^{n} \frac{\partial \tilde{x}_i}{\partial x_j}(x(p)) \frac{\partial}{\partial \tilde{x}_i}\bigg|_p.$$

The transformation of the coefficients can then be written as

$$\sum_{j=1}^{m} \xi_j \frac{\partial}{\partial x_j}\bigg|_p = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \xi_j \frac{\partial \tilde{x}_i}{\partial x_j}(x(p)) \right) \frac{\partial}{\partial \tilde{x}_i}\bigg|_p$$

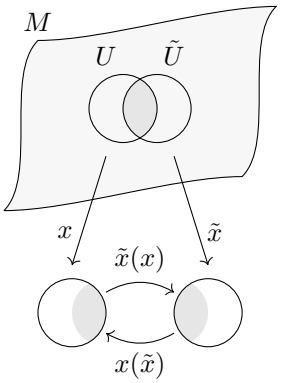


Figure II.3: Suggestive notation for coordinate changes on M.

and bundle coordinate changes as

$$x(U \cap \tilde{U}) \times \mathbb{R}^n \xrightarrow{T\tilde{x} \circ (Tx)^{-1}} \tilde{x}(U \cap \tilde{U}) \times \mathbb{R}^n$$
$$(x, \xi) \mapsto (\tilde{x}(x), \tilde{\xi}(x, \xi)),$$

where

$$\tilde{\xi}_i(x, \xi) = \sum_{j=1}^{n} \xi_j \frac{\partial \tilde{x}_i}{\partial x_j}(x).$$

Next, we can write the change of the standard basis on $U \cap F^{-1}(V)$ under differentials of maps (see Fig. II.4) $M^m \xrightarrow{F} N^n$:

$$\mathrm{d}F_p \left(\frac{\partial}{\partial x_j}\bigg|_p \right) = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_j}(x(p)) \frac{\partial}{\partial y_i}\bigg|_{F(p)}.$$

For general tangent vectors, we have

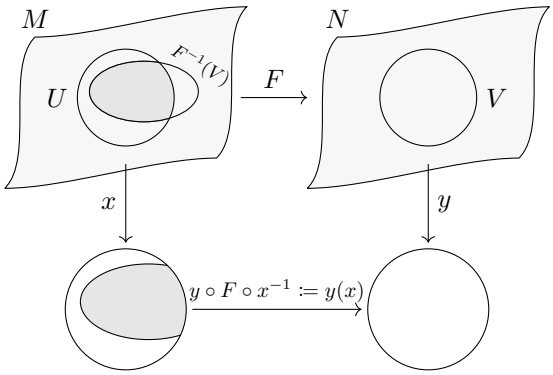


Figure II.4: Local coordinate representation of the differential of a map F.

$$\mathrm{d}F_p\left(\sum_{j=1}^{m}\xi_j\left.\frac{\partial}{\partial x_j}\right|_p\right)=\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\xi_j\frac{\partial y_i}{\partial x_j}(x(p))\right)\left.\frac{\partial}{\partial y_i}\right|_{F(p)}.$$

The differential in local bundle coordinates is

$$\begin{aligned}x(U\cap F^{-1}(V))\times\mathbb{R}^m &\to y(V)\times\mathbb{R}^n\\ (x,\xi)&\mapsto (y(x),\eta(x,\xi))\end{aligned}$$

with

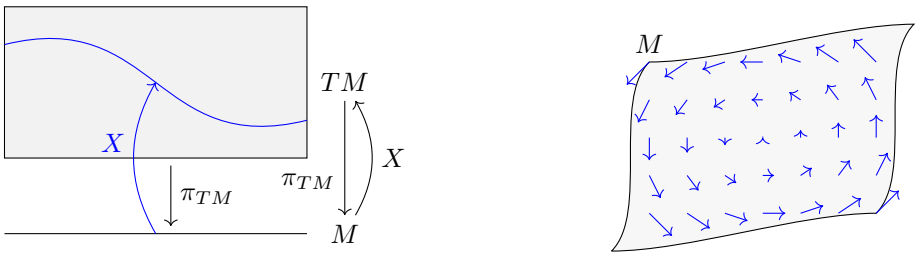
$$\eta_i(x,\xi)=\sum_{j=1}^{m}\xi_j\frac{\partial y_i}{\partial x_j}(x).$$

II.3 Vector fields

Let M be a $\mathcal{C}^k$-manifold with corners.

Definition II.1. A *vector field* X on M is a *section* of the tangent bundle of M, i.e. a map $M \xrightarrow{X} TM$ such that (see Fig. II.5a)

$$\pi_{TM}\circ X=\mathrm{id}_M.$$



(a) Representation of a vector field as a section of the tangent bundle TM.

(b) Representation of a vector field as a vector at each point of M.

Figure II.5: Two representations of a vector field.

This means that $X(p)\in T_pM$ for all $p\in M$. Geometrically, it means that a vector field is a family of tangent vectors at all points (Fig. II.5b). Analytically, a vector field corresponds to a first-order differential operator. For a differentiable function f on M, one obtains by point-wise taking directional derivatives the function

$$Xf\coloneqq\pi_{T\mathbb{R}}\circ\mathrm{d}f\circ X.$$

$$\begin{array}{ccc} TM & \xrightarrow{\mathrm{d}f} & T\mathbb{R} \\ {\scriptstyle X}\uparrow\downarrow{\scriptstyle \pi_{TM}} & & \downarrow{\scriptstyle \pi_{T\mathbb{R}}} \\ M & \xrightarrow{f} & \mathbb{R}. \end{array}$$

Writing it out more explicitly,

$$(Xf)(p)=\mathrm{d}f_p(X(p))\in T_{f(p)}\mathbb{R}\cong\mathbb{R}.$$

It is called a *derivative* of f in the direction of v on the field X. The function $f \mapsto Xf$ is $\mathbb{R}$-linear and satisfies the *product rule*

$$X(fg) = (Xf)g + f(Xg),$$

where f and g are differentiable. In particular, if X is $\mathcal{C}^{k-1}$ (as a map), we obtain the $\mathbb{R}$-linear first order differential operator $\mathcal{C}^k(M) \xrightarrow{X} \mathcal{C}^{k-1}(M)$ satisfying the product rule. Here, $\mathcal{C}^k(M)$ is an $\mathbb{R}$-algebra of $\mathcal{C}^k$-functions.

Remark. In the $\mathcal{C}^\infty$ case, these are *derivations* of the $\mathbb{R}$-algebra $\mathcal{C}^\infty(M)$ and one can show that every derivation comes from a $\mathcal{C}^\infty$-vector field.

In local coordinates (see Fig. II.6), the chart (U, x) vector field to local coordinate vector fields

$$\frac{\partial}{\partial x_i} : p \mapsto \left.\frac{\partial}{\partial x_i}\right|_p$$

on U. The vector filed X on M can be written locally on U as

$$X|_U = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i},$$

where ξ_i are coefficient functions on U.

$$\begin{array}{ccc} M \supset U & \xrightarrow{\quad X|_U \quad} & TU \subset TM \\ X \downarrow \cong & & \cong \downarrow Tx \\ X(U) & \xrightarrow[\text{loc. coo. rep. of } X]{} & X(U) \times \mathbb{R}^n \end{array}$$

$$X(p) \longmapsto (x(p), \xi(p))$$

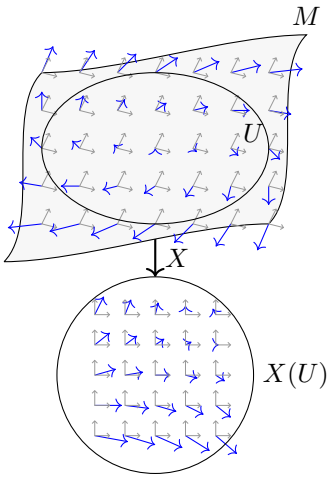


Figure II.6: Local representation of vector field.

The dimension of manifold M is denoted by n. The function $\xi(p)$ can also be expressed as $(\xi \circ X^{-1})(X(p))$, where it is more explicit that $\xi \circ X^{-1}$ is the local coordinate representation of coefficient functions. We see that X is $\mathcal{C}^l$ on U if and only if ξ_i are $\mathcal{C}^l$ for every i. We denote

$$\Gamma^l(TM) = \mathscr{C}^l(M, TM) \coloneqq \{\mathcal{C}^l\text{-vector fields } M \to TM.\}$$

Algebraically speaking, this is a $\mathcal{C}^l(M)$-module

$$(\varphi X)(p) \coloneqq \varphi(p) X(p)$$

point-wise.

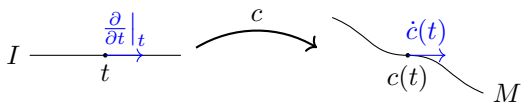
Now, we will go back to the geometric view of tangent vectors as velocities (see Fig. II.5b). Consider "motion" in the direction of the vector field, which describes a dynamic system. Let M be a $\mathcal{C}^k$-manifold with corners.

Definition II.2. An *integral curve* or *trajectory* or *flow line* on a vector field X on M is a curve

$$\mathbb{R} \supset \overset{\text{open}}{I} \xrightarrow[\text{diff}]{c} M$$

satisfying the ODE

$$\dot{c} = X \circ c. \tag{II.1}$$



Let's take a closer look at Eq. (II.1): $\dot{c}(t) = X(c(t))$. The left side of the equation is

$$\mathrm{d}c_t \frac{\partial}{\partial t}\bigg|_{\mathbf{t}} \in T_{c(t)}M,$$

where $\mathbf{t}$ represents the standard basis vector and $\frac{\partial}{\partial t}\big|_{\mathbf{t}} \in T_t\mathbb{R}$. In local coordinates (Fig. II.7) we can write

$$\dot{c} = \sum_{i=1}^{n} (\dot{x_i \circ c}) \left(\frac{\partial}{\partial x_i} \circ c \right),$$

so Eq. (II.1) can be equivalently written as

$$\dot{\gamma_i} = (\dot{x_i \circ c}) = \xi_i \circ c = (\xi_i \circ x^{-1}) \circ (x \circ c) = g_i \circ \gamma.$$

The symbol g_i is the local coordinate representation of coefficient functions and γ is the local

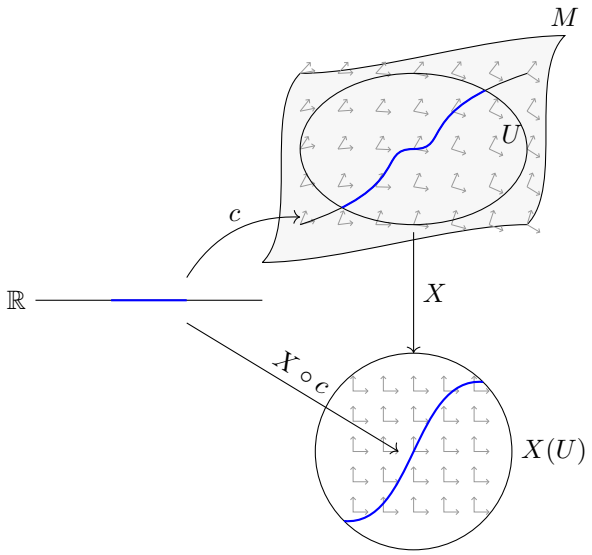


Figure II.7: The trajectory c maps an interval on the real line to the manifold M. The map $c \circ X$ maps a subinterval to the local model.

coordinate representation of a curve. More compactly:

$$\dot{\gamma_i} = g_i(\gamma_1, \ldots, \gamma_n), \quad i \in \{1, \ldots, n\}.$$

This is a system of first-order ODEs.

II.4 Some local ODE theory (digression)

Let X be locally Lipschitz continuous vector field on an open interval $U \subset \mathbb{R}^n$. Being locally Lipschitz is implied by $\mathcal{C}^1$. Consider *initial value problem* (IVP):

$$\begin{aligned} \dot{c} &= X \circ c \\ c(0) &= x_0 \end{aligned}$$

for $x_0 \in U$. The initial condition is of zeroth order because the differential equation is of first order. Our first aim is to find the local existence and uniqueness of solutions (commonly known as *Picard-Lindelöf theorem*).

Remark. The regularity assumption "locally Lipschitz" on the vector field makes precise the physical intuition that solutions of equations of motion with given initial conditions should exist and be unique (determinism).

The counterexample for the uniqueness would be in the α-Hölder case for $0 < \alpha < 1$, where solution branching may occur.

Remark. A function is *α-Hölder continuous* (or satisfies Hölder condition) if there exist $C, \alpha > 0$ such that

$$|f(x) - f(y)| \leq C\|x - y\|^\alpha$$

for all x and y. The continuity of functions can be sorted hierarchically as

$$\text{cotinuously differentiable} \subset \text{Lipschitz continuous} \subset \alpha\text{-Hölder continuous}$$

for $\alpha \in (0, 1)$ and also

$$\text{Lipschitz continuous} \subset \text{absolutely continuous} \subset \text{uniformly continuous} \subset \text{continuous}.$$

It is an exercise for the reader to find a function for each class that is not included in one class lower.

An example of a differential equation that is not Lipschitz would be $\mathrm{d}y/\mathrm{d}x = 3y^{2/3}$ with the initial condition $y(0) = 0$. The reader can check (and correct me if I am wrong) that possible solutions are:

$$y(x) = \begin{cases} (x+a)^3, & x \leq -a; \\ 0, & -a < x < b; \\ (x-b)^3, & x \geq b; \end{cases}$$

for any $a, b \geq 0$.

Since we work locally, we may assume for X without loss of generality that

- it is *bounded*, $\|X\| \leq C$
- (globally) *Lipschitz*, $\|X(x_1) - X(x_2)\| \leq L\|x_1 - x_2\|$, $\forall x_1, x_2 \in U$,

for some constants $C, L \in \mathbb{R}$. That also means that we should expect that the curve stays in a ball $B_r(x_0) \subset U$ at least for time $\varepsilon_0 := r/C$.

The approach to finding the full solution is to successively improve "approximate" solutions in $\overline{B}_r(x_0) \in U$:

$$\begin{array}{c} c : (-\varepsilon_0, \varepsilon_0) \xrightarrow{\mathcal{C}^0} B_r(x_0) \\ 0 \mapsto x_0 \\ \downarrow \\ \tilde{c} : (-\varepsilon_0, \varepsilon_0) \xrightarrow{\mathcal{C}^1} B_r(x_0) \\ 0 \mapsto x_0 \end{array}$$

The new curve is moving as "predicted by" or "expected" by c: $\tilde{x} = X \circ c$. This equation denotes the velocity which c should have, and it is integrated to

$$\tilde{c}(t) = x_0 + \int_0^t X(c(\tau))\,\mathrm{d}\tau.$$

We define the integral operator I on

$$S \coloneqq \{(-\varepsilon_0 \to \varepsilon_0) \xrightarrow[\mathcal{C}^0]{c} \overline{B}_r(x_0)\}.$$

This is a complete metric space with respect to *supremum norm*

$$\|c_1 - c_2\|_\infty \coloneqq \sup_t \|c_1(t) - c_2(t)\|.$$

Note that the initial condition is unnecessary here. The operator $I : S \to S$ is defined by

$$(Ic)(t) = x_0 + \int_0^t X(c(\tau))\,\mathrm{d}\tau.$$

It stays within open ball $B_r(x_0)$. Then c is a *fixed point* of I if and only if $c \in \mathcal{C}^1$ and solves the initial value problem. Since we want to use the Banach fixed point theorem, we want to first check, whether I is a contraction. Let $c_1, c_2 \in S$. We can estimate

$$(Ic_1)(t) - (Ic_2)(t) = \int_0^t (X(c_1(\tau)) - X(c_2(\tau)))\,\mathrm{d}\tau$$

and

$$\|(Ic_1)(t) - (Ic_2)(t)\|_\infty \leq \left| \int_0^t \|X(c_1(\tau)) - X(c_2(\tau))\|\,\mathrm{d}\tau \right| \leq \left| \int_0^t L\|c_2(\tau) - c_1(\tau)\|\,\mathrm{d}\tau \right|.$$

Finally, we can conclude

$$\|Ic_1 - Ic_2\|_\infty \leq \varepsilon_0 L \|c_1 - c_2\|.$$

In order for I to be a contraction, ε_0 should be chosen such that $\varepsilon_0 L < 1$. Now that I is a strong contraction and since S is complete, we can use the *Banach fixed point theorem*. (The achievement of this theorem is not its complexity but the recognition of the usefulness of the elementary fact.) Hence, the initial value problem has a unique solution for $\varepsilon_0 < \min(r/C, 1/L)$. But the $1/L$ bound is only a temporary artificial bound that we need for the contraction. We can remove this bound by the local-to-global gluing procedure. There are several consequences of the existence and uniqueness of the global solution:

- For every compact K we have a guaranteed lifetime $\varepsilon_0(K) > 0$ for local solutions with the starting point in K.
- These local solutions can be extended (inductively) forwards and backwards for the unique *maximal* global solution of the initial value problem on U (considering X is only locally Lipschitz):

$$\begin{aligned} 0 \in (\alpha(x_0), \omega(x_0)) &\xrightarrow[\mathcal{C}^1]{c_{x_0}} U, \\ 0 &\mapsto x_0, \end{aligned}$$

  where $-\infty \leq \alpha(x_0) < 0 < \omega(x_0) \leq \infty$. The time $\alpha(x_0)$ is called the *birth time* (the choice of letters α and ω is no coincidence). If the birth time is finite $\alpha(x_0) > -\infty$, then $c(t)$ leaves every compact eventually as $t \searrow \alpha$. If $\omega(x_0) < \infty$ then $c(t)$ leaves every compact eventually as $t \nearrow \omega$. In other words, maximality and uniqueness imply that any solution to the IVP is a restriction of c_{x_0} to a subinterval containing 0.

The next aim is to check if there is a *continuous dependence* of the solution on initial conditions. Let's again work within the ball $\overline{B}_r(x_0) \in U$. Consider solutions with the initial starting point in a smaller ball $\overline{B}_{r/2}(x_0)$. These solutions exist up to time $\pm\varepsilon_0 \coloneqq \pm\frac{r}{2C}$ and remain in $B_r(x_0)$. We want to estimate how fast any two solutions can diverge. Take two solutions

$$(-\varepsilon_0, \varepsilon_0) \xrightarrow{c_1} B_r(x_0), \quad (-\varepsilon_0, \varepsilon_0) \xrightarrow{c_2} B_r(x_0)$$

with initial conditions $x_1, x_2 \in B_{r/2}(x_0)$. We can estimate the relative velocity of the two solutions

$$\begin{aligned} \dot{c}_1(t) - \dot{c}_2(t) &= X(c_1(t)) - X(c_2(t)) \\ \|\dot{c}_1 - \dot{c}_2\| &\leq L\|c_1 - c_2\| \quad \text{(point-wise)} \end{aligned}$$

This suggests at most exponential growth at rate L for the deviation $\|c_1 - c_2\| \coloneqq u$. For $t \geq 0$ (and equivalently for $t \leq 0$), we have

$$\begin{aligned} \|c_1(t) - c_2(t)\| - \|c_1(0) - c_2(0)\| &\leq \|(c_1 - c_2)_t - (c_1 - c_2)_0\| \\ u(t) - u(0) &\leq \|(c_1 - c_2)_t - (c_1 - c_2)_0\| \\ u(t) - u(0) &\leq \left\| \int_0^t (\dot{c}_1(\tau) - \dot{c}_2(\tau)) \,\mathrm{d}\tau \right\| \\ u(t) - u(0) &\leq \int_0^t \|\dot{c}_1(\tau) - \dot{c}_2(\tau)\| \,\mathrm{d}\tau \\ u(t) - u(0) &\leq L \int_0^t u(\tau) \,\mathrm{d}\tau \\ u(t) &\leq u(0) + L \int_0^t u(\tau) \,\mathrm{d}\tau \coloneqq U(t). \end{aligned}$$

This is an upper bound for deviation of solutions. They satisfy the differential inequality $\dot{U} \leq LU$. If $U > 0$, it follows (in both directions)

$$U(t) \leq e^{L|t|} U(0)$$

for $-\varepsilon_0 < t < \varepsilon_0$. If we relax the condition to $U \geq 0$, we modify the argument slightly. For any $\eta > 0$ it holds $(\eta + U)^{\cdot} \leq L(\eta + U)$, so

$$U(t) \leq e^{L|t|}(\eta + U(0)) - \eta.$$

By sending $\eta \to 0$, we get the same estimate as before.

Combine now the local solutions up to time $\pm\varepsilon_0$ with initial point in $B_{r/2}(x_0)$ to the *local flow* of X:

$$\begin{aligned} B_{r/2}(x_0) \times (-\varepsilon_0, \varepsilon_0) &\xrightarrow{\Phi} B_r(x_0), \\ (x,t) &\mapsto c_x(t) = \Phi(x,t) = \Phi_t(x). \end{aligned}$$

By the above,

$$\|\Phi(x_1,t) - \Phi(x_2,t)\| \le e^{L|\varepsilon_0|}\|x_1 - x_2\|.$$

This shows that the flow is Lipschitz in the x variable. Since the differential equation takes the form

$$\frac{\partial \Phi}{\partial t} = X \circ \Phi$$

and $\|X\| \le C$ on $B_r(x_0)$, we also have this estimate

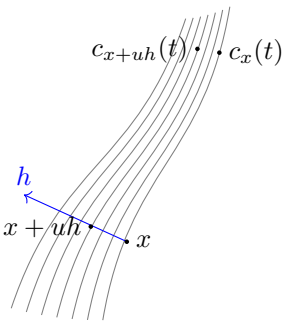
$$\|\Phi(x,t_1) - \Phi(x,t_2)\| \le C|t_1 - t_2|.$$

So the flow is Lipschitz in t as well. Hence, local flow is Lipschitz in both variables.

Next, we will check the *differentiable dependence* on initial conditions. Suppose now $X \in \mathcal{C}^1$ in case of a differentiable flow vector field. We want to show that the local flow is differentiable in x, i.e. that

$$(\partial_{(h,0)}\Phi)(x) = \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} c_{x+uh}(t) = \lim_{h\to 0} \frac{c_{x+uh}(t) - c_x(t)}{u}$$

exists for $h \in \mathbb{R}^n$. One should think of $(c_{x+uh}(t) - c_x(t))/u$ as a vector field along c_x. We will deduce the convergence of the functions $(c_{x+uh} - c_x)/u$ as $u \to 0$ from the fact that they satisfy the "convergent" family of (homogenous) linear ODEs. We thereby reduce our task to the question of the stability of solutions on linear ODEs. Now, we will use the assumption that X is $\mathcal{C}^1$ to examine more closely the deviation of trajectories and establish an ODE for it.



Let $x_1, x_2 \in B_{r/2}(x_0)$ and $-\varepsilon_0 < t < \varepsilon_0$ for $\varepsilon_0 = r/(2C)$. The estimation is

$$\begin{aligned} \dot{c}_{x_2}(t) - \dot{c}_{x_1}(t) &= X(c_{x_2}(t_2)) - X(c_{x_1}(t)) \\ &= \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}u} X((1-u)c_{x_1}(t) + uc_{x_2}(t))\right) \mathrm{d}u \\ &= \underbrace{\left(\int_0^1 \mathrm{d}X_{(1-u)c_{x_1}(t)+uc_{x_2}(t)}\mathrm{d}u\right)}_{\mathrm{End}(\mathbb{R}^n)} \underbrace{(c_{x_2}(t) - c_{x_1}(t))}_{\mathbb{R}^n}. \end{aligned}$$

For $x, x' \in B_r(x_0)$ define

$$A(x,x') = \int_0^1 \mathrm{d}X_{(1-u)x+ux'}\,\mathrm{d}u$$

which is an *endomorphism* field along the line segment $ux + (1-u)x'$ and is uniformly bounded. In particular, $A(x,x) = \mathrm{d}X_x$. Then $c_{x_2} - c_{x_1}$ satisfies the homogenous ODE

$$\dot{(c_{x_2} - c_{x_1})} = A(c_{x_1}, c_{x_2}) \cdot (c_{x_2} - c_{x_1}).$$

Hence, the difference quotients of trajectories solve the IVP:

$$\begin{cases} \dot{\left(\frac{c_{x+uh}-c_x}{u}\right)} = A(c_x, c_{x+uh}) \cdot \left(\frac{c_{x+uh}-c_x}{u}\right), \\ \frac{c_{x+uh}(0)-c_x(0)}{u} = h. \end{cases} \tag{II.2}$$

We want to use the stability of solutions of such linear ODEs under variations of coefficients to show that $(c_{x+uh} - c_x)/u$ converges uniformly as $u \to 0$ to a solution of the IVP

$$\begin{cases} \dot{V} = \mathrm{d}X_{c_x} \cdot V \\ V(0) = h. \end{cases}$$

This is a linearization of ODE $\dot{c} = X \circ c$ along c_x. We then know that the variational vector field

$$\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} c_{x+uh} = (\partial_{h,0}\Phi)(x, \bullet)$$

exists and satisfies an ODE.

II.4.1 Interlude into stability of solutions of linear ODEs

Suppose $(-\varepsilon_0, \varepsilon_0) \overset{v}{\underset{w}{\rightrightarrows}} \mathbb{R}^n$ are $\mathcal{C}^1$ and bounded functions that satisfy ODEs

$$\begin{cases} \dot{v} = Av, \\ \dot{w} = Bw, \end{cases}$$

where $(-\varepsilon_0, \varepsilon_0) \overset{A}{\underset{B}{\rightrightarrows}} \operatorname{End}\mathbb{R}^n$ are $\mathcal{C}^0$ and bounded. The aim is to estimate the deviation $\|v - w\|_\infty$ in terms of $\|v(0)\|$, $\|A - B\|_\infty$ and initial deviation $\|v(0) - w(0)\|$. We have

$$\begin{aligned} \|v(t) - w(t)\| &\le \|v(0) - w(0)\| + \|v(t) - w(t) - v(0) + w(0)\| \\ &\le \|v(0) - w(0)\| + \left\| \int_0^t (\dot{v} - \dot{w}) \, \mathrm{d}\tau \right\| \\ &\le \|v(0) - w(0)\| + \int_0^t \|(\dot{v} - \dot{w})\| \, \mathrm{d}\tau \\ &\le \|v(0) - w(0)\| + \int_0^t \|(Av - Bw)\| \, \mathrm{d}\tau \\ &\le \|v(0) - w(0)\| + \int_0^t (\|A - B\| \cdot \|v\| + \|B\| \cdot \|v - w\|) \, \mathrm{d}\tau \\ &\le \|v(0) - w(0)\| + \varepsilon_0 \|A - B\|_\infty \cdot \|v\|_\infty + \|B\|_\infty \cdot \int_0^t \|v - w\| \, \mathrm{d}\tau. \end{aligned}$$

Norms on B and $A-B$ in the second to last row are point-wise *operator norms* ($\|Bx\| \le \|B\| \, \|x\|$, also $\|B\| \le \|B\|_\infty$). The last row is denoted by

$$\mathrm{d}(t) \coloneqq \|v(0) - w(0)\| + \varepsilon_0 \|A - B\|_\infty \cdot \|v\|_\infty + \|B\|_\infty \cdot \int_0^t \|v - w\| \, \mathrm{d}\tau,$$

which is an upper bound for the deviation and is $\mathcal{C}^1$. It also satisfies the differential inequality

$$\dot{\mathrm{d}}(t) = \|B\|_\infty \cdot \|v(t) - w(t)\| \le \|B\|_\infty \cdot \mathrm{d}(t).$$

As above, the estimate for the difference is

$$\mathrm{d}(t) \leq \mathrm{d}(0) e^{\|B\|_\infty |t|}$$

for $-\varepsilon_0 < t < \varepsilon_0$. First estimate growth of $\|v\|$ by choosing $A = B$ and $w = 0$:

$$\|v(t)\| \leq \|v(0)\| e^{\|A\|_\infty |t|}$$

and

$$\|v\|_\infty \leq \|v(0)\| e^{\|A\|_\infty \varepsilon_0}.$$

Next we estimate $\|v - w\|$:

$$\|v(t) - w(t)\| \leq \left(\|v(0) - w(0)\| + \varepsilon_0 \|A - B\|_\infty e^{\varepsilon_0 \|A\|_\infty} \|v(0)\| \right) e^{\|B\|_\infty |t|}$$

and finally

$$\|v - w\|_\infty \leq C' \left(\|v(0) - w(0)\| + \|A - B\|_\infty \cdot \|v(0)\| \right). \tag{II.3}$$

Remark. This estimate implies the uniqueness of the solutions of the IVP

$$\begin{cases} \dot{v} = Av, \\ v(0) = v_0. \end{cases}$$

We return to the question of convergence of the difference quotients $(c_{x+uh} - c_x)/u$ of trajectories. Since they solve IVP (Eq. (II.2)) with always the same initial condition h and converging coefficients. Since the local flow Φ is Lipschitz, curves c_{x+uh} converge *uniformly* to a curve c_x as $u \to 0$. As a consequence, the matrix valued coefficient functions $A(c_x, c_{x+uh})$ converge uniformly to $A(c_x, c_X) = \mathrm{d}x_{c_x}$ as $u \to 0$. The condition Eq. (II.3) implies that $(c_{x+uh} - c_x)/u$ converges uniformly to a continuous limit function $V_{x,h}$ with $V_{x,h}(0) = h$ as $u \to 0$. In view of the ODE Eq. (II.2), it then follows furthermore that also the derivations converge uniformly

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{c_{x+uh} - c_x}{u} \right) \xrightarrow{\text{uniformly}} \mathrm{d}x_{c_x} V_{x,h}$$

as $u \to 0$. As a consequence (of the fundamental theorem of calculus), the limit function $V_{x,h}$ is $\mathcal{C}^1$ and solves the IVP:

$$\begin{cases} \dot{V}_{x,h} = \mathrm{d}X_{c_x} V_{x,h}, \\ V_{x,h}(0) = h \end{cases}$$

with $x \in B_{r/2}(x_0)$ and $h \in \mathbb{R}^n$. We used the fact that if the sequence of functions g_n converges uniformly to g and the sequence of its derivatives g_n' converges uniformly to h, we can write

$$g\big|_{t_1}^{t_2} = \int_{t_1}^{t_2} h \, \mathrm{d}t$$

which implies $g' = h$.

Now we apply the stability estimate Eq. (II.3) again to this family of IVPs parametrized by (x, h): Lipschitz property of local flow Φ implies that curves c_x vary continuously with x (with respect to supremum metric on curves $(-\varepsilon_0, \varepsilon_0) \to B_r(x_0)$). Hence, the coefficient endomorphism

fields $\mathrm{d}X_{c_x}$ vary continuously with x (with respect to supremum norm on endomorphism fields $(-\varepsilon_0, \varepsilon_0) \to \operatorname{End}\mathbb{R}^n$ induced by the operator norm on $\operatorname{End}\mathbb{R}^n$). Finally, because of Eq. (II.2) variational vector fields $V_{x,h}$ vary continuously with (x, h) (with respect to supremum norm on vector fields $(-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$) and are $\mathcal{C}^1$ in t (i.e. $\dot{V}_{x,h}$ is $\mathcal{C}^0$ in (x, h, t)). In view of $c_x = \Phi(x, \cdot)$ and

$$V_{x,h} = \lim_{u\to 0} \frac{c_{x+uh} - c_x}{u} = (\partial_{h,0}\Phi)(x, \cdot)$$

this shows that the spatial directional derivatives $\partial_{h,0}\Phi$ of Φ:

- exist and are continuous,
- are $\mathcal{C}^1$ in t and satisfy the linear ODE

$$\dot{(\partial_{h,0}\Phi)} = \mathrm{d}X_\Phi \partial_{h,0}\Phi = \mathrm{d}X_\Phi V_{(\cdot,h)}.$$

For the flow equation $\dot{\Phi} = X \circ \Phi$ we know that the partial time derivative of Φ exists and is $\mathcal{C}^0$. It follows (from the continuity of partial derivatives) that the local flow Φ is $\mathcal{C}^1$. We proved that if X is $\mathcal{C}^1$ then local flows Φ are $\mathcal{C}^1$ as well.

We can, therefore, lift the local flow Φ on U by taking its spatial differential to a local flow on the tangent bundle $TU \cong U \times \mathbb{R}^n$. Namely, for $x \in B_{r/2}(x_0)$ and $h \in \mathbb{R}^n$ the variational vector fields

$$\tilde{c}_{x,h} \coloneqq (c_X, V_{x,h})$$

are curves $(-\varepsilon_0, \varepsilon_0) \to TB_r(x_0) \cong B_r(x_0) \times \mathbb{R}^n$ lifting the local integral curves c_x. They have velocities

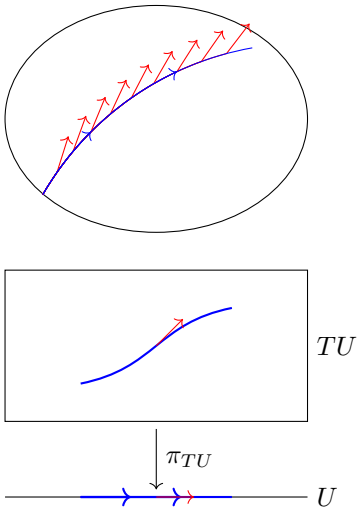
$$\dot{\tilde{c}}_{x,h} = (\dot{c}_x, \dot{V}_{x,h}) = (X(c_x), \mathrm{d}X_{c_x} V_{x,h})$$

and hence are the integral curves of the vector field

$$\tilde{X}(x, h) = (X(x), \mathrm{d}X_x h)$$

on $TU \cong U \times \mathbb{R}^n$ lifting X,

$$\dot{\tilde{c}}_{x,h} = \tilde{X} \circ c_{x,h}.$$



Thus, the local flow $\Phi(x, t) = c_x(t)$ of X lifts to the local flow

$$\tilde{\Phi}(x, h, t) = \tilde{c}_{x,h} = (c_x(t), V_{x,h}(t)) = (\Phi(x, z), \mathrm{d}_{\text{spatial}}\Phi_{x,t} h).$$

The last step is to show higher differentiability. Suppose now that X is $\mathcal{C}^{k\geq 2}$. We use this induced local flow on the tangent bundle to inductively improve our $\mathcal{C}^1$-regularity result for the local flow ϕ of X by *bootstrapping*. We start with a vector field X in $\mathcal{C}^2$. Then the lifted field $\tilde{X}$ is $\mathcal{C}^1$ and using the $\mathcal{C}^1$-regularity of the local flow Φ of X we obtain that the local flow $\tilde{\Phi}$ of $\tilde{X}$ is $\mathcal{C}^1$. Lastly, the flow itself Φ must be $\mathcal{C}^2$. More precisely:

$$\tilde{\Phi}(\dots, e_i, \dots) = (\dots, \frac{\partial \Phi}{\partial x_i}, \dots)$$

is $\mathcal{C}^1$ and using $\partial\Phi/\partial t = X \circ \phi$ we know Φ must be $\mathcal{C}^2$. We can use this to show

$$X \text{ is } \mathcal{C}^3 \implies \tilde{X} \text{ is } \mathcal{C}^2 \implies \tilde{\Phi} \text{ is } \mathcal{C}^2 \implies \Phi \text{ is } \mathcal{C}^3.$$

In conclusion, if X is $\mathcal{C}^k$, then the local flow Φ is also $\mathcal{C}^k$.

II.5 Flows of vector fields

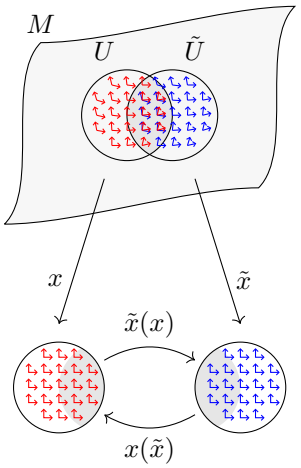
We start this topic with a preliminary technical question of minor importance: how well does the differentiable structure on a manifold need to be so that there is a notion of (locally) Lipschitz continuity of vector fields?

Recall the transition formulas for local coordinate frames and coefficients of vector fields on $\mathcal{C}^1$-manifolds:

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{n} \left(\frac{\partial \tilde{x}_i}{\partial x_j} \circ x \right) \frac{\partial}{\partial \tilde{x}_i}$$

on $U \cap \tilde{U}$ and $\partial \tilde{x}_i / \tilde{x}$ defined on $x(U \cap \tilde{U})$. The full vectors transform as

$$\sum_{j=1}^{n} \xi_j \frac{\partial}{\partial x_j} = \sum_{i=1}^{n} \underbrace{\left(\sum_{j=1}^{n} \xi_j \left(\frac{\partial \tilde{x}_i}{\partial x_j} \circ x \right) \right)}_{\tilde{\xi}_i} \frac{\partial}{\partial \tilde{x}_i}.$$



The coordinate coefficients transform as

$$\tilde{\xi} = \sum_{i=1}^{n} \xi_j \left(\frac{\partial \tilde{x}_i}{\partial x_j} \circ x \right)$$

on $U \cap \tilde{U}$. If we rearrange the symbols a bit, we get

$$\tilde{\xi}_i \circ \tilde{x}^{-1} = \sum_{j=1}^{n} \left(\left(\xi_j \circ x^{-1} \right) \circ \left(x \circ \tilde{x}^{-1} \right) \right) \cdot \left(\left(\frac{\partial \tilde{x}_i}{\partial x_j} \circ x^{-1} \right) \circ \left(x \circ \tilde{x}^{-1} \right) \right).$$

These are the coefficient functions on chart $\tilde{x}(\tilde{U})$. In the formula there are terms $(x \circ \tilde{x}^{-1})$ which are coordinate changes, so $\mathcal{C}^1$ functions and consequently bilipschitz. The term $\xi_j \circ x^{-1}$ is a coefficient on the chart $x(U)$. We see that if coordinate changes are $\mathcal{C}^{1,1}$ (i.e. $\mathcal{C}^1$ with locally Lipschitz differentials), then we have a well-defined notion of locally Lipschitz continuity or vector fields on the manifold:

$$\tilde{\xi}_i \circ \tilde{x}^{-1} \text{ locally Lipschitz } \forall i \iff \xi_j \circ x^{-1} \text{ locally Lipschitz } \forall j.$$

Next we define a $\mathcal{C}^{1,1}$-atlas as an atlas, whose coordinate changes are $\mathcal{C}^{1,1}$ and accordingly a $\mathcal{C}^{1,1}$-manifold. Of course $\mathcal{C}^2$ implies $\mathcal{C}^{1,1}$.

Consider now a $\mathcal{C}^{1,1}$-manifold M without boundary and $\mathcal{C}^k$ locally Lipschitz vector field X on M. We know that the local existence and uniqueness of integral curves and the local-to-global extension of these results imply the IVP

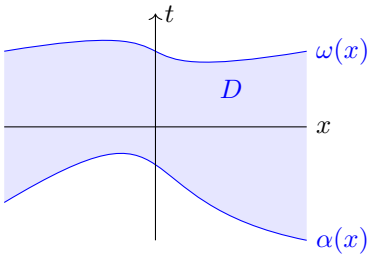
$$\begin{cases} \dot{c} = X \circ c, \\ c(0) = x \end{cases} \tag{II.4}$$

for curves $0 \in I \xrightarrow[\mathcal{C}^1]{c} M$ has a unique maximal solution

$$c_x : (\alpha(x), \omega(x)) \to M$$

where $\alpha(x) \in [-\infty, 0)$ and $\omega(x) \in (0, \infty]$ in the sense that every solution of Eq. (II.4) is a restriction of c_x. Combine now all maximal integral curves c_x for all initial conditions $x \in M$ to the map

$$\begin{aligned} M \times \mathbb{R} \supset D \coloneqq \{(x,t) | \alpha(x) < t < \omega(x)\} &\xrightarrow{\Phi} M \\ (x,t) &\mapsto \Phi(x,t) = c_x(t). \end{aligned}$$



This is called the *flow* of the vector field X. However, D is a priori not a "nice" (i.e. open) topological subset of $M \times \mathbb{R}$.

Define a *time-t map*

$$\Phi_t \coloneqq \Phi(\bullet, t), \quad t \in \mathbb{R}$$

between (possibly empty) subsets of M

$$M \supset \{\alpha < t < \omega\} \underset{\Phi_{-t}}{\overset{\Phi_t}{\rightleftarrows}} \{\alpha < -t < \omega\} \subset M.$$

We used the notation $\{\alpha < t < \omega\} = \{x | \alpha(x) < t < \omega(x)\}$. Due to the uniqueness of maximal integral curves, the following claim holds.

Claim II.1. *The time-t maps satisfy the group law, i.e. or $t_1, t_2 \in \mathbb{R}$ it holds that*

$$\Phi_{t_2+t_1} = \Phi_{t_2} \circ \Phi_{t_1}$$

wherever the right-hand side is defined.

Proof. If $\Phi_{t_1} = c_x(t_1)$ is defined, then $c_x(\bullet + t_1)$ is a maximal integral curve with the initial condition $c_x(t_1)$ and hence by the uniqueness of maximal integral curves

$$c_x(\bullet + t_1) = c_{c_x(t_1)}.$$

If also $\Phi_{t_2}(\Phi_{t_1}(x)) = c_{c_x(t_1)}(t_2)$ is defined, it follows that

$$\Phi_{t_2+t_1}(x) = c_x(t_2 + t_2) = x_{c_x(t_1)}(t_2) = \Phi_{t_2}(\Phi_{t_1}(x))$$

where all expressions are defined if the right-hand side is defined. □

Now we use the regularity (Lipschitz property and $\mathcal{C}^k$) of local flows to show that the domain of definition D is open and the (global) flow map has the corresponding regularity (locally Lipschitz and $\mathcal{C}^k$).

Note.

$$D \text{ open} \iff \begin{cases} \alpha \text{ upper semicontinuous} \\ \omega \text{ lower semicontinuous.} \end{cases}$$

We know that for all $x \in M$ there exists an open neighbourhood U_x of x in M and $\varepsilon_x > 0$ such that $\Phi|_{U_x \times (-\varepsilon_x, \varepsilon_x)}$ is defined and is Lipschitz ($\mathcal{C}^k$). Consequently, (by covering argument) for each compact $K \subset M$ there exists open neighbourhood U_K of K in M and $\varepsilon_K > 0$ such that $\Phi|_{U_K \times (-\varepsilon_K, \varepsilon_K)}$ is defined and locally Lipschitz ($\mathcal{C}^k$). Fix now $x \in M$ and $\alpha(x)\tau < 0 < \tau'\omega(x)$. There exists a compact subset $K \subset M$ such that $c_x([\tau, \tau']) \subset K$ (we could just use the curve itself, as it is already compact). Take a subdivision $\tau = t_0 < t_1 < \cdots < t_l = 0 < \cdots < t_m = \tau'$ with fineness $\max_i(t_{i+1} - t_i) < \varepsilon_K$. Let's look at a point y close to x and time t close to τ'. By group law, we have

$$\Phi_t(y) = (\Phi_{t-t_{m-1}} \circ \cdots \circ \Phi_{t_{l+1}-0})(y)$$

where each subdivision $\Phi_{t_i - t_{i-1}}$ is Lipschitz near $c_x(t_{i-1})$. In particular, $\Phi_{t_{l+1}-0}$ is Lipschitz near $c_x(t_l) = x$ and $\Phi_{t-t_{m-1}}$ is Lipschitz near $(c_x(t_{m-1}), t_m)$. Using this it is easy to determine that the flow is defined and locally Lipschitz ($\mathcal{C}^k$) for points (y, t) near (x, τ'). We can use similar steps to find that the flow is defined and locally Lipschitz ($\mathcal{C}^k$) near (x, τ).

This shows that D is open and Φ is locally Lipschitz $\mathcal{C}^k$.

A particularly nice situation is when the flow is defined everywhere at all times.

Definition II.3.
(i) A maximal integral curve c_x is called *complete* if it is defined on all of $\mathbb{R}$.
(ii) The vector field X is called *complete* if all maximal integral curves are complete, i.e. if the flow Φ is defined on all of $M \times \mathbb{R}$.

In case (ii), all time-t maps are defined on all of M and hence are locally bilipschitz homeomorphisms ($\mathcal{C}^k$-diffeomorphisms) $\Phi_t : M \to M$. We will give some sufficient conditions for completeness.

Proposition II.2. *If a maximal integral curve c_x runs/remains inside a compact subset $K \subset M$, then it is complete.*

Proof. With the notation above ($\varepsilon_K > 0$), it holds in view of the group law that

$$\omega(x) > t \implies \omega(x) > t + \varepsilon_K.$$

Hence, $\omega(x) \geq \omega(x) + \varepsilon_K$ and therefore $\omega(x) = \infty$. Similarly, $\alpha(x) = -\infty$. □

Lemma II.3. *If $\omega \geq \varepsilon_0$ and $\alpha \leq -\varepsilon_0$ on M for some $\varepsilon_0 > 0$, then the vector field X is complete.*

Proof. The same as for the previous proposition. □

This immediately yields the following.

Proposition II.4.
(i) If vector field X has compact support, then X is complete.
(ii) If M is compact, then X is complete.

II.6 Lie bracket and Lie derivative

We will now use the analytic interpretation of vector field to define their *Lie bracket*. Let M be a $\mathcal{C}^{k\geq 2}$-manifold. For vector fields $X, Y \in \Gamma^{k-1}(TM)$ we form the linear differential operator

$$\begin{aligned}[X,Y] : \mathcal{C}^k(M) &\to \mathcal{C}^{k-2}(M)\\ f &\mapsto [X,Y]f \coloneqq X(Yf) - Y(Xf).\end{aligned}$$

A priori, it is of the second order. However, the second-order terms cancel due to skew-symmetrization. To see this, we write the differential operator in local coordinates:

$$X|_U = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}, \qquad Y|_U = \sum_{i=1}^n \eta_i \frac{\partial}{\partial x_i},$$

where η_i and ξ_i are $\mathcal{C}^{k-1}$ on U. We then have

$$XYf|_U = \sum_{i=1}^n \sum_{j=1}^n \xi_i \frac{\partial}{\partial x_i} \eta_j \frac{\partial f}{\partial x_j} = \sum_{i=1}^n \sum_{j=1}^n \left(\xi_i \frac{\partial \eta_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \xi_i \eta_j \frac{\partial^2 f}{\partial x_i \partial x_j} \right).$$

The second term will cancel due to skew-symmetrization and the combined result is

$$[X,Y]|_U = \sum_{j=1}^n \sum_{i=1}^n \left(\xi_i \frac{\partial \eta_j}{\partial x_i} - \eta_i \frac{\partial \xi_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}.$$

On any chart, the differential operator $[X,Y]$ is given by a $\mathcal{C}^{k-2}$-vector field. Since these vector fields are uniquely determined, they must match on the overlaps of the charts. The differential operator of first order $[X,Y]$ is given globally by uniquely determined $\mathcal{C}^{k-2}$-vector field, also denoted by $[X,Y]$. The skew-symmetric $\mathbb{R}$-bilinear map

$$[\bullet,\bullet] : \Gamma^{k-1}(TM) \times \Gamma^{k-1}(TM) \to \Gamma^{k-2}(TM).$$

This map is called the *Lie bracket* of vector fields. This "product" is *not* associative, but instead satisfies the *Jacobi identity* (if $k \geq 3$):

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0.$$

Let's see why this is true. We have

$$XYZ - YXZ + YZX - ZYX + ZYX - XZY = 0.$$

Definition II.4. A *Lie algebra* over a field K is a K-vector space V together with a skew-symmetric bilinear map $[\cdot,\cdot] : V \times V \to V$ satisfying the Jacobi identity.

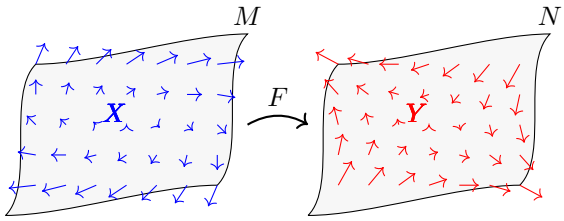
Hence, for $k = \infty$ the space $\Gamma^\infty(TM)$ of smooth vector fields on M carries a natural structure as a Lie algebra over $\mathbb{R}$. In other words, it is also a $\mathcal{C}^\infty(M)$-module over the ring $\Gamma^\infty(TM)$.

Note. The Lie bracket is not bilinear over $\mathcal{C}^\infty(M)$:

$$\begin{aligned}[fX, gY] &= (fX)(gY) - (gY)(fX) \\ &= f(Xg)Y + fgXY - g(Yf)X - gfYX \\ &= f(Xg)Y - g(Yf)X + fg[X, Y].\end{aligned}$$

The first two terms are deviations from being bilinear over $\mathcal{C}^\infty(M)$.

Lie bracket has a *normality property*. It is clearly invariant under diffeomorphisms. We will describe a more general form of compatibility with maps. There is no general procedure for pushing forward or pulling back vector fields with maps. But there is a natural notion of vector fields being related with respect to ($\mathcal{C}^1$) maps $F : M \to N$.



Definition II.5. Vector fields X on M and Y on N are called *F-realted* if

$$\mathrm{d}F \circ X = Y \circ F,$$

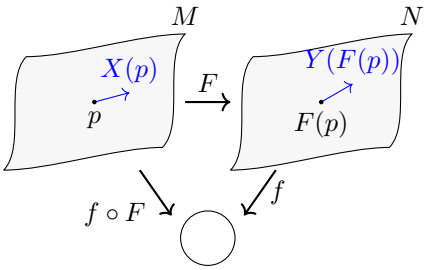
i.e. if $\mathrm{d}F_p(X(p)) = Y(F(p))$ for all $p \in M$.

There is an equivalent way of expressing this analytically:

$$X(f \circ F) = (Yf) \circ F$$

for any $f \in \mathcal{C}^1(N)$ (see figure on the right). Indeed, with the chain rule:

$$\begin{aligned}(X(f \circ F))(p) &= \mathrm{d}(f \circ F)_p(X(p)) \\ &= \mathrm{d}f_{F(p)}(\mathrm{d}F_p(X(p))) \\ &= (Yf)(F(p))\end{aligned}$$



Now, we can return to the regularity of the Lie bracket.

Claim II.5. *If $X_1, X_2 \in \Gamma^{k-1}(TM)$ are F-related to $Y_1, Y_2 \in \Gamma^{k-1}(TN)$, then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.*

Proof from analytic point of view. Let $f \in \mathcal{C}^2(N)$. We have

$$X_1X_2(f \circ F) = X_1((Y_2f) \circ F) = (Y_1Y_2f) \circ F.$$

After skew-symmetrization we get

$$[X_1, X_2](f \circ F) = ([Y_1, Y_2]f) \circ F.$$

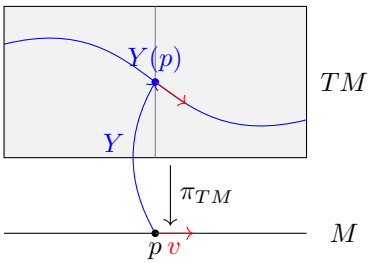
□

From now on, we will assume that manifolds don't have boundaries (as we will be working with flows). Wouldn't it be great if we could take directional derivatives of not only functions but also other objects, say vector fields? Let $Y \in \Gamma^1(TM)$ and $v \in T_pM$. We would like to define $\partial_v Y$ as a tangent vector in T_pM.

Note. The object $\partial_v Y$ indeed exists, it is an element of $T_{Y(p)}TM$ which is a superset of $T^{\text{vert}}_{Y(p)}TM \coloneqq T_{Y(p)}T_pM \cong T_pM$. $Y(p)$ is an element a fibre T_pM, which is a submanifold of TM.

We are even more greedy – we want to differentiate vector fields along curves. Let $Y \in \Gamma^1(TM)$ be a vector field and $c : I \to M$ be a $\mathcal{C}^1$ curve. We would like to define $(Y \dot{\circ} c)$ as a vector field along c. We could try

$$(Y \dot{\circ} C)(t) \overset{??}{=} \lim_{h \to 0} \frac{Y(c(t+h)) - Y(c(t))}{h}.$$



But this is not defined! Values of Y live in different tangent spaces for different h and there is no canonical way to identify different tangent spaces along c. In order to carry this out, we need additional structure on M inducing identifications of tangent spaces along c. There are at least two ways to do this:

- *linear connection* on TM,
- or flow on M having c as the integral curve.

Differentials on time-t maps can be used to identify tangent spaces along integral curves

$$(\mathrm{d}\Phi_t)_x : T_xM \to T_{c_x(t)}M.$$

We can now define the derivative of a vector field along the direction of another vector field using the flow of the latter.

Definition II.6. Let $X, Y \in \Gamma^1(TM)$ and Φ flow of X. Define a vector field

$$L_X Y \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{d}\Phi_{-t}(Y \circ \Phi_t)$$

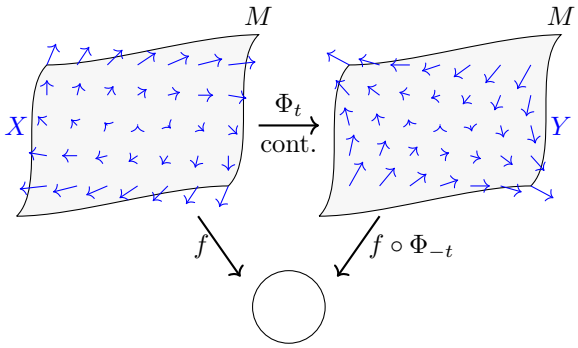
called the *Lie derivative* of Y in the direction of X.

Note that $Y \circ \Phi_t \in T_{c_p(t)}M$ and $\mathrm{d}\Phi_{-t}(Y \circ \Phi_t) \in T_pM$.

Remark. Even if Φ is only $\mathcal{C}^1$ and $\mathrm{d}\Phi$ only $\mathcal{C}^0$, it still holds that $(x,t) \mapsto (\mathrm{d}\Phi_t)_x$ is $\mathcal{C}^1$ in t since it satisfies the linearized flow ODE.

II.6.1 Relation between Lie bracket and Lie derivative

Let's look at the Lie derivative analytically as a differential operator. Let Φ_t be the flow of X and $Y \in \Gamma^1(TM)$. Two vectors are Φ_t-related by the picture below.



We have

$$\begin{aligned}
(L_X Y)f &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{d}\Phi_{-t}(Y \circ \Phi_t)\right) f \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\mathrm{d}\Phi_{-t} \circ Y \circ \Phi_t)\, f \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (Y(f \circ \Phi_{-t}) \circ \Phi_t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} Y(f \circ \Phi_{-t}) + \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (Yf) \circ \Phi_t \\
&= Y\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (f \circ \Phi_{-t})\right) + XYf \\
&= -YX(f) + XY(f) = [X,Y]f.
\end{aligned}$$

In the fourth line, we used the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \varphi(t,t) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \varphi(t,0) + \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \varphi(0,t).$$

This shows that $L_X Y$ is a geometric way of viewing the Lie bracket $[X,Y]$. The product rule for the Lie derivative is

$$L_X(fY) = (Xf)Y - f(L_X Y)$$

for $f \in \mathcal{C}^\infty(M)$ and $X, Y \in \Gamma^\infty(TM)$. See exercise sheet 10 for the proof.

II.7 Distributions and foliations

Consider a smooth manifold M^m without boundary. Informally, a distribution on manifold M is just a family of linear subspaces with equal dimensions of the tangent spaces. If the fields X and Y are complete, then $L_X Y = 0$ implies $\Phi_s^X \circ \Phi_t^Y = \Phi_t^Y \circ \Phi_s^X$ for all $t, s \in \mathbb{R}$ (exercise sheet 9).

Definition II.7. A smooth d-dimensional *distribution* or *d-plane field* on M is a family $D = (D_p)_{p\in M}$ of d-dimensional linear subspaces $D_p \subset T_pM$ which vary smoothly in the following sense: near every point $p \in M$ there exist smooth vector fields $X_1, \dots, X_d \in \Gamma^\infty(TM)$ such that $(X_1(q), \dots, X_d(q))$ forms a linear basis of D_q for q near p.

In bundle language, a distribution is a d-dimensional $\mathcal{C}^\infty$-subbundle of the tangent bundle.

Example 7.

1) Let X be a nowhere-vanishing smooth vector field on M. This vector field generates a smooth line field (1-dimensional distribution) $(\mathbb{R}\cdot X_p)_{p\in M}$.

2) Let $F : M^m \to N^n$ be a smooth submersion. Then we get an $m-n$-dimensional distribution $(\ker \mathrm{d}F_p)_{p\in M}$ tangent to the level set of F. More generally, a smooth d-dimensional foliation on M gives a smooth *tangent distribution*, i.e. a family of tangential spaces to leaves (the reader should return to this case after studying foliations (which we didn't cover)).

3) a) Let $M^m \subset \mathbb{R}^n$ be an embedded smooth manifold (e.g. open subset) or, more generally, a Riemannian manifold. Starting from a smooth distribution $D = (D_p)_{p\in M}$ on M we can form the *orthogonal distribution*, which is defined as $D^\perp = (D_p^{\perp(T_pM)})_{p\in M}$ an orthogonal complement of D_p inside the tangent space T_pM for each p. It is an exercise to show that $D^\perp$ is a smooth distribution.

   b) *Hopf fibration* of S^{2n-1} by great circles is

$$S^1 \cdot z = \mathbb{C}z \cap S^{2n-1} = (\pi|_{S^{2n-1}})^{-1}([z]),$$

   where $[z] = \mathbb{C}z$ and π is a smooth submersion. Tangent to it is a smooth line field $(i\mathbb{R}z)_{z\in S^{2n-1}}$. Orthogonal to it is a smooth hyperplane field (codimension-1 distribution) $D = (D_z)_{z\in S^{2n-1}}$, where $D_z = (\mathbb{C}z)^{\perp(\mathbb{C}^n)}$. This is exactly the orthogonal complement of $i\mathbb{R}z$ in $T_zS^{2n-1} = (\mathbb{R}z)^{\perp(\mathbb{C}^n)}$. Alternatively, we can get this subspace as $D_z = T_zS^{2n-1} \cap iT_zS^{2n-1}$. It is the maximal complex linear subspace of T_zS^{2n-1} and a *standard contact structure* on S^{2n-1}, which refers to "maximally generic" hyperplane field (an odd-dimensional cousin of symplectic structure). More generally, a $\mathcal{C}^\infty$-hyperplane field can be constructed like this on any $\mathcal{C}^\infty$-hypersurface in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ (in general not constant).

$$\begin{array}{ccc} S^{2n-1} & \subset & \mathbb{C}^n - \{0\} \\ & \searrow^{\text{Hopf fibration}} \quad \swarrow_{\pi} & \\ & \mathbb{C}\mathrm{P}^{n-1} & \end{array}$$

   If for a form $\alpha \in \Omega^1(M)$ on an odd-dimensional manifold M^{2n-1} the form $\alpha\wedge(\mathrm{d}\alpha)^{n-1}$ is nowhere vanishing (i.e. is volume form), then α is called a *contact form* and the distribution $\ker\alpha$ is called a *contact structure*. For a smooth contact structure on a $(2n-1)$-dimensional smooth manifold, there exist no n-dimensional embedded smooth submanifolds tangent to it (see exercise sheet 12). There is a unique vector field $X \in \Gamma(TM)$ such that $\alpha(X) = 1$ and $\iota_X(\mathrm{d}\alpha) = \mathrm{d}\alpha(X, \cdot) = 0$. This is called the *Reeb vector field*.

Our next aim is to integrate distributions. Either we can try to locally find submanifolds tangent on a given distribution, or we can try to find infinitesimal obstructions to their existence. Let's consider a smooth manifold M^m equipped with smooth d-dimensional distribution D.

Definition II.8. A smooth map $F : N \to M$ is called:
(i) *tangent* to D if $\operatorname{Im} \mathrm{d}F_y \subset D_{F(y)}$ for all $y \in N$,
(ii) an immersed/embedded *integral manifold* for D if F is an immersion/embedding and $\operatorname{Im} \mathrm{d}F_y = D_{F(y)}$ for all $y \in N$.

Definition II.9. The distribution is called *completely integrable* if there exists a local integral manifold for the distribution D through every point $p \in M$

To approach the question of integrability, consider flows on vector fields tangential to the distribution. Let us denote

$$\Gamma^\infty(D) \coloneqq \{X \in \Gamma^\infty(TM) | X(p) \in D_p \ \forall p \in M\}.$$

This space is large! The distribution D is locally generated by smooth base fields, and we can glue local vector fields tangent to D using a partition of unity. We make two observations for vector fields $X \in \Gamma^\infty(D)$.

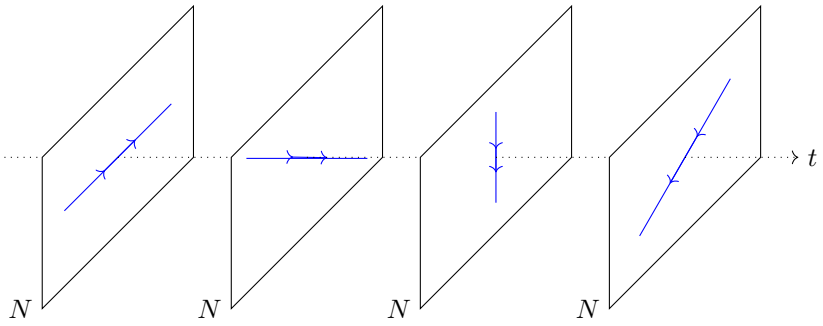
1) Non-constant trajectories are 1-dimensional submanifolds tangent to D. In particular, in the case of a line field D, they are integral manifolds, and smooth line fields are completely integrable – there is an abundance of 1-manifolds tangent to such distribution.

2) Integral manifolds of D are locally preserved by the flow Φ^X, because X is tangent to them, and the flow Φ^X restrict to flows on the integral manifolds. However, in general, the flow Φ^X doesn't preserve the distribution D, because X only determines a line field in D.

Example 8 (Propeller)**.** Let $M = N \times \mathbb{R}$ and $X = \partial/\partial t \in \Gamma^\infty(TM)$. It is locally a general case of a non-vanishing vector field. Choose

$$D_{(y,t)} \subset T_y N \subset T_y N \oplus \mathbb{R} \cong T_{(y,t)} M.$$

This is a smooth distribution in the N-direction.



Next, we enlarge the distribution by adding t-direction:

$$\hat{D}_{y,t} = D_{y,t} \oplus \mathbb{R} \subset T_{(y,t)} M.$$

Then X is tangent to $\hat{D}$, but the translation flow Φ^X doesn't preserve D if $D_{(y,\cdot)}$ vary with t. This shows that smooth distributions of dimension at least 2 are, in general, not completely integrable. It also suggests (which turns out to be true) that, in general, there do not even exist 2-dimensional submanifolds tangent to a distribution.

Definition II.10. A smooth distribution is called *involutive* if the following holds:

$$X, Y \in \Gamma^\infty(D) \implies [X, Y] \in \Gamma^\infty(D).$$

Lemma II.6 (Necessary condition for complete integrability)**.** *For a smooth distribution D on M, it holds that D is involutive if D is completely integrable.*

Proof. Let $X, Y \in \Gamma^\infty(D)$. Then for any embedded integral manifold $N \subset M$, it holds that

$$[X, Y]|_N \equiv [X|_N, Y|_N]$$

which follows from the naturality of the Lie bracket applied to inclusion $\iota : N \hookrightarrow M$ ($X|_N$ being ι-related to X). Hence, $[X, Y](p) \in T_pN = D_p$ for all $p \in N$. Since D is completely integrable it follows that $[X, Y] \in \Gamma^\infty(D)$. □

Alternative argument would be to take $f \in \mathcal{C}^\infty(M)$ such that $f|_N = \text{const}$. Then $(XY - YX)f|_N = 0$, so the Lie bracket must be tangent to N.

Lemma II.7 (Sufficient condition for complete integrability)**.** *For a smooth distribution D^d consider the following properties:*

(i) D is locally generated by commuting smooth vector fields,

(ii) D is tangent to a d-dimensional smooth foliation (equivalently, locally tangent to level sets of a smooth submersion), i.e. everywhere there exists an extrinsic local chart (U, x) such that $D|_U$ is generated by d of the coordinate vector fields $\frac{\partial}{\partial x_i}$ on U.

(iii) D is completely integrable.

It holds (i) $\implies$ (ii) $\implies$ (iii).

Proof.

(i) $\implies$ (ii) Suppose $X_1, \dots, X_d \in \Gamma^\infty(D)$ pairwise commute near p and generate D near p. Choose a d-codimensional embedded smooth submanifold $S \subset M$ through p and transversal to D near p. This means that

$$T_pM = T_pS \oplus D_p.$$

We can shrink S so that there exists $\varepsilon > 0$ such that

$$\begin{aligned} S \times (-\varepsilon, \varepsilon)^d &\xrightarrow{\psi} M \\ (q, t_1, \dots, t_d) &\mapsto \Phi^{X_d}_{t_d} \circ \cdots \circ \Phi^{X_1}_{t_1}(q). \end{aligned}$$

The result is independent of the order of flows since if X_i commute for small times then Φ^{X_i} commute for small times (see exercises). The map $\psi(\bullet, 0, \dots, 0)$ is an inclusion $S \hookrightarrow M^m$, so

$$\mathrm{d}(\psi(\bullet, 0, \dots, 0))_q : T_qS \to T_qM$$

is also an inclusion for $q \in S$. Because the order of flows can be interchanged the following differential equation is satisfied:

$$\frac{\partial \psi_i}{\partial t_i} = X_i \circ \psi.$$

In particular $\mathrm{d}\psi_{(p,0,\dots,0)}$ is invertible. After shrinking S and ε and using the Inverse function theorem ψ is smooth diffeomorphism onto an open subset $U \in M$. Then

$$\mathrm{d}(\psi^{-1})(X_i) = \frac{\partial}{\partial t_i} \circ \psi^{-1}$$

where $\frac{\partial}{\partial t_i}$ are coordinate vector fields on $S \times (-\varepsilon, \varepsilon)^d$. By smoothly identifying S with an open subset of $\mathbb{R}^{m-d}$ we obtain a chart with respect to which X_i are vector fields (d of them).

(ii) $\implies$ (iii) Apparently very easy to prove.

□

Theorem II.8 (*Frobenius theorem*)**.** *For a smooth distribution D the following statements are equivalent:*
(i) D is completely integrable,
(ii) D is locally generated by commuting smooth vector fields,
(iii) D is involutive.

Proof. We have already shown (ii) $\implies$ (i) $\implies$ (iii). What we need is (iii) $\implies$ (ii). Consider a d-dimensional involutive smooth distribution D on M^m. We can work locally. Without loss of generality, there exists a smooth submersion $s : M \to \mathbb{R}^d$ with fibres (level sets) transversal to D. Then for $q \in M$

$$\begin{array}{ccccc} T_qM & \xrightarrow{\mathrm{d}s_q} & T_{s(q)}\mathbb{R}^d & \cong & \mathbb{R}^d \\ \cup & \overset{\cong}{\nearrow}_{\text{linear isomorphism}} & & & \\ U & & & & \end{array}$$

Next we lift the coordinate vector fields $\frac{\partial}{\partial x_i}$ on $\mathbb{R}^d$ to vector fields $X_i \in \Gamma^\infty(D)$:

$$\mathrm{d}s \circ X_i = \frac{\partial}{\partial x_i} \circ s.$$

This is the definition of X_i and $\frac{\partial}{\partial x_i}$ being s-related. Vector fields X_i are uniquely defined by the above equation as vector fields on M tangent to D. Regarding their smoothness, we work even more locally and consider smooth vector fields $Y_1, \dots, Y_d \in \Gamma^\infty(D)$ generating D. Then we can write:

$$Y_j = \sum_{i=1}^{d} b_{ij} X_i.$$

In order to prove that X_i are smooth, we must show that the coefficient functions b_{ij} are smooth (then the entries of the inverse matrix will also be smooth). By construction of X, we have

$$\mathrm{d}s \circ Y_j = \sum_{i=1}^{d} b_{ij} \left(\frac{\partial}{\partial x_i} \circ s \right)$$

since the right side of the equation is smooth and the terms in the parenthesis are smooth, the coefficient functions must also be smooth. With this, we conclude that X_i are smooth vector fields.

Now we continue the argument. The vector fields X_i generate D and by the naturality of the Lie bracket we have

$$\mathrm{d}s \circ [X_i, X_j] = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] \circ s = 0.$$

It tells us that the Lie bracket $[X_i, X_j]$ must be *tangent* to fibres of s. Since D is involutive, it follows that $[X_i, X_j]$ is tangent to D. This can only mean that the Lie bracket vanishes and X_i, X_j commute. This completes the proof. □

Remark. As a consequence of the last lemma, the local integral manifolds of a completely integrable distribution vary smoothly.

Now we move to infinitesimal *obstructions* to integrability. Deviation from involutivity can be quantified by the following tensorial invariant. Let D^d be a smooth distribution on M, $X, Y \in \Gamma^\infty(D)$ and $f, g \in \mathcal{C}^\infty(M)$. Then

$$[fX, gY] = fg[X, Y] + \text{terms tangent to } D.$$

We can write this as

$$[fX, gY] \equiv fg[X, Y] \bmod D.$$

Furthermore, $[X, Y](p) \bmod D_p$ depends only on $X(p)$ and $Y(p)$ and not on the derivatives. Indeed, D is locally generated by smooth vector fields $V_1, \ldots, V_d \in \Gamma^\infty(D)$ and

$$\left[\underbrace{\sum_{i=1}^{d} f_i V_i}_{X}, \underbrace{\sum_{j=1}^{d} g_j V_j}_{Y}\right] = \sum_{i=1}^{d}\sum_{j=1}^{d} f_i g_j [V_i, V_j] \bmod D.$$

Vector fields V_i are fixed and f_i, g_j are taken pointwise. This gives a smooth family of skew-symmetric $\mathbb{R}$-bilinear forms β_p:

$$\begin{aligned} D_p \times D_p &\xrightarrow{\beta_p} T_pM/D_p \\ (X_p, Y_p) &\mapsto [X, Y](p) \bmod D_p \end{aligned}$$

In tensor language, β is TM/D-valued 2-form on M, where TM/D is considered as a smooth vector bundle over M. It is the *full obstruction* to complete integrability:

$$D \text{ involutive} \implies \beta = (\beta_p)_{p \in M} \text{ vanishes.}$$

Note. β vanishes on every smooth submanifold tangent to D.

Remark. The curvature tensor of a connection (on a smooth fibre bundle) arises in this way as the obstruction to flatness.

Now we turn to foliations, a type of additional geometrical structure on a differentiable manifold. A local model for a d-dimensional foliations on n-dimensional manifolds ($d \leq n$) is the partition

$$\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d} = \bigsqcup_{y \in \mathbb{R}^{n-d}} \mathbb{R}^d \times \{y\}$$

of $\mathbb{R}$ into the level sets of the canonical projection $\mathbb{R}^n \to \mathbb{R}^{n-d}$ to the second factor (see Fig. II.8).

Definition II.11. A d-dimensional smooth *foliation* on an n-dimensional smooth manifold (without boundary) is a maximal (sub)atlas of smooth charts (belonging to the given smooth differentiable structure on the manifold) such that the local coordinate changes preserve the model partition, i.e. are locally of the form

$$(x, y) \mapsto (\varphi(x, y), \psi(y)).$$

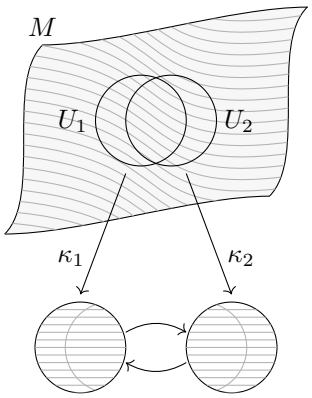


Figure II.8: A foliation of a manifold M.

Unfortunately, we didn't have enough time to delve into foliations more thoroughly.

Vector bundle

We have been using certain notions from the theory of vector bundles, such as sections and fibres, without having properly defined them. This short section is here to at least partially mend this situation and to collect all the relevant terminology in one place. The content of this section was not explicitly discussed in the lecture and is mostly a short summary of the topic – this also means that the probability of mistakes is much higher than usual and (as always) the reader is encouraged to consult other sources for more information.

Definition II.12. A (smooth) *vector bundle* of rank r is a triple (E, π, M) consisting of (smooth) manifolds E and M (or more generally topological spaces) and a continuous (smooth) surjective map $\pi : E \to M$ such that:

1. for each $p \in M$, the *fibre* $E_p \coloneqq \pi^{-1}(p)$ is a r-dimensional vector space,
2. for all $p \in M$, there exists an open neighbourhood $U \subset M$ of p and a fibre-preserving diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that the for all $q \in U$, the restriction $\varphi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^r$ is a linear isomorphism.

The space E is called the *total space*, M the *base space*, and π the *projection map* of the vector bundle. The map φ is called a *local trivialization* of E over U and the neighbourhood U a *trivializing neighbourhood*.

The rank of the vector bundle is the dimension of the fibres and is constant on the connected components of M. We might abuse the language and say that E is a vector bundle over M. We have already encountered a vector bundle. Namely, the tangent bundle is a vector bundle

(TM, π, M) of rank $\dim M$. The fibres of the tangent bundle are the tangent spaces T_pM. If the vector bundle has one-dimensional fibres, it is called a *line bundle*. Another example of a vector bundle is *trivial bundle* $(M \times \mathbb{R}^r, \pi, M)$, where $\pi : M \times \mathbb{R}^r \to M$ is the projection to the first factor. An example of a non-trivial line bundle over a circle is the Möbius strip.

Definition II.13. A *section* of a vector bundle (E, π, M) is a continuous (smooth) map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. The set of smooth sections of E is denoted by $\Gamma^\infty(M, E) = \Gamma^\infty(E)$ or sometimes $\mathcal{C}^\infty(M, E)$.

Vector fields are nothing but sections of that tangent bundle! We will see in the next chapter that 1-forms are sections of the cotangent bundle. It turns out that sums and products of sections are also sections. So $\Gamma^\infty(E)$ is a vector space over $\mathbb{R}$ and a module over the ring $\mathcal{C}^\infty(M)$ of smooth functions on M. Sometimes, we only consider sections of E over an open set U. Set of all such smooth sections is denoted by $\Gamma^\infty(U, E)$. If the section is defined on the whole manifold M, then it is called *global section*.

Definition II.14. A (smooth) *frame* of a vector bundle (E, π, M) over an open set $U \subseteq M$ is a collection of sections $\{s_1, \dots, s_r\} \subset \Gamma^\infty(U, E)$ such that at each point $p \in U$, the vectors $\{s_1(p), \dots, s_r(p)\}$ form a basis of the fibre E_p.

Chapter III

Review of differential forms

III.1 Linear algebra: alternating multilinear forms

Let V be a finite-dimensional real vector space. We denote a dual vector space with $V^* = \mathrm{Hom}(V, \mathbb{R})$. Next we define the space of alternating k-linear forms on V (also known as k-th exterior power of V) as

$$\bigwedge^k V \coloneqq \mathrm{Alt}_k(V).$$

Let $\alpha \in \mathrm{Alt}_k(V)$. Then

$$\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma)\alpha(v_1, \ldots, v_n)$$

for $v_1, \ldots, v_n \in V$ and $\sigma \in S_k$. Geometrically, α assigns k-dimensional oriented volumes of k-parallelotopes. Convention for $k = 0$ is $\bigwedge^0 V = \mathbb{R}$. The dimension of $\bigwedge^k V$ is

$$\dim \bigwedge^k V = \binom{n}{k}.$$

Definition III.1. A *wedge product* is a bilinear map

$$\wedge : \bigwedge^k V \times \bigwedge^l V \to \bigwedge^{k+l} V$$

with

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot \beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

for $\alpha \in \bigwedge^k V$ and $\beta \in \bigwedge^l V$.

The normalization comes from the fact that $|S_k| = k!$. One can verify that:

- $\wedge$ is associative:

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) = \alpha \wedge \beta \wedge \gamma,$$

- $\wedge$ is graded commutative:
$$\beta \wedge \alpha = (-1)^{kl} \alpha \wedge \beta$$
for $\alpha \in \bigwedge^k V$ and $\beta \in \bigwedge^l V$.

For multiple wedge products of $\alpha_1 \in \bigwedge^{k_1} V, \dots, \alpha_r \in \bigwedge^{k_r} V$ one has

$$(\alpha_1 \wedge \cdots \wedge \alpha_r)(v_1, \dots, v_{k_1+\cdots+k_r}) =$$
$$= \frac{1}{k_1! \dots k_r!} \sum_{\sigma \in S_{k_1+\cdots+k_r}} \operatorname{sgn}(\sigma) \alpha_1(v_{\sigma(1)}, \dots, v_{\sigma(k_1)}) \dots \alpha_r(v_{\sigma(k_1+\cdots+k_{r-1}+1)}, \dots, v_{\sigma(k_1+\cdots+k_r)}).$$

In particular, for wedge products of linear forms $\lambda_1, \dots, \lambda_r \in V^*$, one has

$$(\lambda_1, \dots, \lambda_r)(v_1, \dots, v_r) = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \lambda_1(v_1) \cdot \dots \cdot \lambda_r(v_r) = \det(\lambda_i(v_j))_{i,j=1,\dots,r}.$$

One sees that if $(e_i)_{i=1,\dots,n}$ is a basis of V, then (e_i^*) is a basis of $V^* = \bigwedge^1 V$ and hence

$$(e_{i_1}^*, \dots, e_{i_k}^*)_{1 \le i_1 < \cdots < i_k \le n}$$

forms a basis for $\bigwedge^k V$. It is an exercise to show that this is indeed a basis.

Remark. This shows that we can canonically identify

$$\bigwedge^k V \cong \bigwedge_k V^*,$$

where $\bigwedge_k$ is the k-th exterior power of the dual space.

Indeed, we can join the spaces of alternating linear forms for all dimensions with a direct sum to obtain the space of all alternating multilinear maps on V:

$$\bigwedge^\bullet V = \bigoplus_{k=0}^{n} \bigwedge^k V.$$

This is the *exterior algebra* with the multiplication given by the $\wedge$-product. It is also a *graded algebra* with the grading given by the degree of the forms. There is also another, more general and abstract way to define exterior algebra, which involves taking the quotient of the tensor algebra by the ideal generated by all elements of the form $v \otimes v$ for $v \in V$. But this exceeds the scope of this course.

What does *graded* even mean? Typically, graded property refers to a ring R, which can be written as a direct sum

$$R = \bigoplus_{k=1}^{\infty} R_k$$

such that $R_m R_n \subseteq R_{m+n}$. A polynomial ring is a good example of a graded ring. If we build an associative algebra over a graded ring, it is called graded algebra.

III.2 Differential forms

Let M be a smooth manifold with corners. Remember from before that we constructed the tangent bundle

$$TM = \bigsqcup_{p \in M} T_p M$$

with the footpoint projection $\pi_{TM} : TM \to M$, which is a smooth submersion. Now, we apply the following linear algebra construction fibre-wise: replace the fibre T_pM with its dual $(T_pM)^* \coloneqq T_p^*M$ (called *cotangent space*) and join them in the *cotangent bundle* of M:

$$T^*M = \bigsqcup_{p\in M} T_p^*M.$$

The footpoint projection is $\pi_{T^*M} : T^*M \to M$. More generally, we can consider a *bundle of alternating k-linear forms* on M:

$$\bigwedge^k TM = \bigsqcup_{p\in M} \bigwedge^k T_pM = \bigsqcup_{p\in M} \mathrm{Alt}_k(T_pM).$$

This is just a bundle of vector spaces of alternating k-linear forms on T_pM. The footpoint projection is again very similar $\pi_{\bigwedge^k TM} \to M$. This bundle is homeomorphic to the k-th exterior power of the tangent bundle:

$$\bigwedge^k TM \cong \bigwedge_k T^*M.$$

Note. For $k = 0$ we have $\bigwedge^0 TM \cong M \times \mathbb{R}$, which is a trivial line bundle over M.

Definition III.2. A *differential form* of degree k in M is a section of $\pi_{\bigwedge^k TM}$, i.e. a map

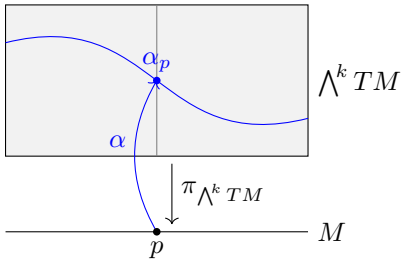
$$\begin{aligned} \alpha : M &\to \bigwedge^k TM \\ p &\mapsto \alpha_p \end{aligned}$$

such that $\pi_{\bigwedge^k TM} \circ \alpha = \mathrm{id}_M$.

Geometrically it assigns a k-dimensional oriented volume to each point $p \in M$.

The total spaces $\bigwedge^k TM$ again inherit smooth structures making the footpoint projections $\pi_{\bigwedge^k TM}$ smooth submersions. Let (U, x) be a local chart for M. Earlier we have constructed the induced bundle chart (TU, Tx) for TM as

$$\begin{aligned} TU = TM|_U &\xrightarrow{Tx} x(U) \times \mathbb{R}^m \\ \sum_{i=1}^m \xi_i \frac{\partial}{\partial x_i}\bigg|_p &\mapsto \left(x(p), \sum_{i=1}^m \xi_i e_i\right), \end{aligned}$$



where $(e_i)_{i=1,\dots,m}$ is the standard basis of $\mathbb{R}^m$.

Similarly, the induced bundle charts (T^*U, T^*x) for the cotangent bundle T^*M are

$$\begin{aligned} T^*U = T^*M|_U &\xrightarrow{T^*x} x(U) \times (\mathbb{R}^m)^* \\ \sum_{i=1}^m a_i (\mathrm{d}x_i)_p\bigg|_p &\mapsto \left(x(p), \sum_{i=1}^m a_i e_i^*\right), \end{aligned}$$

where $(e_i^*)_{i=1,\dots,m}$ is the dual standard basis for $\mathbb{R}^m$. The basis of $T_p^*M = (T_pM)^*$ induced by the chart (U, x) is $((\mathrm{d}x_i)_p)_{i=1,\dots,m}$ and is dual to the basis $((\partial/\partial x_i)_p)_{i=1,\dots,m}$ of T_pM. More

generally, the induced bundle chart for $\bigwedge^k TM$ is

$$\bigwedge^k TU = \bigwedge^k TM|_U \xrightarrow{\bigwedge^k Tx} x(U) \times \bigwedge^k \mathbb{R}^m$$
$$\sum_{1\le i_1<\cdots<i_k\le m} a_{i_1\dots i_k}(\mathrm{d}x_{i_1})_p \wedge \cdots \wedge (\mathrm{d}x_{i_k})_p \mapsto \left(x(p), \sum_{1\le i_1<\cdots<i_k\le m} a_{i_1\dots i_k} e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}\right).$$

As in the case of the tangent bundle, the charts $(\bigwedge^k TU, \bigwedge^k Tx)$ induced by the charts (U, x) of the smooth structure M generate the natural smooth structure on $\bigwedge^k TM$.

A differential k-form $\alpha : M \to \bigwedge^k TM$ can be written locally with respect to a chart (U, x) on M as

$$\alpha|_U = \sum_{1\le i_1<\cdots<i_k\le m} a_{i_1\dots i_k}\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k},$$

where $a_{i_1\dots i_k}$ are coefficient on U and $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ is a pointwise $\wedge$-product of coordinate 1-forms on U ($\mathrm{d}x_i : p \mapsto (\mathrm{d}x_i)_p$). Hence, α is smooth (or continuous or k-times differentiable) on U if and only if the coefficient functions $a_{i_1\dots i_k}$ are smooth (or continuous or k-times differentiable) on U.

Denote $\Omega^k(M) \coloneqq \Gamma^\infty(\bigwedge^k TM)$ the space of smooth k-forms on M. This is not only a vector space but also a module over the ring of smooth functions (a $\mathcal{C}^\infty(M)$-module). Note that $\Omega^0(M) = \mathcal{C}^\infty(M)$, i.e. 0-forms are smooth functions. The wedge product of forms is a smooth map

$$\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M).$$

Of course, if $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, then $(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p$ which is again smooth. This definition of wedge product is equivalent to the one before, so it is associative and graded commutative. The direct sum

$$\Omega^\cdot(M) = \bigoplus_{k=1}^{\infty} \Omega^k(M)$$

is a graded algebra with the multiplication given by the wedge product. Of course, the sum is, in fact, finite because $\Omega^k(M) = 0$ for $k > \dim M$.

Next, we will discuss the *pull-back* of differential forms. Let $F : M \to N$ be a smooth map between smooth manifolds. Then $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ (we have already seen this). But it also works for linear forms $\mathrm{d}F_p^* : T^*_{F(p)}N \to T^*_pM$ and even more generally

$$\mathrm{d}F_p^* : \bigwedge^k T_{F(p)}N \to \bigwedge^k T_pM.$$

Applying this pointwise to differential k-forms α on N yields differential k-forms on M:

$$(F^*\alpha)_p(v_1, \dots, v_k) = \alpha_{F(p)}(\mathrm{d}F_p(v_1), \dots, \mathrm{d}F_p(v_k))$$

where $v_1, \dots, v_k \in T^*_pM$. The pull-back commutes with wedge product:

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta).$$

Let's look explicitly at the pull-back of coordinate 1-forms (see Fig. III.1):

$$F^*\mathrm{d}y_j = \sum_{i=1}^m \left((F^*\mathrm{d}y_j)\frac{\partial}{\partial x_i}\right)\mathrm{d}x_i = \sum_i^m \left(\mathrm{d}y_j\left(\mathrm{d}F\frac{\partial}{\partial x_i}\right)\right)\mathrm{d}x_i = \sum_{i=1}^m \left(\frac{\partial y_j}{\partial x_i} \circ x\right)\mathrm{d}x_i.$$

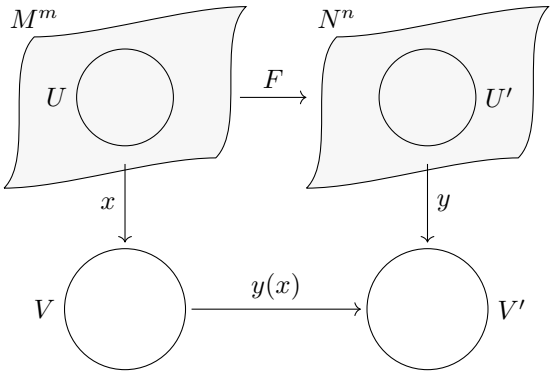


Figure III.1: Map $F : M \to N$ and its local representation in coordinates.

It is clear from this computation and the compatibility of the pull-back with $\wedge$-product that pull-backs of smooth forms are smooth, so

$$\begin{aligned} F^* : \Omega^k(N) &\to \Omega^k(M) \\ \alpha &\mapsto F^*\alpha. \end{aligned}$$

Pull-backs also commute with the exterior derivative:

$$F^* \circ \mathrm{d}_N = \mathrm{d}_M \circ F^*.$$

For proof see exercise sheet 12.

III.3 Total differential and exterior derivative

The cotangent bundle is the natural habitat for the total differential of functions. Let $f \in \mathcal{C}^\infty(M)$. The *total differential* $\mathrm{d}F$ is a section of the tangent bundle:

$$\begin{aligned} \mathrm{d}f : M &\to TM \\ p &\mapsto \mathrm{d}f_p \in T_p^*M. \end{aligned}$$

In local coordinates with respect to a chart (U, x) it can be written as

$$\mathrm{d}f|_U = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} \mathrm{d}x_i.$$

Clearly, the total differential of a smooth function is a smooth 1-form. The total differential is an $\mathbb{R}$-linear first-order differential operator

$$\begin{aligned} \mathrm{d} : \Omega^0 = \mathcal{C}^\infty(M) &\to \Omega^1(M) \\ f &\mapsto \mathrm{d}f \end{aligned}$$

satisfying the *product rule*

$$\mathrm{d}(f \cdot g) = (\mathrm{d}f) \cdot g + f \cdot (\mathrm{d}g).$$

Now we ask an important question: when is a smooth 1-form α the total differential of a smooth function $f \in \mathcal{C}^\infty(M)$, at least locally? This is really a question of solving the partial differential

equation $\mathrm{d}f = \alpha$ for f locally. There is a precise integrability condition for the existence of local solutions. Namely, writing the PDE locally in the chart (U, x) as

$$\sum_{j=1}^{m} \frac{\partial f}{\partial x_i} \mathrm{d}x_i = \sum_{i=1}^{m} a_i \mathrm{d}x_i$$

one gets a system of m linear PDEs

$$\frac{\partial f}{\partial x_i} = a_i, \qquad \text{for } i = 1, \dots, m.$$

One can show that the system is locally solvable if and only if the integrability condition

$$\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}, \qquad \text{for } i, j = 1, \dots, m$$

is satisfied (necessity is clear, sufficiency is given by *Poincaré lemma* which we will prove later).

The exterior derivative is a first order linear differential operator on differential forms extending the total differential. It will be motivated a posteriori by

- Poincaré lemma (saying that d gives the full obstruction to local exactness) and
- Stokes' theorem.

It is described by the following theorem.

Theorem III.1. *There exists a unique extension of the total differential* $\mathrm{d} : \Omega^0(M) \to \Omega^1(M)$ *to a family of* $\mathbb{R}$*-linear operators*

$$\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M), \qquad 0 \leq k \leq m$$

such that the following holds

(i) product rule (also called an antiderivation property)

$$\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathrm{d}\beta)$$

for $\alpha \in \Omega^k(M)$ *and* $\beta \in \Omega^l(M)$*,*

(ii) chain complex property $\mathrm{d} \circ \mathrm{d} = 0$.

This operator d is called the *exterior derivative*. It is indeed a first-order differential operator (as we will show in the proof).

Proof. We will first prove *locality*. Suppose that d is an operator with the required properties. Let $\alpha, \tilde{\alpha} \in \Omega^k(M)$ be two smooth k-forms such that $\alpha = \tilde{\alpha}$ near on a neighbourhood A of p. There exists a cut-off function $\psi \in \mathcal{C}^\infty(M)$ such that $\psi = 1$ on a neighbourhood $B \subset A$ of p and $B \subset \operatorname{supp} \psi \subset U \subset A$. Then $\psi \cdot (\alpha - \tilde{\alpha}) = 0$ and we obtain with the product rule for d:

$$0 = \mathrm{d}(\psi \cdot (\alpha - \tilde{\alpha})) = \mathrm{d}(\psi \wedge (\alpha - \tilde{\alpha})) = (\mathrm{d}\psi) \wedge (\alpha - \tilde{\alpha}) + \psi \wedge (\mathrm{d}\alpha - \mathrm{d}\tilde{\alpha}) = \mathrm{d}\alpha - \mathrm{d}\tilde{\alpha}.$$

The last step is the consequence that $\psi = 1$ near p and that $\mathrm{d}\psi = 0$ near p as it is the total differential of a constant function. This shows locality in the following sense: $(\mathrm{d}\alpha)_p$ depends only on the values of α arbitrarily close to p.

Next, we will show *localization*. For any open subset $U \subset M$, the exterior derivative d on M induces an exterior derivative d_U on U as follows. For $\alpha_U \in \Omega^k(U)$ and $p \in U$ there exists $\alpha \in \Omega^k(M)$ (with compact support) such that $\alpha_U = \alpha$ near p. We can achieve this by multiplication with an appropriate cut-off function. Define

$$(\mathrm{d}_U \alpha_U)(p) \coloneqq (\mathrm{d}\alpha)_p.$$

This is well-defined due to the locality of d. Then properties (i) and (ii) carry over from d to d_U.

Near p we can equate $\mathrm{d}_U(\alpha_U \wedge \beta_U) = \mathrm{d}(\alpha \wedge \beta)$ and $(\mathrm{d}_U\alpha_U) \wedge \beta_U + (-1)^k \alpha_U \wedge (\mathrm{d}_U\beta_U) = (\mathrm{d}\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathrm{d}\beta)$. Since

$$\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathrm{d}\beta)$$

it locally holds

$$\mathrm{d}_U(\alpha_U \wedge \beta_U) = (\mathrm{d}_U\alpha_U) \wedge \beta_U + (-1)^k \alpha_U \wedge (\mathrm{d}_U\beta_U).$$

For the localized exterior derivatives it holds that $\mathrm{d}_U(\alpha_U) = (\mathrm{d}\alpha)_U$ for $\alpha \in \Omega^k(M)$.

Now we will show local *existence* and *uniqueness* of local exterior derivatives on chart domains. Let (U, x) be a chart for M. An exterior derivative d_U on U must satisfy:

$$\mathrm{d}_U \left(\sum_{1 \le i_1 < \cdots < i_k \le m} a_{i_1 \ldots i_k} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \right) \overset{(i)+(ii)}{=} \sum_{1 \le i_1 < \cdots < i_k \le m} \mathrm{d}_U a_{i_1 \ldots i_k} \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

Functions $a_{i_1 \ldots i_k}$ are smooth on U and are a unique representation of $\alpha_U \in \Omega^k(U)$. On the right side, $\mathrm{d}_U a$ refers to the total derivative of a as a function on U. There should also be additional terms for the exterior derivative of the $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$, but they are all zero, since $\mathrm{d}_U(\mathrm{d}x_i) = \mathrm{d}^2 x_i = 0$. The exterior derivative is thus uniquely determined on U. On the other hand, the last formula can be used as a definition of d_U. We verify that the operator d_U defined by it satisfies (i) and (ii). Let's first look at (i). For $a, b \in \mathcal{C}^\infty(U)$ we have

$$\begin{aligned}
&\mathrm{d}_U \left((a\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) \wedge (b\,\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l}) \right) = \\
&\qquad = \mathrm{d}_U \left(ab\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l} \right) \\
&\qquad = \mathrm{d}(ab) \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l} \\
&\qquad = ((\mathrm{d}a) \cdot b + a \cdot (\mathrm{d}b)) \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l} \\
&\qquad = \mathrm{d}a \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \wedge b\,\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l} + \\
&\qquad\quad + (-1)^k a\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \wedge \mathrm{d}b \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l} \\
&\qquad = \mathrm{d}_U(a\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) \wedge (b\,\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l}) + \\
&\qquad\quad + (-1)^k (a\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) \wedge \mathrm{d}_U(b\,\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l})
\end{aligned}$$

This shows that the definition obeys the product rule. Now we will show (ii). For function

$f \in \mathcal{C}^\infty(U)$ we have

$$\begin{aligned}
\mathrm{d}_U^2 f &= \mathrm{d}_U(\mathrm{d}_U f) \\
&= \mathrm{d}_U \left(\sum_{i=1}^m \frac{\partial f}{\partial x_i} \mathrm{d}x_i \right) \\
&= \sum_{i=1}^m \left(\mathrm{d} \frac{\partial f}{\partial x_i} \right) \wedge \mathrm{d}x_i \\
&= \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^2 f}{\partial x_j \partial x_i} \mathrm{d}x_j \wedge \mathrm{d}x_i \\
&= \sum_{i<j} \left(\frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \mathrm{d}x_j \wedge \mathrm{d}x_i \\
&= 0.
\end{aligned}$$

For smooth k-forms it follows that

$$\begin{aligned}
\mathrm{d}_U^2(a\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) &= \mathrm{d}_U(\mathrm{d}a \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) \\
&= (\mathrm{d}_U^2 a) \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} - \mathrm{d}a \wedge \mathrm{d}_U(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) = 0.
\end{aligned}$$

Hence, d_U is indeed an exterior derivative on U.

Lastly, we will show *global existence* on M (uniqueness is already clear). Combine the exterior derivatives d_U on all chart domains $U \subset M$. For a global form $\alpha \in \Omega^k(M)$ we define $(\mathrm{d}\alpha)_U \coloneqq \mathrm{d}_U(\alpha_U)$. We have to check that the restrictions on two overlapping charts are equal at the intersection:

$$\mathrm{d}_U(\alpha_U)|_{U\cap\tilde{U}} = \mathrm{d}_{U\cap\tilde{U}}(\alpha_{U\cap\tilde{U}}) = \mathrm{d}_{\tilde{U}}(\alpha_{\tilde{U}})|_{U\cap\tilde{U}}.$$

The equalities hold because of the uniqueness of exterior derivative on $U \cap \tilde{U}$. Properties (i) and (ii) carry over from the d_U to d. □

When dealing with differentials of 1-forms, the following formula comes in handy: for $\alpha \in \Omega^1(M)$ and $X, Y \in \Gamma^\infty(TM)$ it holds

$$\mathrm{d}\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]).$$

Proof of this statement is on exercise sheet 12.

Exterior derivative increases the degree of forms by one. However, there also exists an operation that decreases the degree of forms by one. This is the *interior product* or the contraction with a vector field X, and it is defined as

$$\iota_X : \Omega^p(M) \to \Omega^{p-1}(M)$$

which maps a p-form α to a $(p-1)$-form $\iota_X\alpha$ by

$$(\iota_X\alpha)(X_1, \ldots, X_{p-1}) = \alpha(X, X_1, \ldots, X_{p-1})$$

for any vector fields $X_1, \ldots, X_{p-1}$. On 1-form α, it acts as $\iota_X\alpha = \alpha(X)$, and by convention, it is defined as zero on 0-forms. It obeys graded Leibniz rule, namely for forms $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ it holds

$$\iota_X(\alpha \wedge \beta) = (\iota_X\alpha) \wedge \beta + (-1)^l \alpha \wedge (\iota_X\beta).$$

There also exists a *Lie derivative* on forms. For a vector field $X \in \Gamma^\infty(M)$ and a form $\alpha \in \Omega^k(M)$ it is defined as

$$L_X \alpha = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Phi_t^* \alpha$$

where Φ_t^* is the pull-back of the flow of vector field X. For $\alpha, \beta \in \Omega^\bullet(M)$ it satisfies the product rule

$$L_X(\alpha \wedge \beta) = (L_X \alpha) \wedge \beta + \alpha \wedge (L_X \beta).$$

It also commutes with the exterior derivative

$$\mathrm{d} \circ L_X = L_x \circ \mathrm{d}.$$

There is a nice formula connecting the Lie derivative and the interior product called the *Cartan's formula*:

$$L_X \alpha = \mathrm{d}(\iota_X \alpha) + \iota_X(\mathrm{d}\alpha).$$

For proof of these statements, see exercise sheet 13.

Chapter IV

Integration of differential forms

IV.1 Linear algebra: orientation

Let V be an n-dimensional $\mathbb{R}$-vector space with $n \in \mathbb{N}_0$. The set of basis

$$\mathrm{Bases}(V) = \underbrace{V \times \cdots \times V}_{n}$$

is open and dense. If we have two bases of (e_i) and (e'_j) bases of V, we can express one in terms of the other by a unique *transition matrix* $A = (a_{ij}) \in \mathrm{GL}(n, \mathbb{R})$:

$$e'_j = \sum_{i=1}^{n} a_{ij} e_i$$

or written a bit differently:

$$(e_1, \ldots, e_n) = (e'_1, \ldots, e'_n) \cdot A.$$

The tuples $(e_1, \ldots, e_n)$ are to be understood as matrices as each e_i is a column vector, and $\cdot$ is a matrix product. This is the natural right action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathrm{Bases}(V)$

$$\mathrm{Bases}(V) \curvearrowleft \mathrm{GL}(n, \mathbb{R}),$$

and it is simply transitive. The *orbit maps*

$$\begin{aligned} O_{e_1,\ldots,e_n} : \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{Bases}(V) \\ A &\mapsto (e_1, \ldots, e_n) \cdot A \end{aligned}$$

are smooth diffeomorphisms (and hence also homeomorphisms). We can split $\mathrm{GL}(n, \mathbb{R})$ into two connected components $\mathrm{GL}_+(n, \mathbb{R})$ and $\mathrm{GL}_-(n, \mathbb{R})$, that are characterized by the sign of the determinant. From this, it follows that $\mathrm{Bases}(V)$ has two connected components, namely the $\mathrm{GL}_+(n, \mathbb{R})$-orbits.

Definition IV.1. An *orientation* of V is a choice of one of the two connected components of $\mathrm{Bases}(V)$. The bases of this component are then called positively oriented.

There is a description in terms of volume forms $\bigwedge^n V = \mathrm{Alt}_n(V)$ which is a 1-dimensional vector space (but we cannot identify it with $\mathbb{R}$).

Definition IV.2. A *volume form* on V is a multilinear form $\omega \in \bigwedge^n V - \{0\}$.

Every volume form ω induces an orientation: we say a base $(e_1, \dots, e_n) \in \mathrm{Bases}(V)$ is positively oriented if and only if $\omega(e_1, \dots, e_n) > 0$.

Note (Note regarding well-definedness). If we take two bases $(e_1, \dots, e_n)$ and $(e'_1, \dots, e'_n)$, and a transition matrix A between them, then $\omega(e_1, \dots, e_n) = \det A \cdot \omega(e'_1, \dots, e'_n)$. Two volume forms on V induce the same orientation if and only if they are positive multiples of each other. Hence,

$$\text{connected components of } \bigwedge^n V - \{0\} \leftrightarrow \text{orientations of } V.$$

For $n = 0$, we use the convention that an orientation on a trivial vector space $\{0\}$ is defined as the assignment of a sign $+$ or $-$. The natural choice is $+$.

IV.2 Orientations of differentiable manifolds

Let M be an m-dimensional smooth manifold (with corners). A *local base field* for TM on an open subset $U \subset M$ is an m-tuple $(X_1, \dots, X_m)$ of vector fields X_i on U, such that $(X_1(p), \dots, X_m(p))$ is a basis of T_pM for all $p \in U$. For example a standard base field

$$\left(\frac{\partial}{\partial x_i}, \dots, \frac{\partial}{\partial x_m}\right)$$

is associated to a chart (U, x).

Definition IV.3. An *orientation* of M is a family of orientations of all tangent spaces T_pM for $p \in M$ which is locally constant in the following sense. For any continuous base field (X_i) on any open subset $U \subset M$ the sign of the orientation of the basis $(X_1(p), \dots, X_n(p))$ of T_pM for $p \in U$ is locally constant in p.

Remark. For any two continuous local base fields (X_i) and (X_j) defined on open subsets $U, U' \subset M$ the determinant of the (continuous) transition matrix (field) $A : U \cap U' \to \mathrm{GL}(n, \mathbb{R})$ has a locally constant sign.

In 0 dimensions, the orientation on a discrete space is a $\{\pm\}$-valued function, i.e. an assignment of signs to the points. Let's look also at the description in terms of volume forms.

Definition IV.4. A *volume form* on M^m is a nowhere vanishing differential m-form.

Continuous volume form ω on M induces (pointwise) an orientation of M. Let (X_i) be a continuous local base field on $U \subset M$. Then $\omega|_U(X_1, \dots, X_m)$ is a continuous $\mathbb{R}^*$-valued function that has locally constant sign. Two volume forms ω and ω' on M induce the same orientation on M if and only if $\omega' = f\omega$ for some continuous function $f : M \to \mathbb{R}^+$.

Example 9. On the local model $\overline{O}^m = \{x_1, \dots, x_m \leq 0\} \subset \mathbb{R}^m$ the standard volume form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$ induces the standard orientation and a standard base field

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right)$$

is positively oriented.

Proposition IV.1. *An orientation of a smooth manifold is always induced by a smooth volume form.*

Proof sketch. We first pick a family of local smooth volume forms inducing the given orientation. If this form induces the given orientation in a point, then it induces the orientation also nearby. Using a smooth partition of unity, we can glue these local forms together to obtain a global smooth form inducing the given orientation: $\omega = \sum \phi_\iota \omega_\iota$, where $\omega_\iota \in \Omega^m(M)$ are volume forms on neighbourhoods of the supports of ϕ_ι. □

Note. For continuous volume forms ω and ω' on M it holds that they induce the same orientation if and only if $\mathcal{C}(M) \ni \omega'/\omega > 0$.

Definition IV.5. A manifold M is *orientable* if and only if there exists an orientation of M. By the previous proposition, it is equivalent to the existence of smooth volume forms.

Remark. We can look at the orientation from the point of view of the geometrical structures. Namely, there is a one-to-one correspondence between orientations of M and the maximal sub-atlases of smooth structure with orientation-preserving coordinate changes.

IV.3 Induced orientation of boundary

Let M^m be a smooth manifold with boundary ∂M. If we have a manifold with corners, we can always remove the strata of codimension more than 2 and apply this discussion to the rest. The local model for a manifold with boundary is $\overline{H}^m = \{x_1 \leq 0\} \subset \mathbb{R}^n$. For a point on the boundary $p \in \partial M$ the tangent space $T_p\partial M \subset T_pM$ is a linear subspace of codimension 1 and divides T_pM into the half-spaces of outward/inward pointing tangent vectors. An orientation of T_pM induces an orientation of $T_p\partial M$ as follows. A basis $(v_2, \dots, v_m)$ of $T_p\partial M$ is positively oriented if and only if for any outward pointing tangential vector $\nu \in T_pM$ the basis $(\nu, v_2, \dots, v_m)$ of T_pM is positively oriented.

This is well-defined. Indeed, note that "outward-pointing" is a *convex property* – any convex combination of two outward-pointing vectors is also outward-pointing. If we take another outward-pointing vector $\nu' \in T_pM$ then

$$((1-t)\nu + t\nu', v_2, \dots, v_n), \quad t \in [0,1]$$

is a continuous family, and these have, therefore, all the same orientation. Hence, the orientation of M induces the orientation on ∂M. To summarize:

orientation sign of basis $(v_2, \dots, v_m)$ of $T_p\partial M \coloneqq$ orientation sign of basis $(\nu, v_2, \dots, v_m)$ of T_pM

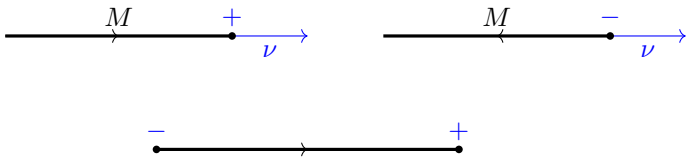
for some outward-pointing vector $\nu \in T_pM$. Applying this pointwise, the orientation on M induces orientation of ∂M. That the family of induced orientations of the tangent spaces $T_p\partial M$ of ∂M is locally constant as a function of a point follows from its description in terms of the volume form. For this, we choose $X \in \Gamma^\infty(TM)$ such that it is outward pointing along ∂M. Construct X by gluing local outward pointing smooth fields using a smooth partition of unity subordinate to a cover of ∂M by coordinate charts:

$$X = \sum_\iota \varphi_\iota X_\iota$$

with $X_\iota \in \Gamma^\infty(TM)$ being outward pointing on the neighbourhood of $\operatorname{supp} \varphi_L$. Outward-pointing everywhere follows from the convex property of outward-pointing. Let vol_M be a smooth volume form on M inducing given orientation. Then

$$\operatorname{vol}_{\partial M} = \iota_X \operatorname{vol}_M |_{\partial M} = \operatorname{vol}_M(X, \bullet, \dots, \bullet)|_{\partial M}$$

is a smooth volume form of ∂M inducing the orientation of ∂M as defined above. Convention in the dimension $m = 1$ is



Example 10.

1. Let $\overline{H}^m = \{x_1 \leq 0\}$ with standard orientation or orientation induced by $\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_m$. This means that the basis
$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right)$$
is positively oriented. The induced orientation on $\partial\overline{H}^m \cong \mathbb{R}^{m-1}$ is given by
$$\frac{\partial}{\partial x_1}(\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_m) = \mathrm{d}x_2 \wedge \dots \wedge \mathrm{d}x_m.$$
In other words, the basis
$$\left(\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_m}\right)$$
is positively oriented.
2. Let $\overline{B} = \overline{B}_1(0)$ be a unit ball with standard orientation. The field
$$X = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \in \Gamma^\infty(\overline{B})$$
is outward-pointing along $\partial\overline{B}$. Induced orientation on $\partial\overline{B} \cong S^{m-1}$ is given by the form
$$\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}(\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_m) = \sum_{i=1}^n (-1)^{i-1} x_i \mathrm{d}x_1 \wedge \dots \wedge \hat{\mathrm{d}x_i} \wedge \dots \wedge \mathrm{d}x_m.$$

IV.4 Integration of differential forms

The basic task we will try to perform is to integrate top-degree forms over oriented manifolds. Let M be an oriented m-dimensional smooth manifold with corners and let $\alpha \in \Omega^m_{\mathrm{cpt}}(M)$. The index "cpt" refers to compact support. Define

$$\int_M \alpha$$

to be *α-volume* of M. The intuition behind this is that α assigns infinitesimal volumes, i.e. yields volume measurements on tangent spaces. This means that for $v_1, \dots, v_m \in T_pM$, $\alpha(v_1, \dots, v_m)$ is an oriented α-volume of *parallelotope* in T_pM spanned by $v_1, \dots, v_m$.

Let's start very easy. A 0-dimensional oriented manifold M^0 is a discrete collection of points with assigned signs. The orientation can be thought of as a function $\varepsilon : M \to \{+1, -1\}$. Then $f : M \to \mathbb{R}$ is a 0-form (function) with finite compact support (it is also always smooth). Then the α-volume is:

$$\int_M \alpha \coloneqq \sum_{p \in M} \varepsilon(p) f(p).$$

For $m \geq 1$, we will first integrate in the local model. Let U be a relatively open subset of orthant $\overline{O}^m$ with standard orientation and let $\alpha \in \Omega^m_{\mathrm{cpt}}(U)$. We can write α as

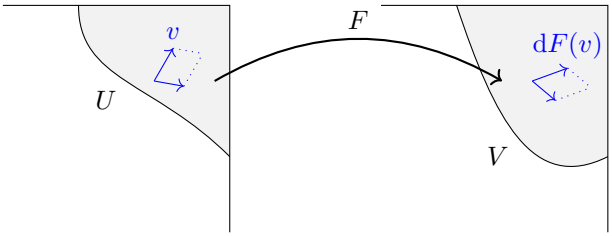
$$\alpha = f \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$$

where f is a smooth function with compact support in U, $f \in \mathcal{C}^\infty_{\mathrm{cpt}}(U)$. Then we define

$$\int_U \alpha \coloneqq \int_U f \, \mathrm{d}x_1 \dots \mathrm{d}x_m,$$

where the right side is the *Lebesgue integral*. Technically, we don't need it in the current case (normal Riemann integral would be just fine), but it is nice to have if we want to extend the definition to less regular forms. In order to carry this local construction to oriented manifolds, we need this integral (volume measurement) to be invariant under orientation preserving diffeomorphisms:

$$\overline{O}^m \supset \overset{\text{rel. open}}{U} \xrightarrow[\mathcal{C}^\infty-\text{diffeo}]{\text{orient. pres.}} \overset{\text{rel. open}}{V} \subset \overline{O}^m.$$



For a form $\beta \in \Omega^m_{\mathrm{cpt}}(V)$ the corresponding form on U with respect to the identification F is $F^*\beta \in \Omega^m_{\mathrm{cpt}}(U)$. The pullback preserves infinitesimal volumes by construction

$$(F^*\beta)_p(v_1, \dots, v_m) = \beta(\mathrm{d}F_p(v_1), \dots, \mathrm{d}F_p(v_m))$$

with $v_1, \dots, v_m \in T_pU$. On the left is $F^*\beta$-volume form of parallelotope in T_pU spanned by the v_i and on the right is β-volume form of parallelotope in $T_{F(p)}V$ spanned by $\mathrm{d}F_p(v_i)$. It is, therefore, not surprising that it preserves volumes.

Lemma IV.2. *Let $F : U \to V$ be an orientation preserving diffeomorphism between relatively open subsets of $\overline{O}^m$. Then for any $\beta \in \Omega^m_{\mathrm{cpt}}(V)$ it holds*

$$\int_U F^*\beta = \int_V \beta.$$

Proof. It is a consequence of the transformation formula. Write

$$\beta = g\,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_m$$

with $g \in \mathcal{C}^\infty_{\mathrm{cpt}}(V)$. We have to compute $F^*\beta$. Pullback of coordinate 1-forms is

$$F^*\mathrm{d}y_j = \sum_{i=1}^m (F^*\mathrm{d}y_j)\frac{\partial}{\partial x_i}\mathrm{d}x_i = \sum_{i=1}^m \frac{\partial F_j}{\partial x_i}\mathrm{d}x_i.$$

The pullback of a standard volume form is:

$$\begin{aligned}
F^*(\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_m) &= F^*\mathrm{d}y_1 \wedge \cdots \wedge F^*\mathrm{d}y_m \\
&= \sum_{i_1,\ldots,i_m} \frac{\partial F_1}{\partial x_{i_1}} \cdots \frac{\partial F_m}{\partial x_{i_m}}\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_m} \\
&= \sum_{\sigma \in S_m} \frac{\partial F_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial F_m}{\partial x_{\sigma(m)}}\,\mathrm{d}x_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}x_{\sigma(m)} \\
&= \sum_{\sigma \in S_m} \frac{\partial F_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial F_m}{\partial x_{\sigma(m)}}\,\mathrm{sgn}(\sigma)\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m \\
&= \det\left(\frac{\partial F_i}{\partial x_j}\right)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m.
\end{aligned}$$

Pullback of a general m-form is then

$$F^*(g\,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_m) = (g \circ F)\det\left(\frac{\partial F_i}{\partial x_j}\right)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m.$$

We can also put an absolute value around the determinant of F since we know F is orientation-preserving and its determinant is positive. Hence, we can use the usual formula for coordinate transformation in the integral

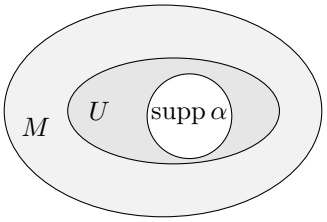
$$\int_U F^*\beta = \int_U (g \circ F)\left|\det\left(\frac{\partial F_i}{\partial x_j}\right)\right|\mathrm{d}x_1 \ldots \mathrm{d}x_m = \int_V g\,\mathrm{d}y_1 \ldots \mathrm{d}y_m = \int_V \beta.$$

□

We are now ready for the definition of integration of forms on oriented manifolds. Let M^m be an oriented smooth manifold with corners. Due to the previous lemma, we can define the integral of m-forms, which are supported in a chart. Let $\alpha \in \Omega^m_{\mathrm{cpt}}(M)$ such that $\operatorname{supp}\alpha \subset U$ for some chart (U, x). Define

$$\int_M \alpha \coloneqq \int_{x(U)} (x^{-1})^*\alpha.$$

Observe that $(x^{-1})^*\alpha \in \Omega^m_{\mathrm{cpt}}(x(U))$.

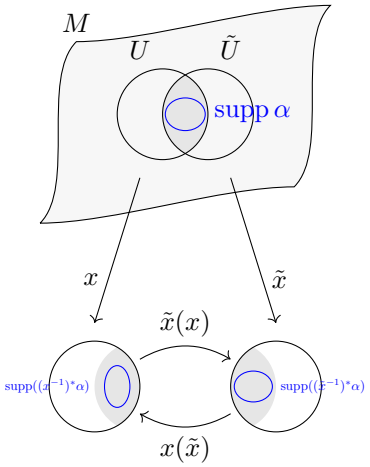


Let $\alpha \in \Omega^m_{\mathrm{cpt}}(U \cap \tilde{U})$ for two charts (U, x) and $(\tilde{U}, \tilde{x})$. Then, the integral is independent of the chart:

$$\int\limits_{x(U)} (x^{-1})^*\alpha = \int\limits_{x(U\cap\tilde{U})} (x^{-1})^*\alpha = \int\limits_{x(U\cap\tilde{U})} (\tilde{x}\circ x^{-1})^*(\tilde{x}^{-1})^*\alpha = \int\limits_{\tilde{x}(U\cap\tilde{U})} (\tilde{x}^{-1})^*\alpha = \int\limits_{\tilde{x}(\tilde{U})} (\tilde{x}^{-1})^*\alpha.$$

Now, define the integral of a general m-form using a partition of unity. Let $\alpha \in \Omega^m_{\mathrm{cpt}}(M)$ and $\sum_\iota \varphi_\iota = 1$ be a smooth partition of unity subordinate to the open cover of M by chart domains. It holds

$$\int_M \alpha \coloneqq \sum_\iota \int_M \varphi_\iota \alpha.$$



Note that the sum in the above expression is finite, and the integral on the right-hand side is well-defined as it is compactly supported on chart domains. We want to check that the result is independent of the partition of unity. Let $\sum_j \psi_j = 1$ be another partition of unity subordinate to the open cover. Then

$$\sum_\iota \int_M \varphi_\iota \alpha = \sum_{\iota,j} \int_M \varphi_\iota \psi_j \alpha = \sum_j \int_M \psi_j \alpha$$

since all sums are finite. This finishes the construction of the integral. We will state one final remark on the local version of diffeomorphism invariance (the last lemma). Let $F : M_1^m \to M_2^m$ be an orientation preserving smooth diffeomorphism between oriented manifolds and let $\beta \in \Omega^m_{\mathrm{cpt}}(M_2)$. Then

$$\int_{M_1} F^*\beta = \int_{M_2} \beta.$$

The lemma yields the assertion for forms compactly supported on a chart domain. The general case follows by linearity.

We want to extend the integral to the integration of forms on smooth maps. Let N^n be an oriented manifold and M^m not necessarily an oriented manifold. Consider a smooth map $F : N \to M$. Let $\alpha \in \Omega^n(M)$ such that $\operatorname{supp}(F^*\alpha)$ is compact (one possibility is that N is compact or if α has compact support and F is *proper*). We define the integral as

$$\int_F \alpha \coloneqq \int_N F^*\alpha.$$

Let's look at some important special cases:

1) *Integration over oriented submanifolds*: let N^n be a smooth oriented submanifold embedded in not necessarily oriented smooth manifold M^m. Let $\alpha \in \Omega^n(M)$ such that $\operatorname{supp}(\alpha|_N)$ is compact. There is a smooth inclusion $\iota : N \to M$. Then

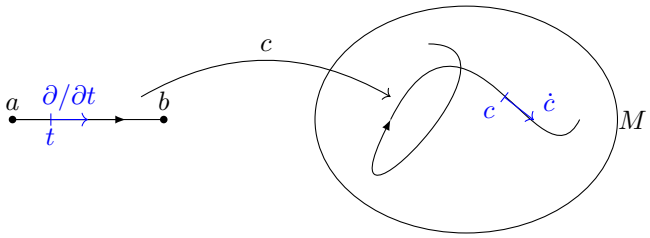
$$\int_N \alpha = \int_\iota \alpha = \int_N \iota^* \alpha = \int_N \alpha|_N.$$

This is the integral of α over the submanifold N.

2) *Line (or curve) integral* of 1-forms over curves ($n = 1$): let $c : [a, b] \to M^m$ be a smooth map between oriented interval and (not necessarily oriented) smooth manifold and let $\alpha \in \Omega^1(M)$. We don't need to worry about compactness as the interval is already compact. Then

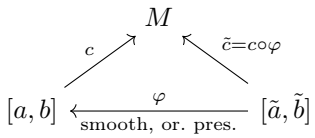
$$\int_c \alpha = \int_{[a,b]} c^* \alpha = \int_a^b \alpha_{c(t)}(\dot{c}(t)) \, \mathrm{d}t.$$

This is an integral of 1-form over the curve c.



Note. The invariance of the integral of forms under orientation preserving diffeomorphism φ implies the invariance of the curve integral under orientation preserving reparametrisation:

$$\int_{\tilde{c}} \alpha = \int_{[\tilde{a},\tilde{b}]} \tilde{c}^* \alpha = \int_{[\tilde{a},\tilde{b}]} \phi^*(c^* \alpha) = \int_{[a,b]} c^* \alpha = \int_c \alpha.$$



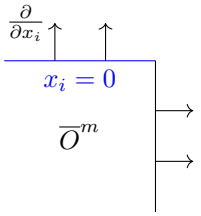
Addendum. Let M be m-dimensional oriented smooth manifold with corners and $\alpha \in \Omega^{m-1}_{\mathrm{cpt}}(M)$. We want to define the integral of α along the boundary. In the case when M is a smooth manifold with a boundary, this has already been defined as

$$\int_{\partial M} \alpha = \int_{\partial M} \alpha|_{\partial M}.$$

Note that ∂M is $(m-1)$-dimensional smooth manifold without boundary, which inherits the orientation from M. However, if there are boundary strata of codimension more than 1 (i.e. boundary points which in a chart correspond o points in $\partial \overline{O}^m$ with at least two coordinates vanishing), then ∂M is not a smooth manifold (only a topological manifold).

Remark. The codimension-1 boundary stratum $\partial^{(m-1)}M \subset \partial M$ is an $(m-1)$-dimensional smooth manifold without boundary and inherits an orientation from M. However, it is generally not a closed subset of M and hence compactly supported forms on M in general don't restrict to compactly supported forms on $\partial^{m-1}M$.

We wish to extend the definition of $\int_{\partial M}\alpha$ to the case with corners. On the local model, let $\overline{O}^m$ be an orthant with the standard orientation given by the standard volume form $\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_m$. The vector field $\partial/\partial x_i$ is outward pointing along the component $b_i=\{x_i=0, x_j<0, j\neq i\}$ of $\partial^{(m-1)}\overline{O}^m$ (denoted with blue colour on the sketch right). The induced orientation of this component is given by the form



$$\iota_{\partial/\partial x_i}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_m = (-1)^{i-1}\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_{i-1}\wedge\mathrm{d}x_{i+1}\wedge\cdots\wedge\mathrm{d}x_m.$$

For $f_i\in\mathcal{C}^\infty_{\mathrm{cpt}}(\overline{O}^m)$ and

$$\alpha=\sum_{i=1}^m f_i\,\mathrm{d}x_1\wedge\ldots\widehat{\mathrm{d}x_i}\cdots\wedge\mathrm{d}x_m\in\Omega^{m-1}_{\mathrm{cpt}}(\overline{O}^m)$$

define

$$\int_{\partial\overline{O}^m}\alpha\coloneqq\sum_{i=1}^m\int_{b_i\subset\overline{O}^m} f_i\,\mathrm{d}x_1\wedge\ldots\widehat{\mathrm{d}x_i}\cdots\wedge\mathrm{d}x_m.$$

Each individual term in the sum is a restriction of α to the closure of the i-th component of $\partial^{(m-1)}\overline{O}^m$. Also note that $b_i=\{x_i=0, x_j<0, i\neq j\}\cong\overline{O}^{m-1}$. We rewrite the wedge product of 1-forms as a positively oriented volume form on $\overline{O}^{m-1}$:

$$\begin{aligned}\int_{\partial\overline{O}^m}\alpha&=\sum_{i=1}^m(-1)^{i-1}\int_{b_i} f_i(-1)^{i-1}\,\mathrm{d}x_1\wedge\ldots\widehat{\mathrm{d}x_i}\cdots\wedge\mathrm{d}x_m\\&=\sum_{i=1}^m(-1)^{i-1}\int_{b_i} f_i\,\mathrm{d}x_1\ldots\widehat{\mathrm{d}x_i}\ldots\mathrm{d}x_m.\end{aligned}$$

The last line is the Lebesgue integral. The Lebesgue measure absorbed one of the minus signs. Again, we have the invariance under orientation preserving smooth diffeomorphisms of open subsets of $\overline{O}^m$. The integral $\int_{\partial M}\alpha$ for forms $\alpha\in\Omega^{m-1}_{\mathrm{cpt}}(M)$ compactly supported on charts is well-defined. Using the partition on unity, we can extend the definition to general forms $\alpha\in\Omega^{m-1}_{\mathrm{cpt}}(M)$.

IV.5 Stokes' theorem

This is a generalization of the fundamental theorem of calculus. If f is a smooth function $f:[a,b]\to\mathbb{R}$ then the following holds:

$$\int_a^b f'(x)\,\mathrm{d}x=f(b)-f(a).$$

We can reinterpret the function as 0-form on an oriented 1-dimensional smooth manifold $[a,b]$. The statement would then read

$$\int_{[a,b]} \mathrm{d}f = \int_{\partial[a,b]} f.$$

Theorem IV.3 (*Stokes' theorem*)**.** *Let M be an m-dimensional oriented smooth manifold ($m \geq 1$) with corners and $\alpha \in \Omega^{m-1}_{\mathrm{cpt}}(M)$. Then*

$$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha.$$

Proof. We first prove the theorem on the local model $\overline{O}^m$ with standard orientation. The form can be written as

$$\alpha = \sum_{i=1}^m f_i \,\mathrm{d}x_1 \wedge \ldots \widehat{\mathrm{d}x_i} \cdots \wedge \mathrm{d}x_m = \sum_{i=1}^m \alpha_i \in \Omega^{m-1}_{\mathrm{cpt}}(\overline{O}^m).$$

Using the definition from above, the integral of this form is

$$\int_{\partial \overline{O}^m} \alpha = \sum_{i=1}^m \int_{b_i} \alpha_i = \sum_{i=1}^m (-1)^{i-1} \int_{b_i} f_i \,\mathrm{d}x_1 \ldots \widehat{\mathrm{d}x_i} \ldots \mathrm{d}x_m.$$

The last term is a Lebesgue integral. Now

$$\begin{aligned}
\int_{b_i} f_i \,\mathrm{d}x_1 \ldots \widehat{\mathrm{d}x_i} \ldots \mathrm{d}x_m &= \int_{b_i} f_i(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m) \,\mathrm{d}x_1 \ldots \widehat{\mathrm{d}x_i} \ldots \mathrm{d}x_m \\
&\overset{(1)}{=} \int_{b_i} \left(\int_{-\infty}^0 \frac{\partial f_i}{\partial x_i}(x_1, \ldots, x_m) \,\mathrm{d}x_i \right) \mathrm{d}x_1 \ldots \widehat{\mathrm{d}x_i} \ldots \mathrm{d}x_m \\
&\overset{(2)}{=} \int_{\overline{O}^m} \frac{\partial f_i}{\partial x_i} \,\mathrm{d}x_1 \ldots \mathrm{d}x_m \\
&= \int_{\overline{O}^m} \frac{\partial f_i}{\partial x_i} \,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m \\
&= \int_{\overline{O}^m} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} \,\mathrm{d}x_i \wedge \mathrm{d}x_1 \wedge \ldots \widehat{\mathrm{d}x_i} \cdots \wedge \mathrm{d}x_m \\
&= \int_{\overline{O}^m} (-1)^{i-1} \,\mathrm{d}f_i \wedge \mathrm{d}x_1 \wedge \ldots \widehat{\mathrm{d}x_i} \cdots \wedge \mathrm{d}x_m \\
&= (-1)^{i-1} \int_{\overline{O}^m} \mathrm{d}(f_i \,\mathrm{d}x_1 \wedge \ldots \widehat{\mathrm{d}x_i} \cdots \wedge \mathrm{d}x_m) \\
&= (-1)^{i-1} \int_{\overline{O}^m} \mathrm{d}\alpha_i.
\end{aligned}$$

In step (1), we used the fundamental theorem of calculus and the compact support of the form,

and in step (2), we used *Fubini's theorem*. We now have

$$\begin{aligned}\int_{\partial\overline{O}^m} \alpha &= \sum_{i=1}^{m} (-1)^{i-1} \int_{b_i} f_i \, \mathrm{d}x_1 \ldots \widehat{\mathrm{d}x_i} \ldots \mathrm{d}x_m \\ &= \sum_{i=1}^{m} ((-1)^{i-1})^2 \int_{\overline{O}^m} \mathrm{d}\alpha_i \\ &= \int_{\overline{O}^m} \mathrm{d}\alpha.\end{aligned}$$

Using invariance of integrals under orientation preserving diffeomorphisms implies we can generalize Stokes' theorem to forms with compact support on charts, and the linearity of both sides of the equation and the partition of unity allow us to extend the theorem to general forms on manifolds. □

Chapter V

De Rham cohomology

V.1 Definition

De Rham cohomology is an algebraic invariant attached to smooth manifolds.

Remark (Smoothing of differentiable manifolds). According to Whitney we can make $\mathcal{C}^1$-approximations of $\mathcal{C}^1$-maps (diffeomorphisms) by $\mathcal{C}^\infty$-maps.

- Every $\mathcal{C}^{k\geq 1}$-structure contains a $\mathcal{C}^\infty$-structure (as a sub-atlas). In other words, every $\mathcal{C}^{k\geq 1}$-manifold is $\mathcal{C}^k$-diffeomorphic to a $\mathcal{C}^\infty$-manifold:

$$(M, \mathcal{A}_{\mathcal{C}^k}) \xrightarrow{\mathrm{id}_M} (M, \mathcal{A}_{\mathcal{C}^\infty}), \qquad \varphi : (M, \mathcal{A}_{\mathcal{C}^k}) \xrightarrow[\cong]{\mathcal{C}^k\text{-diffeo}} (M', \mathcal{A}'_{\mathcal{C}^\infty}).$$

- If two smooth manifolds are $\mathcal{C}^{k\geq 1}$-diffeomorphic, then they are $\mathcal{C}^\infty$-diffeomorphic. More on that can be found in the second chapter of Differential topology by M. Hirsch.

We can also view this as a functor from the category of $\mathcal{C}^k$-manifolds to the category of $\mathcal{C}^\infty$-manifolds – more specifically as a *forgetful functor* between classes of smooth diffeomorphisms on smooth manifolds and $\mathcal{C}^k$-diffeomorphisms on $\mathcal{C}^k$-manifolds. Whitney says that this functor is $1:1$.

Smoothing of topological manifolds is difficult and not always possible.

Let M be an m-dimensional smooth manifold with corners. De Rham cohomology is constructed in two steps (functors). The first step brings us from geometry (smooth manifolds) to *homological algebra*. One exploits the smooth structure by using the exterior derivative. It gives rise to a sequence of linear maps of $\mathbb{R}$-vector spaces:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} \Omega^m(M) \longrightarrow 0 \longrightarrow \ldots$$

with the property $\mathrm{d}^2 = 0$:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} \Omega^m(M) \longrightarrow 0.$$

This makes the sequence $(\Omega^\bullet(M), \mathrm{d})$ a *complex*, a basic object of homological algebra. It is called the *de Rham complex* of M.

Definition V.1. A differential form $\alpha \in \Omega^k(M)$ is called:
(i) *closed*, if $\mathrm{d}\alpha = 0$ and
(ii) *exact*, if $\alpha = \mathrm{d}\beta$ for some $\beta \in \Omega^{k-1}(M)$.

The property of the complex $\mathrm{d}^2 = 0$ is equivalent to

$$\operatorname{Im}|_{\Omega^{k-1}(M)} \subset \mathrm{d}|_{\Omega^k(M)}$$

and hence the implication

$$\text{exact} \implies \text{closed}.$$

The converse generally fails; that is, not every closed form is exact. This leads to the invariance of smooth manifolds, which will be introduced and studied here.

The second step is purely algebraic. The discrepancy between closedness and exactness is measured by taking the *homology* of the complex, a basic process in homological algebra. The *de Rham cohomology groups* (indeed this is a vector space (or module), but the name "group" stuck from algebraic topology where integer groups are used as they give more information) are defined as the cohomology groups of the de Rham complex:

$$H^k_{\mathrm{dR}}(M) \coloneqq \frac{\ker \mathrm{d}|_{\Omega^k(M)}}{\operatorname{Im} \mathrm{d}|_{\Omega^{k-1}(M)}}$$

for $k \geq 0$. One denotes by $[\alpha] \in H^k_{\mathrm{dR}}(M)$ the *cohomology class* represented by the closed form $\alpha \in \Omega^k(M)$. If finite, the dimensions of the de Rham groups (or $\mathbb{R}$-vector spaces) are called the *Betti numbers*:

$$\beta_k(M) \coloneqq \dim_{\mathbb{R}} H^k_{\mathrm{dR}}(M)$$

(denoting Betti numbers by β is a very natural thing to do). De Rham cohomology carries more than just a linear structure. The $\wedge$-product on forms descents to the *cup product* on cohomology, providing cohomology with a multiplicative structure:

$$\begin{aligned} \smile \colon H^k(M) \times H^l(M) &\to H^{k+l}(M) \\ ([\alpha], [\beta]) &\mapsto [\alpha \wedge \beta] \end{aligned}$$

for $k, l \geq 0$ (funny enough, the LaTeX command for $\smile$ is `\smile`). Definedness and well-definedness follow from the product rule for the exterior derivative. It implies:

$$\text{closed} \wedge \text{closed} = \text{closed},$$

$$\text{exact} \wedge \text{closed} = \text{exact}.$$

We can see this if we take two closed forms α and β ($\mathrm{d}\alpha = 0$ and $\mathrm{d}\beta = 0$). Then

$$\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathrm{d}\beta) = 0.$$

If $\alpha = \mathrm{d}\gamma$ and $\mathrm{d}\beta = 0$, then

$$\alpha \wedge \beta = (\mathrm{d}\gamma) \wedge \beta = \mathrm{d}(\gamma \wedge \beta) - (-1)^k \gamma \wedge (\mathrm{d}\beta) = \mathrm{d}(\gamma \wedge \beta).$$

The $\smile$-product makes the graded $\mathbb{R}$-vector space

$$H^\bullet(M) = \bigoplus_{k\geq 1} H^k_{\mathrm{dR}}(M)$$

into a graded-commutative, associative $\mathbb{Z}$-graded $\mathbb{R}$-algebra (represented by $1 \in \Omega^0(M)$) the *de Rham cohomology ring* of M (again the naming comes from algebraic topology, in our case it is an algebra, which is also a ring).

We want to discuss the *functoriality*. De Rham cohomology behaves well with respect to smooth maps. Let $F : M \to M$ be a smooth map. Then

$$F : M \to N \qquad \xrightarrow{\textit{contravariant process}} \qquad F^* : \Omega^k(N) \to \Omega^k(M)$$

the pull-back of forms commutes with $\wedge$ and d. Due to the commutation of pull-back and exterior derivative, we, in fact, obtain the *commutative ladder*.

$$\begin{array}{ccccccc}
\ldots & \xrightarrow{\mathrm{d}} & \Omega^k(N) & \xrightarrow{\mathrm{d}} & \Omega^{k+1}(N) & \xrightarrow{\mathrm{d}} & \ldots \\
 & & \downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} & & \\
\ldots & \xrightarrow{\mathrm{d}} & \Omega^k(M) & \xrightarrow{\mathrm{d}} & \Omega^{k+1}(M) & \xrightarrow{\mathrm{d}} & \ldots
\end{array}$$

which in the language of homological algebra is a homomorphism of complexes of $\mathbb{R}$-vector spaces. Since closedness and exactness are preserved by pull-back

$$\begin{array}{ccc}
\bullet & \xmapsto{\mathrm{d}} & 0 \\
\downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} \\
* & \xmapsto{\mathrm{d}} & 0
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & \xmapsto{\mathrm{d}} & \bullet \\
\downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} \\
 & \xmapsto{\mathrm{d}} & *
\end{array}$$

the homomorphism of complexes descends to linear maps on their homology

$$\begin{aligned}
H^kF : H^k(N) &\to H^k(M) \\
[\alpha] &\mapsto [F^*\alpha].
\end{aligned}$$

$$\begin{array}{ccc}
\ker \mathrm{d}|_{\Omega^k(N)} & \supset & \mathrm{Im}\, \mathrm{d}|_{\Omega^{k-1}(N)} \\
\downarrow & & \downarrow \\
\ker \mathrm{d}|_{\Omega^k(M)} & \supset & \mathrm{Im}\, \mathrm{d}|_{\Omega^{k-1}(M)}
\end{array}$$

Since all pull-back of forms commutes with the wedge product, the multiplicative structure is respected. If $\alpha \in \Omega^k(N)$ and $\beta \in \Omega^l(N)$, then

$$H^\bullet F([\alpha] \smile [\beta]) = H^\bullet F([\alpha \wedge \beta]) = [F^*(\alpha \wedge \beta)] = [F^*\alpha] \smile [F^*\beta] = H^\bullet F([\alpha]) \smile H^\bullet F([\beta]).$$

The linear maps H^kF combine to a homomorphism of graded $\mathbb{R}$-algebras

$$H^\bullet F : H^\bullet(N) \to H^\bullet(M).$$

The composition of maps is respected. If

$$M \underset{G\circ F}{\overset{F}{\xrightarrow[\mathcal{C}^\infty]{}} N \xrightarrow[\mathcal{C}^\infty]{G}} P$$

then

$$H^\bullet(P) \xrightarrow{H^\bullet G} H^\bullet(N) \xrightarrow{H^\bullet F} H^\bullet(M).$$
$$H^\bullet F \circ H^\bullet G$$

In other words

$$\boxed{H^\bullet(G \circ F) = H^\bullet F \circ H^\bullet G.}$$

Indeed,

$$(H^\bullet(G \circ H))([\alpha]) = [(G \circ F)^* \alpha] = [F^*(G^* \alpha)] = H^\bullet F([G^* \alpha]) = (H^\bullet F \circ H^\bullet G)([\alpha])$$

and identity maps of manifolds induce identity maps on cohomology:

$$\boxed{H^\bullet \operatorname{id}_M = \operatorname{id}_{H^\bullet(M)}.}$$

The computation for this is a bit shorter

$$H^\bullet \operatorname{id}_M([\alpha]) = [\operatorname{id}_M^* \alpha] = [\alpha] = \operatorname{id}_{H^\bullet(M)}([\alpha]).$$

Note that the pull-back of forms behaves well with respect to composition ($(F \circ G)^* = F^* \circ G^*$). This good behaviour under smooth maps makes de Rham cohomology into a *contravariant functor* from smooth manifolds to graded $\mathbb{R}$-algebras. In particular, de Rham cohomology is a smooth diffeomorphism invariant of smooth manifolds, i.e. smooth diffeomorphisms of smooth manifolds induce isomorphisms in de Rham cohomology. For a deeper introduction to functors, you can consult any introductory book on category theory. Statements $H^\bullet \operatorname{id}_M = \operatorname{id}_{H^\bullet(M)}$ and $H^\bullet(G \circ F) = H^\bullet F \circ H^\bullet G$ are exactly the defining properties of a contravariant functor.

Let's look at the situation more abstractly. Let $\mathcal{C}, \mathcal{C}'$ be two *categories*. Consider $A, B \in \mathcal{C}$ be two objects from category $\mathcal{C}$ and let $F : A \to B$ and $G : B \to A$ be *morphisms* in this category. Moreover, let $A', B' \in \mathcal{C}'$ be two objects from a different category $\mathcal{C}'$ and $F' : A' \to B'$ and $G' : B' \to A'$ be morphisms in $\mathcal{C}'$. *Functors* are structure-preserving maps between categories and morphisms.

$$\begin{array}{ccc} A & & A' \\ {\scriptstyle F}\big(\ \big){\scriptstyle G} & \xrightarrow[\text{functor}]{\text{covariant}} & {\scriptstyle F'}\big(\ \big){\scriptstyle G'} \\ B & & B' \end{array}$$

A functor is *covariant* if

$$\begin{array}{l} G \circ F = \operatorname{id}_A \\ F \circ G = \operatorname{id}_B \end{array} \implies \begin{array}{l} G' \circ F' = \operatorname{id}_{A'} \\ F' \circ G' = \operatorname{id}_{B'}. \end{array}$$

$$\begin{array}{ccc} A & & A' \\ {\scriptstyle F}\big(\ \big){\scriptstyle G} & \xrightarrow[\text{functor}]{\text{contravariant}} & {\scriptstyle G'}\big(\ \big){\scriptstyle F'} \\ B & & B' \end{array}$$

A functor is *contravariant* if

$$\begin{array}{l} G \circ F = \operatorname{id}_A \\ F \circ G = \operatorname{id}_B \end{array} \implies \begin{array}{l} F' \circ G' = \operatorname{id}_{A'} \\ G' \circ F' = \operatorname{id}_{B'}. \end{array}$$

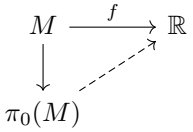
In our case, we have the category of smooth manifolds where morphisms are smooth diffeomorphisms between manifolds and the category of graded $\mathbb{R}$-algebras where morphisms are just homomorphisms between algebras. Conditions $H^\bullet \mathrm{id}_M = \mathrm{id}_{H^\bullet(M)}$ and $H^\bullet(G \circ F) = H^\bullet F \circ H^\bullet G$ then exactly define a contravariant functor.

We will see later that de Rham cohomology has much stronger invariance properties, namely that it is a *homotopy invariant*. Thus, even though de Rham cohomology is constructed using the smooth structure in an essential way, it has *topological* meaning. Now come some good news and some bad news:

- The bad news is that there is no distinction of smooth structure on the same topological manifold.
- The good news is that the smooth structure is a useful tool that helps understand the underlying manifold's topology.

V.2 Topological meaning in degree 0

Let's get straight to the point: H^0 detects connected components. Indeed, for a smooth function $f \in \mathcal{C}^\infty(M) = \Omega^0(M)$. This form is closed exactly when $\mathrm{d}f = 0$, i.e. f is locally constant.



Function f descends to a function on set $\pi_0(M)$ of connected components of M, where $\pi_0(M)$ is the homotopy invariant. Thus,

$$H^0(M) = \{f \in \mathcal{C}^\infty | f \text{ loc. constant}\} \cong \mathbb{R}^{\pi_0(M)} = \{\text{maps } \pi_0(M) \to \mathbb{R}\}.$$

This is an $\mathbb{R}$-vector space and the Betti number is $\beta_0(M) = |\pi_0(M)|$.

V.3 Non-vanishing in degree ≥ 1

We will first look at cohomology in the top degree. Let $M^{m \geq 1}$ be an oriented closed (compact without boundary) smooth manifold. All smooth m-forms are closed, since $\Omega^{m+1}(M) = 0$. However, they are not all exact. The natural linear functional

$$\int_M : \Omega^m(M) \to \mathbb{R}$$
$$\alpha \mapsto \int_M \alpha$$

given by the integration of the manifold vanishes on exact forms:

$$\int_M \mathrm{d}\eta \overset{\text{Stokes}}{=} \int_{\partial M} \eta = 0$$

as $\partial M = \varnothing$. Note that M is compact, so we can indeed integrate. Hence, it descends to a linear functional on cohomology

$$\begin{aligned} \int_M : H^m(M) &\to \mathbb{R} \\ [\alpha] &\mapsto \int_M \alpha. \end{aligned}$$

It is non-trivial! Indeed, any volume form $\mathrm{vol} \in \Omega^m(M)$ has a non-vanishing integral and, therefore, represents a non-zero cohomology class. This observation generalizes to submanifolds. Let N^n be an embedded closed smooth oriented submanifold of M, so an inclusion $\iota : N \to M$ exists. If there exists a closed form $\alpha \in \Omega^n(M)$ with $\int_N \alpha \neq 0$ (meaning it restricts to a volume form on N) then α represents a non-trivial cohomology class $0 \neq [\alpha] \in H^n(M)$ because its image under the restriction map

$$\begin{aligned} H^n\iota : H^n(M) &\to H^n(N) \\ [\alpha] &\mapsto [\alpha|_N] \end{aligned}$$

is non-zero. In particular $H^n(M) \neq 0$ in this case. However, clearly, such a form α cannot always exist. For instance, if N can be "filled in", i.e. if there exists a compact embedded oriented smooth submanifold W^{n+1} with $\partial W = N$, then any closed form $\alpha \in \Omega^n(M)$ integrates over N to 0 by Stokes:

$$\int_N \alpha = \int_W \mathrm{d}\alpha = 0.$$

The existence of a closed form $\alpha \in \Omega^n(M)$ with $\int_N \alpha \neq 0$ thus reflects that the topological position of the submanifold N inside the ambient manifold M is non-trivial.

Example 11 (Punctured plane)**.** Consider the plane without the origin $\mathbb{R}^2 - \{0\} = \mathbb{C}^*$. The function $z = re^{i\theta} \mapsto \theta$ is multivalued and hence well-defined only up to integer multiples of 2π. Fortunately, additive constants are annihilated by differentiation. Hence, $\mathrm{d}\theta \in \Omega^1(\mathbb{C}^*)$ is well-defined and locally exact (hence closed). However, it is not globally exact. Non-exactness follows from the fact that $\mathrm{d}\theta$ restricts to a volume form on the unit circle $S^1 \subset \mathbb{C}^*$. The integral of $\mathrm{d}\theta$ over S^1 (oriented in counterclockwise direction) is non-zero, hence $0 \neq [\mathrm{d}\theta] \in H^1(\mathbb{C}^*)$. Note that the position of S^1 in $\mathbb{C}^*$ is topologically non-trivial! It "winds" around the "hole" of $\mathbb{C}^*$. We can generalize this to a plane that is punctured multiple times.

The following topological notion describes a situation when a subspace has a non-trivial position in the ambient space.

Definition V.2. Let X be a topological space and $Z \subset X$ a subspace. The inclusion is $i : Z \to X$. The *retraction* from X to Z is a continuous map $r : X \to Z$ such that $r \circ i = \mathrm{id}_Z$. The subspace is called a *retract*.

Intuitively, we can "fold" the space onto its retract and this process generally simplifies the topology of space (the topology of the ambient space is "at least as reach" as the topology of a retract). The retract carries a part of the topology of ambient space.

We can use retracts to estimate the cohomology from below in the smooth setting. Let $N \subset M$ be an embedded smooth manifold, which is a smooth retract. Then

$$N \underset{i}{\overset{r}{\leftrightarrows}} M$$

with $r \circ i = \mathrm{id}_N$. We pass these maps to cohomology

$$H^\bullet(N) \underset{H^\bullet i}{\overset{H^\bullet r}{\rightleftarrows}} H^\bullet(M)$$

with $H^\bullet i \circ H^\bullet r = \mathrm{id}_{H^\bullet(N)}$. This means that $H^\bullet i$ is surjective and $H^\bullet r$ is injective. Then $H^\bullet(M)$ is a least as rich as $H^\bullet(N)$ on the level of differential forms and $\alpha \in H^\bullet(N)$ extends to $r^*\alpha \in \Omega^\bullet(M)$. Indeed,

$$r^*\alpha|_N = i^* r^* \alpha = (r \circ i)^* \alpha = \alpha.$$

Note. It holds that $r^*\alpha$ is closed/exact if and only if $\alpha = i^*(r^*\alpha)$ is closed/exact. Also, remember that pull-backs i^* and r^* commute with d.

Example 12. Consider $\mathbb{R}^m - \mathbb{R}^k \times \{0_{\mathbb{R}^{m-k}}\}$. This set retracts on $S^{m-k-1} \subset \{0_{\mathbb{R}^k}\} \times \mathbb{R}^{m-k}$. Since the $(m-k-1)$-th cohomology of S^{m-k-1} is non-zero it follows by the extension

$$H^{n-k-1}(S^{m-k-1}) \xrightarrow{H^\bullet r} H^{n-k-1}(\mathbb{R}^m - \mathbb{R}^k)$$

that the $(m-k-1)$-th cohomology of $\mathbb{R}^m - \mathbb{R}^k$ is non-zero.

One can use the product structure on cohomology to detect non-zero cohomology classes. Let M be a smooth manifold and $\alpha, \beta \in \Omega^\bullet(M)$ closed. On the level of differential forms, we know

$$\alpha \wedge \beta \text{ non-exact} \implies \alpha, \beta \text{ non-exact}.$$

Equivalently, on the level of cohomology, if $[\alpha \wedge \beta] \neq 0$ in $H^\bullet(M)$ then $[\alpha] \smile [\beta] \neq 0$ and thus $[\alpha], [\beta] \neq 0$.

Example 13. A *symplectic* form β is a 2-form that is closed and pointwise non-degenerate. Since $\beta : V \times V \to \mathbb{R}$ we have a canonical diffeomorphism $V \cong V^*$ with $V \ni v \mapsto \beta(v, \bullet) \in V^*$, i.e. if $\beta(x, y) = 0$ for all $y \in V$, then $x = 0$. If $\beta(e_1, e_2) \neq 0$, then $\langle e_1, e_2 \rangle \oplus \langle e_1, e_2 \rangle^{\perp_\beta} = V$ and $e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + \cdots + e_{2n-1}^* \wedge e_{2n}^* = \beta$. It is easy to check that

$$\beta^n = n! e_1^* \wedge e_2^* \wedge \cdots \wedge e_{2n}^*.$$

Let (M^{2n}, ω) be a smooth closed symplectic manifold (and ω a symplectic form on M). The previous discussion shows that ω^n is a volume form. Hence, $[\omega^n] \neq 0$ in $H^{2n}(M)$ and $[\omega] \neq 0$ in $H^2(M)$. Even more, $[\omega^{2i}] \neq 0$ for $i \leq n$. For example, we know that $H^2(S^4)$ is trivial, so S^4 admits no symplectic structure.

Example 14. Consider a torus $T = S_1^1 \times S_2^1$. Let $\pi_1 : T \to S_1^1$ and $\pi_2 : T \to S_2^1$ be projection maps to corresponding factors. Moreover, let $\overline{\alpha}_i \in \Omega^1(S_i^1)$, $i = 1, 2$ be volume forms on factors. Define $\alpha_i = \pi_i^*(\overline{\alpha}_i) \in \Omega^1(T)$. It follows that $\alpha_1 \wedge \alpha_2 \in \Omega^2(T)$ is a volume form and thus $[\alpha_1 \wedge \alpha_2] = [\alpha_1] \smile [\alpha_2] \neq 0$ in $H^2(T)$. To prove that $[\alpha_1]$ and $[\alpha_2]$ are linearly independent in $H^1(T)$, consider $c_1\alpha_1 + c_2\alpha_2 = \mathrm{d}f$ for some $f \in \mathcal{C}^\infty(T)$. Then it holds

$$0 = \int_{S_i^1} \mathrm{d}f = \int_{S_i^1} (c_1\alpha_1 + c_2\alpha_2) = \sum_j c_j \int_{S_i^1} \alpha_j = \sum_j c_j \delta_{ij} \int_{S_i^1} \alpha_i = c_i \int_{S_i^1} \alpha_i.$$

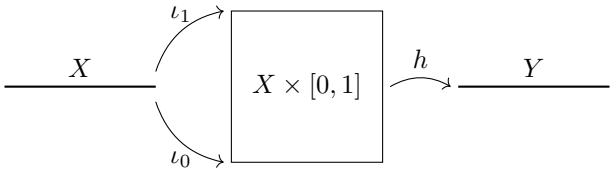
Since α_i is a volume form and its integral is non-zero, it must hold that $c_i = 0$. So $[\alpha_1]$ and $[\alpha_2]$ are indeed independent, and we can conclude that $\beta_1(T^2) \geq 2$. More generally, an analogous argument yields that $\beta_k(T^n) \geq \binom{n}{k}$.

V.4 Homotopy invariance

Consider a topological space X and another space $X \times [0,1]$, which we can think of as a cylinder or a prism with base X. A *cross-sectional embedding* is a map

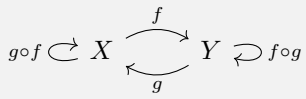
$$\begin{aligned} \iota_t^X : X &\to X \times [0,1] \\ x &\mapsto (x,t). \end{aligned}$$

Definition V.3. A *homotopy* between continuous maps $f_0, f_1 : X \to Y$ between two topological spaces is a continuous map $h : X \times [0,1] \to Y$ such that $f_i = h \circ \iota_i^X$ for $i = 0, 1$.



Being homotopic is an equivalence relation on the set of continuous maps between topological spaces. We denote $[f]$ the homotopy class represented by f.

Definition V.4. A continuous map $f : X \to Y$ is a *homotopy equivalence* if there exists a homotopy inverse, i.e. a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. The symbol $\simeq$ denotes homotopic equivalence.



Being homotopy equivalent is an equivalence relation on topological spaces. An equivalence class is called a *homotopy type* (e.g. "contractible" is defined to be homotopically equivalent to the point). In the smooth case there is an analogous notion between smooth maps of smooth homotopy. It doesn't matter whether one works with smooth or continuous homotopies due to

$$\{\mathcal{C}^\infty\text{-homotopy classes of } \mathcal{C}^\infty\text{-maps } M \to N\} \xrightarrow{1:1} \{\mathcal{C}^0\text{-homotopy classes of } \mathcal{C}^0\text{-maps } M \to N\}$$

for M, N smooth manifolds. This means that we can use the smooth structure to study the homotopy properties of manifolds since every smooth homotopy invariant is actually a (continuous) homotopy invariant. In order to establish that de Rham cohomology is a homotopy invariant, the key step is the following result on the level of differential forms.

Lemma V.1. *Let $f_0, f_1 : M \to N$ be smooth homotopic smooth maps. Then $f_1^* - f_0^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ maps closed forms to exact forms.*

Proof. There exists a smooth homotopy $h : M \times [0,1] \to N$ from f_0 to f_1, i.e. $f_i = h \circ \iota_i$ for $i = 0, 1$. It follows $f_i^* = \iota_i^* \circ h^*$ so $f_1^* - f_0^* = (\iota_1^* - \iota_0^*) \circ h^*$. Since h^* preserves closedness, it suffices to show that $\iota_1^* - \iota_2^*$ maps closed forms to closed forms, i.e. the assertion reduces to the

"model homotopy" $\mathrm{id}_{M\times[0,1]}$ from ι_0 to ι_1. This is in turn equivalent, modulo integration, to the infinitesimal assertion that the linear maps

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=\tau} \iota_t^* : \Omega^\bullet(M\times[0,1]) \to \Omega^\bullet(M)$$
$$\alpha \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=\tau} \iota_t^*\alpha$$

for $\tau \in [0,1]$ send closed forms to exact forms. Consider on $M\times[0,1]$ the coordinate vector field $\partial/\partial t$. It generates the translation flow $\Phi = \Phi_t$ in t direction: $\Phi_t(x,s) = (x, s+t)$ for $0 \le s \le s+t \le 1$. The flow Φ homotopes cross-sections to cross-sections: $\Phi \circ \iota_s = \iota_{s+t}$. On forms, it acts as $\iota_{s+t}^* = \iota_s^* \circ \Phi_t^*$. Let $\alpha \in \Omega^\bullet(M\times[0,1])$. Then for $0 \le t \le 1$, $\iota_t^*\alpha \in \Omega^\bullet(M)$ is a restriction to cross-sections. The change in time of the form is given by the Lie derivative of α in t direction:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iota_t^*\alpha = \left.\frac{\mathrm{d}}{\mathrm{d}h}\right|_{h=0} \iota^* t + h\alpha = \iota_t^* \left.\frac{\mathrm{d}}{\mathrm{d}h}\right|_{h=0} \Phi_h^*\alpha = \iota_t^*(L_{\partial/\partial t}\alpha).$$

We are interested in the case when α is closed. Then Cartan's formula yields

$$L_{\partial/\partial t}\alpha = i_{\partial/\partial t}(\mathrm{d}\alpha) + \mathrm{d}(i_{\partial/\partial t}\alpha) = \mathrm{d}(i_{\partial/\partial t}\alpha) = \text{exact}.$$

Form α is closed, so by definition $\mathrm{d}\alpha = 0$. It follows then

$$\frac{\mathrm{d}}{\mathrm{d}t}\iota_t^*\alpha = \iota_t^*(L_{\partial/\partial t}\alpha) = \iota_t^*(\mathrm{d}i_{\partial/\partial t}\alpha) = \mathrm{d}(\iota_t^* i_{\partial/\partial t}\alpha).$$

One needs to be careful to distinguish between the pullback of the cross-sectional embedding ι_t^* and the interior product $i_{\partial/\partial t}$. More precisely, the smooth family of forms $\partial/\partial t\, \iota_t^*$ are the exterior derivative of the smooth family of forms $\iota_t^* i_{\partial/\partial t}\alpha$. Integrating yields that

$$\iota_{t_1}^* - \iota_{t_2}^* = \mathrm{d}\left(\int_{t_1}^{t_2} \mathrm{d}t\, \iota_t^* i_{\partial/\partial t}\alpha\right)$$

are exact. □

This lemma immediately implies the homotopy invariance of de Rham cohomology, i.e. for smooth homotopic maps $f, g : N \to M$ the induced cohomology maps agree $H^\bullet f_0 = H^\bullet f_1 : H^\bullet(N) \to H^\bullet(N)$. As a consequence, every *continuous* map $f : M \to N$ of smooth manifolds induces a well-defined map $H^\bullet f : H^\bullet(N) \to H^\bullet(M)$ on the de Rham cohomology, namely one defines $H^\bullet f := H^\bullet \tilde{f}$ where $\tilde{f} : M \to N$ is a *smooth* map homotopic to f. The induced map $H^\bullet f$ depends only on the homotopy class $[f]$ of f. These more general induced maps on cohomology keep behaving functorially. In particular, de Rham cohomology is a homotopy invariant of smooth manifolds.

Corollary V.2. *Homotopy equivalences of smooth manifolds induce isomorphisms on de Rham cohomology.*

Proof. Let $f : M \to N$ and $g : N \to M$ be continuous maps such that $g\circ f \simeq \mathrm{id}_M$ and $f \circ g = \mathrm{id}_N$ (i.e. they are homotopy equivalences). By functoriality, it follows that

$$H^\bullet(N) \underset{H^\bullet f}{\overset{H^\bullet g}{\rightleftarrows}} H^\bullet(M)$$

such that $H^\bullet f \circ H^\bullet g = \mathrm{id}_{H^\bullet(M)}$ and $H^\bullet g \circ H^\bullet f = \mathrm{id}_{H^\bullet(N)}$, i.e. $H^\bullet f$ and $H^\bullet g$ are inverse to each other and hence isomorphisms. □

There is an interesting consequence of this corollary.

Corollary V.3. *Homeomorphisms of smooth manifolds induce isomorphisms on the de Rham cohomology.*

The little we know about the non-vanishing of the de Rham cohomology already enables us to distinguish some manifolds up to homeomorphisms and even up to homotopy equivalence.

Example 15. Take two closed orientable smooth manifolds M^m and N^n and assume homotopy equivalence. Then they must have equal dimensions, $m = n$. This follows from the non-vanishing of their de Rham cohomology in top degree.

As a special case of homotopy equivalence, we have the following strengthening of the notion of retract.

Definition V.5. A retraction $r : X \to Z$ of topological space onto a subspace $Z \subset X$ is called a *deformation retraction* if the retraction is homotopic to the identity, $r \simeq \mathrm{id}_X$.

Remark. One speaks of a *strong deformation retraction* if $r \simeq \mathrm{id}_X$ rel Z, i.e. if there is a homotopy which does not move the points of Z. With other words, for $h : X \times [0,1] \to X$ it holds $h(z,t) = z$ for all $z \in Z$ and $t \in [0,1]$. Intuitively, one can "fold" X onto Z by a continuous process, not just "instantaneously".

Deformation retractions are homotopy equivalences. More precisely, they are homotopy inverses of the inclusions, implying that the inclusions themselves are homotopy equivalences. Thus, whereas a retract of a space carries, in general, only part of its homotopy type, a deformation retract carries all of it.

Example 16. A punctured Euclidean space $\mathbb{R}^M - \{0\}$ is a deformation retract of the sphere S^{M-1}. More generally, $\mathbb{R}^m - \mathbb{R}^n \times \{0_{\mathbb{R}^{m-n}}\}$ is a deformation retract of S^{m-k-1}. Consequently, $\mathbb{R}^m - \mathbb{R}^k \simeq \mathbb{R}^{m'} - \mathbb{R}^{k'}$ if and only if $m - k = m' - k'$.

Interestingly, this observation already suffices to prove the topological invariance of dimensions.

Proposition V.4. *Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be two open subsets (U non-empty) and $F : U \to V$ a homeomorphism. Then $m = n$.*

This implies that two manifolds of different dimensions cannot be locally diffeomorphic.

Proof. Suppose without loss of generality that $1 \leq m < n$. Let $x_0 \in U$ and $y_0 = F(x_0) \in V$. Then $U - \{x_0\}$ and $V - \{y_0\}$ (radially) retract onto a small sphere around x_0 and y_0:

$$\begin{array}{ccc} U - \{x_0\} & \xrightarrow[\text{homeo}]{F|_{U-\{x_0\}}} & V - \{y_0\} \\ {\scriptstyle\text{incl}}\uparrow \quad \downarrow{\scriptstyle\text{retract}} & & {\scriptstyle\text{incl}}\uparrow \quad \downarrow{\scriptstyle\text{retract}} \\ S^{m-1} & & S^{n-1} \end{array}$$

We may suppose furthermore (by passing to open subsets) that U is a ball around x_0 in a geometrical sense (not only in the topological sense). Then $U - \{x_0\}$ deformation retracts onto the small sphere around x_0.

$$\begin{array}{ccc} U - \{x_0\} & \xrightarrow[\text{homeo}]{F|_{U-\{x_0\}}} & V - \{y_0\} \\ {\scriptstyle\text{incl}}\uparrow \simeq \downarrow {\scriptstyle\text{defo. retract}} & & \downarrow {\scriptstyle\text{retract}} \\ S^{m-1} & \xrightarrow{\quad G \quad} & S^{n-1} \end{array}$$

We obtain a continuous map $G : S^{m-1} \to S^{n-1}$ which behaves like a retraction (having a right homotopy inverse).

$$\begin{array}{ccc} H^\bullet(U - \{x_0\}) & \xleftarrow[\cong]{} & H^\bullet(V - \{y_0\}) \\ \simeq \downarrow {\scriptstyle\text{incl}^*} & & {\scriptstyle\text{ret}^*}\uparrow \\ H^\bullet(S^{m-1}) & \xleftarrow[H^\bullet G]{} & H^\bullet(S^{n-1}) \end{array}$$

It is therefore injective on the de Rham cohomology, and in particular, this yields an injective map

$$H^{n-1}G : H^{n-1}(S^{n-1}) \rightarrowtail H^{n-1}(S^{m-1}).$$

However, $H^{n-1}(S^{n-1}) \neq 0$ because volume forms represent non-zero cohomology classes and $H^{n-1}(S^{m-1}) = 0$ because $n - 1 > m - 1$. Since $H^{n-1}G$ should be injective, this leads to a contradiction. □

The simplest homotopy type is the one of a point.

Definition V.6. A space is called *contractible* if it is homotopy equivalent to a point.

Equivalently, the identity is homotopic to a constant map or the space deformation retracts to a point.

Example 17. A star-shaped (e.g. convex) subset of $\mathbb{R}^n$ radially deformation retracts to a point and, therefore, is contractible.

The homotopy equivalence of the de Rham cohomology implies the following.

Theorem V.5. *If M is contractible smooth manifold, then $H^{\geq 1}(M) = 0$.*

Remark. The converse does not hold.

The property that $H^{\geq 1} = 0$ is equivalent to the implication "closed $\implies$ exact" for smooth differential forms of degree ≥ 1. This means that there are no obstructions to the existence of global solutions to certain PDEs. In particular, we obtain the following.

Lemma V.6 (*Poincaré lemma*)**.** *On a star-shaped open subset of $\mathbb{R}^m$, every closed smooth differential form of degree ≥ 1 is exact.*

Consequently, on a smooth manifold, we have the implication

$$\text{closed} \implies \text{locally exact}$$

for smooth differential forms of degree ≥ 1. In this case, there are no local contributions to de Rham cohomology. It is a global invariant.

Remark. From the point of view of differential equations, the Poincaré lemma says that the exterior derivative is the full integrability obstruction to exactness ($\mathrm{d}\alpha = 0$ is the full integrability condition for $\alpha = \mathrm{d}\eta$).

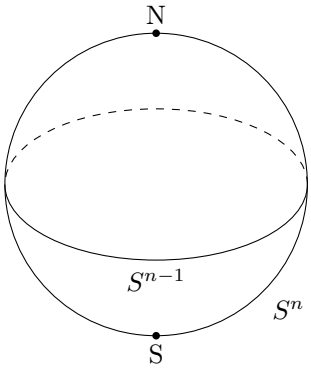
V.5 Cohomology of spheres

We will use a Mayer-Vietoris type arguments for computing the de Rham cohomology of smooth manifolds:

- choose an open cover of the manifold,
- relate the cohomologies of the manifold, the covering sets and their overlaps.

The computation of the cohomology of spheres is an easy example of this method and gives a good taste of it. *Mayer-Vietoris theorem* is an important result in homological algebra, but we don't have enough time to cover it in this course.

Consider $S^n \subset \mathbb{R}^{n+1}$ with an open cover $U = S^n - \{\mathrm{S}\}$, $V = S^n - \{N\}$. The two pieces of the cover are contractible, but the overlap is rather simple $U \cap V \cong S^{n-1} \times (-1, 1) \simeq S^{n-1}$. We will be able to relate the cohomologies of S^n and S^{n-1} and thus determine $H^\bullet(S^n)$ by induction over dimension.



Let's first determine the cohomology of $H^1(S^n)$ for $n \geq 1$. Let $[\alpha] \in H^1(S^n)$. Since U and V are contractible, by Poincaré lemma it follows

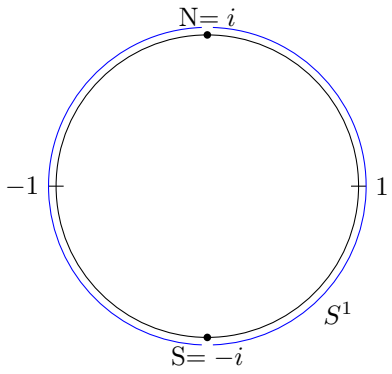
$$\alpha|_U = \mathrm{d}f, \qquad f \in \mathcal{C}^\infty(U),$$

$$\alpha|_V = \mathrm{d}g, \qquad g \in \mathcal{C}^\infty(V).$$

On the overlap $U \cap V$ it holds $\mathrm{d}(f - g) = 0$ so $f - g$ is *locally* constant. For $n \geq 2$ the overlap $U \cap V$ is connected, so $f - g$ is constant. We add a suitable constant to either of the functions, so we can, without loss of generality, assume that $f = g$ on $U \cap V$. Then f and g are both restrictions of a globally defined function $h \in \mathcal{C}^\infty(S^n)$ and hence $\alpha = \mathrm{d}h$ which is exact. With this, we conclude

$$H^1(S^{n\geq 2}) = 0.$$

The case for $n = 1$ is more interesting. Let $S^1 \subset \mathbb{C}$ have a standard orientation. The overlap $U \cap V$ has two components (drawn in blue on the sketch below).



We split the integral of the form α into two hemispheres (semicircles):

$$\begin{aligned}\int_{S^1}\alpha &= \int_{S^1\cap\{\text{Im } z\geq 0\}}\alpha \;+\; \int_{S^1\cap\{\text{Im } z\leq 0\}}\alpha \\ &= \int_{S^1\cap\{\text{Im } z\geq 0\}}\mathrm{d}f \;+\; \int_{S^1\cap\{\text{Im } z\leq 0\}}\mathrm{d}g \\ &= f|_1^{-1} + g|_{-1}^{1} \\ &= (g(1)-f(1)) - (g(-1)-f(-1)).\end{aligned}$$

If $\int_{S^1}\alpha = 0$, then $g = f$ on $S^1\cap\{\text{Re } z > 0\}$ if and only if $f = g$ on $S^1\cap\{\text{Re } z < 0\}$. Hence, in this case, we can modify f or g by adding a constant such that $f = g$ on *both* components of $U\cap V$. This shows that

$$\int_{S^1}\alpha = 0 \implies \alpha \text{ is exact.}$$

Consequently, the natural surjective linear map

$$\begin{aligned}\int_{S^1} : H^1(S^1) &\to \mathbb{R} \\ [\alpha] &\to \int_{S^1}\alpha\end{aligned}$$

is an isomorphism (and well-defined by Stokes). In particular, $H^1(S^1)\cong\mathbb{R}$.

Next, we determine the higher cohomology groups below the top degree. We relate the cohomology of spheres in successive dimensions.

Claim V.7. *For* $2\leq k<n$

$$H^{k-1}(S^{n-1}) = 0 \implies H^k(S^n) = 0.$$

Proof. Let $[\alpha]\in H^k(S^n)$. By Poincaré lemma it holds

$$\alpha|_U = \mathrm{d}\xi, \quad \xi\in\Omega^{k-1}(U),$$

$$\alpha|_V = \mathrm{d}\eta, \quad \eta\in\Omega^{k-1}(V).$$

The difference of $\mathrm{d}\eta$ and $\mathrm{d}\xi$ is closed on the overlap, $\mathrm{d}(\xi-\eta)=0$ on $U\cap V\simeq S^{n.1}$. By the assumption $H^{k-1}(S^{n-1})=0$ we can write $\xi-\eta=\mathrm{d}\zeta$ on $U\cap V$ with $\zeta\in\Omega^{k-2}(U\cap V)$. We would like to match ξ and η by modifying them by adding an exact form (so that $\mathrm{d}\xi$ and $\mathrm{d}\eta$ remain unchanged). However, the most obvious modification of just adding or subtracting $\mathrm{d}\xi$ doesn't work, because ξ is defined only on $U\cap V$ and doesn't necessarily extend smoothly to S or N. We overcome this potential obstacle by "partitioning" ζ into two parts supported away from S and N, respectively. We decompose the unity with $1=\varphi+\psi$ such that $\varphi,\psi\in\mathcal{C}^\infty(S^1)$ and $\operatorname{supp}\varphi\subset V$ and $\operatorname{supp}\psi\subset U$. This is just the smooth partition of unity subordinate to the cover U,V. Now decompose

$$\zeta=\varphi\zeta+\psi\zeta\quad\text{on } U\cap V.$$

The first part smoothly extends to the North Pole (to U) by 0 and the second part extends to the South Pole (to V) by 0. We use the modification

$$\tilde{\xi}\coloneqq\xi-\mathrm{d}(\varphi\zeta)\in\Omega^{k-1}(U),$$

$$\tilde{\eta}\coloneqq\eta+\mathrm{d}(\psi\zeta)\in\Omega^{k-1}(V).$$

Then $\tilde{\xi}=\tilde{\eta}$ on $U\cap V$ and $\tilde{\xi},\tilde{\eta}$ are both restrictions of a globally defined form $\tilde{\beta}\in\Omega^{k-1}(S^n)$. Hence, $\alpha=\mathrm{d}\beta$ is exact. □

Since $H^1(S^{\geq 2})=0$ it follows $H^k(S^n)=0$ for $1\leq k<n$. Finally, we determine $H^n(S^n)$ for $n\geq 2$.

Claim V.8 (S^n)**.** *The natural surjective linear map*

$$\int_{S^n}:H^n(S^n)\to\mathbb{R}$$

is an isomorphism.

Proof. We need to show that for $\alpha\in\Omega^n(S^n)$

$$\int_{S^n}\alpha=0\implies\alpha\text{ is exact.}\tag{$*_{S^n}$}$$

We localize the problem. By Poincaré lemma (closed $\implies$ locally exact) we can write $\alpha=\mathrm{d}\eta$ on some open neighbourhood of the South Pole S, where $\eta\in\Omega^{n-1}(S^n)$ is globally defined obtained by cutting off. Then $\alpha-\mathrm{d}\eta$ is supported away from the South Pole (which means supported on U) and

$$\int_{S^n}(\alpha-\mathrm{d}\eta)\overset{\text{Stokes}}{=}\int_{S^n}\alpha\overset{\text{assu.}}{=}0.$$

We can therefore replace α by $\alpha-\mathrm{d}\eta$, and we are reduced to showing that for $\alpha\in\Omega^n_{\mathrm{cpt}}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n}\alpha=0\implies\alpha=\mathrm{d}\beta$$

with $\beta\in\Omega^{n-1}_{\mathrm{cpt}}(\mathbb{R}^n)$. For the integral we fix the orientation of $\mathbb{R}^n$ and note that $\mathbb{R}^n\cong U$. This will allow us from U to S^n in Eq. ($*_{S^n}$). This amounts to proving the following claim. □

Claim V.9 ($\mathbb{R}^n$)**.** *The natural surjective linear map*

$$\int_{\mathbb{R}^n} : H^n_{\text{cpt}}(\mathbb{R}^n) \to \mathbb{R}$$

is an isomorphism. The symbol H_{cpt} denotes the de Rham cohomology with compact supports.

Our discussion shows that Claim V.9 (on $\mathbb{R}^n$) implies Claim V.8 (on S^n) for $n \geq 1$. Since we have already proven Claim V.8 for S^1, the missing piece is the implication Claim V.8 on S^{n-1} $\implies$ Claim V.9 on $\mathbb{R}^n$ for $n \geq 2$.

Proof. Let $\alpha \in \Omega^n_{\text{cpt}}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \alpha = 0$ and $\operatorname{supp}\alpha \subset B \subset \mathbb{R}^n$ (where B is an open ball). By Poincaré lemma $\alpha = \mathrm{d}\tilde{\eta}$ with $\tilde{\eta} \in \Omega^{n-1}(\mathbb{R}^n)$. However, $\tilde{\eta}$ is not necessarily compactly supported. But $\mathrm{d}\tilde{\eta} = 0$ outside $\operatorname{supp}\alpha$ and

$$\int_{\partial B} \tilde{\eta} \overset{\text{Stokes}}{=} \int_{\overline{B}} \alpha = \int_{\mathbb{R}^n} \alpha \overset{\text{assu.}}{=} 0.$$

It also holds that

$$\tilde{\eta}|_{\partial B} \in \Omega^{n-1}(\partial B) = \Omega^{n-1}(S^{n-1})$$

is closed. The Claim V.8 on S^{n-1} implies

$$[\tilde{\eta}|_{\partial\beta}] = 0$$

in $H^{n-1}(\partial B)$. Choose a slightly smaller ball B', such that

$$\operatorname{supp}\alpha \subset B' \subset \overline{B}' \subset B.$$

Then $\mathbb{R}^n - \overline{B}'$ deformation retracts to ∂B and

$$[\tilde{\eta}|_{\mathbb{R}^n - \overline{B}'}] = 0 \text{ in } H^{n-1}(\mathbb{R}^n - \overline{B}') \cong H^{n-1}(\partial B).$$

Hence,

$$\tilde{\eta}|_{\mathbb{R}^n - \overline{B}'} = \mathrm{d}\zeta \text{ with globally defined form } \zeta \in \Omega^{n-1}(\mathbb{R}^n - \overline{B}')$$

and

$$\tilde{\eta}|_{\mathbb{R}^n - B} = \mathrm{d}\tilde{\zeta} \text{ with } \tilde{\zeta} \in \Omega^{n-1}(\mathbb{R}^n)$$

is globally defined. This is a cut-off in annulus $B - \overline{B}'$ and extension to $\overline{B}'$ by zero. Now modify

$$\eta \coloneqq \tilde{\eta} - \mathrm{d}\overline{\zeta} \in \Omega^{n-1}_{\text{cpt}}(\mathbb{R}^n).$$

Then $\alpha = \mathrm{d}\tilde{\eta} = \mathrm{d}\eta$. □

Altogether, we conclude using induction that Claim V.8 for $S^{n\geq 1}$ and Claim V.9 for $\mathbb{R}^{n\geq 2}$ hold. Furthermore, Claim V.9 for $\mathbb{R}$ is an easy consequence of the fundamental theorem of calculus. Then

$$H^n(S^n) \cong \mathbb{R} \quad \text{and} \quad H^n_{\text{cpt}}(\mathbb{R}^n) \cong \mathbb{R}, \quad \text{for } n \geq 1.$$

Remark. We came close to proving the more general result that if M^n is connected oriented smooth manifold, then

$$\int_M : H^n_{\text{cpt}}(M) \to \mathbb{R}$$

is an isomorphism.

The cohomology of S^n for $n \geq 1$ is thus

$$H^k(S^n) = \begin{cases} \mathbb{R} & \text{for } k \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

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